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LIMINF AND LIMSUP CONTRACTIONS

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ABSTRACT. We give some theorems related to the contraction mapping principle of Banach-Caccioppoli and Edelstein. The contractive conditions we consider involve the quantities $\liminf_{\xi\to\cdot} d(\xi, f\xi)$ and $\limsup_{\xi\to\cdot} d(\xi, f\xi)$ instead of $d(\cdot, f\cdot)$. Some examples are provided to show the difference between our results and the classical ones.

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1. Introduction. One of the simplest and most useful results in the fixed point theory is the Banach-Caccioppoli contraction mapping principle (see [1, 2]), which in the general setting of complete metric spaces reads as follows.

THEOREM 1.1. *Let* (X,d) *be a complete metric space,* $f : X \to X$ *a mapping and* $c \in [0,1[$ *such that*

$$d(fx, fy) \le c\,d(x,y), \quad \forall x,y \in X; \tag{1.1}$$

then

(i) *there exists a point* $a \in X$ *such that for each* $x \in X$, $\lim_{n\to+\infty} f^n x = a$;
(ii) a *is the unique fixed point for* f;
(iii) *for each* $x \in X$, $d(f^n x, a) \le c^n/(1-c)d(x, fx)$.

In 1962, in the case of compact metric spaces, Edelstein in [3] has proved the following generalization of the contraction mapping principle.

THEOREM 1.2. *Let* (X,d) *be a compact metric space and* $f : X \to X$ *be a mapping such that*

$$d(fx, fy) < d(x,y), \quad \forall x,y \in X,\ x \ne y; \tag{1.2}$$

then there exists a unique fixed point for f.

The main results of this paper are related to Theorems 1.1 and 1.2 in which we have replaced the contractive conditions above by similar ones using the mappings $\phi(\cdot) := \liminf_{\xi\to\cdot} d(\xi, f\xi)$ and $\psi(\cdot) := \limsup_{\xi\to\cdot} d(\xi, f\xi)$ instead of $d(\cdot, f\cdot)$.

Before stating our theorems we need some notations and definitions: by $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{R}$, and $\mathbb{R}^+$ we denote, respectively, the sets of integers, nonnegative integers, real numbers and nonnegative real numbers; let now X' be the cluster set of X and $\varphi : X \to \mathbb{R}^+$ be a real-valued mapping, then φ is called *(weak) lower semicontinuous at* $x \in X'$ if and only if

$$\varphi(x) \le \liminf_{\xi\to x} \varphi(\xi) \quad \Big(\varphi(x) \le \limsup_{\xi\to x} \varphi(\xi)\Big); \tag{1.3}$$

if this happens for all $x \in X'$ then we simply say that φ is a *(weak) lower semicontinuous* mapping. Finally, taking a mapping $f : X \to X$ then φ is said to be *f-orbitally (weak) lower semicontinuous at $a \in X$* if and only if for each $x \in X$ and for each sequence $(\xi_n)_{n\in\mathbb{Z}^+}$ in $O_\infty(x) := \{f^n x; n \in \mathbb{Z}^+\}$ (the *orbit* of x) converging to a, one has

$$\varphi(a) \le \liminf_{n\to+\infty} \varphi(\xi_n) \quad \Big(\varphi(a) \le \limsup_{n\to+\infty} \varphi(\xi_n)\Big). \tag{1.4}$$

2. Main results. We are now ready to state and prove our results; the first two are related to Theorem 1.1, while the third one is related to Theorem 1.2.

Theorem 2.1. *Let (X,d) be a complete metric space such that $X' \neq \emptyset$ and let $f : X \to X$ be a mapping such that $f(X') \subseteq X'$. Suppose that there exists a point $x \in X'$ such that*

$$\liminf_{\xi\to x} d(\xi, f\xi) < +\infty \tag{2.1}$$

and that the mapping $\varphi(\cdot) := d(\cdot, f\cdot)$ is lower semicontinuous. Finally, let $c \in [0,1[$ be such that

$$\liminf_{\eta\to fx} d(\eta, f\eta) \le c \liminf_{\xi\to x} d(\xi, f\xi) \quad \forall x \in X'; \tag{2.2}$$

then for all $x \in X'$ satisfying (2.1),

(i) *there exists a point $a \in X$ such that $f^n x \to a$ as $n \to +\infty$;*

(ii) *$fa = a$;*

(iii) *$d(f^n x, a) \le c^n/(1-c) \liminf_{\xi\to x} d(\xi, f\xi)$.*

Proof. Let $x \in X'$ be such that (2.1) holds and consider the sequence $(f^n x)_{n\in\mathbb{Z}^+}$, thus for $n \in \mathbb{Z}^+$ we have

$$\liminf_{\xi\to f^n x} d(\xi, f\xi) \le c \liminf_{\xi\to f^{n-1}x} d(\xi, f\xi) \le \cdots \le c^n \liminf_{\xi\to x} d(\xi, f\xi) < +\infty, \tag{2.3}$$

further, by the lower semicontinuity of φ, one has

$$d(f^n x, f^{n+1}x) \le \liminf_{\xi\to f^n x} d(\xi, f\xi) \le c^n \liminf_{\xi\to x} d(\xi, f\xi), \tag{2.4}$$

thus

$$\sum_{n=0}^{+\infty} d(f^n x, f^{n+1}x) \le \liminf_{\xi\to x} d(\xi, f\xi) \sum_{n=0}^{+\infty} c^n < +\infty; \tag{2.5}$$

this implies that $(f^n x)_{n\in\mathbb{Z}^+}$ is a Cauchy sequence so that, for the completeness of (X,d), there exists $\lim_{n\to+\infty} f^n x = a \in X$ so that (i) is proved. Further, again by the lower semicontinuity of φ, one has

$$d(a, fa) \le \liminf_{\xi\to a} d(\xi, f\xi) \le \lim_{n\to+\infty} d(f^n x, f^{n+1}x) = 0, \tag{2.6}$$

thus $fa = a$ and (ii) is proved. Finally, to see the validity of (iii), we note that

$$\begin{aligned} d(f^n x, f^{n+m} x) &\le d(f^n x, f^{n+1} x) + \cdots + d(f^{n+m-1} x, f^{n+m} x) \\ &\le (c^n + \cdots + c^{n+m-1}) \liminf_{\xi \to x} d(\xi, f\xi) \\ &= c^n (1 + \cdots + c^{m-1}) \liminf_{\xi \to x} d(\xi, f\xi) \end{aligned} \tag{2.7}$$

so that, letting $m \to +\infty$, one has

$$d(f^n x, a) \le \frac{c^n}{1-c} \liminf_{\xi \to x} d(\xi, f\xi), \tag{2.8}$$

which proves item (iii). □

Theorem 2.2. *Let (X,d) be a complete metric space such that $X' \neq \emptyset$ and let $f : X \to X$ be a mapping such that $f(X') \subseteq X'$. Suppose that there exists a point $x \in X'$ such that*

$$\limsup_{\xi \to x} d(\xi, f\xi) < +\infty \tag{2.9}$$

and that the mapping $\varphi(\cdot) := d(\cdot, f\cdot)$ is weak lower semicontinuous. Finally, let $c \in [0,1[$ be such that

$$\limsup_{\eta \to fx} d(\eta, f\eta) \le c \limsup_{\xi \to x} d(\xi, f\xi) \quad \forall x \in X'; \tag{2.10}$$

then for all $x \in X'$ satisfying (2.9)

(i) *there exists a point $a \in X$ such that $f^n x \to a$ as $n \to +\infty$;*

(ii) *$fa = a$ if and only if φ is f-orbitally weak lower semicontinuous at a;*

(iii) *$d(f^n x, a) \le c^n/(1-c) \limsup_{\xi \to x} d(\xi, f\xi)$.*

Proof. We start with a point $x \in X'$ such that (2.9) holds and consider the sequence $(f^n x)_{n \in \mathbb{Z}^+}$, thus as in the previous proof we have

$$\limsup_{\xi \to f^n x} d(\xi, f\xi) \le c \limsup_{\xi \to f^{n-1} x} d(\xi, f\xi) \le \cdots \le c^n \limsup_{\xi \to x} d(\xi, f\xi) < +\infty; \tag{2.11}$$

using the weak lower semicontinuity of φ one now has

$$d(f^n x, f^{n+1} x) \le \limsup_{\xi \to f^n x} d(\xi, f\xi) \le c^n \limsup_{\xi \to x} d(\xi, f\xi) \tag{2.12}$$

thus

$$\sum_{n=0}^{+\infty} d(f^n x, f^{n+1} x) \le \limsup_{\xi \to x} d(\xi, f\xi) \sum_{n=0}^{+\infty} c^n < +\infty; \tag{2.13}$$

hence $(f^n x)_{n \in \mathbb{Z}^+}$ is a Cauchy sequence in the complete metric space (X,d), so that there exists $a = \lim_{n \to +\infty} f^n x$, thus (i) is proved. Now let $a = fa$, then

$$d(a, fa) = 0 \le \limsup_{k \to +\infty} d(f^{n_k} y, f^{n_k+1} y), \tag{2.14}$$

which is true for each subsequence $(f^{n_k}y)_{k\in\mathbb{Z}^+}$ of $(f^n y)_{n\in\mathbb{Z}^+}$ converging to a as $k\to+\infty$, that is, φ is f-orbitally weak lower semicontinuous at a. Conversely, suppose the f-orbitally weak lower semicontinuity of φ at a, then

$$d(a,fa)\le\limsup_{n\to+\infty}d(f^n x,f^{n+1}x)=0, \tag{2.15}$$

that is, $fa=a$, which guarantees (ii). Finally, as in the proof of Theorem 2.1, we have

$$\begin{aligned}d(f^n x,f^{n+m}x)&\le d(f^n x,f^{n+1}x)+\cdots+d(f^{n+m-1}x,f^{n+m}x)\\&\le(c^n+\cdots+c^{n+m-1})\limsup_{\xi\to x}d(\xi,f\xi)\\&=c^n(1+\cdots+c^{m-1})\limsup_{\xi\to x}d(\xi,f\xi)\end{aligned} \tag{2.16}$$

and thus, as $m\to+\infty$, one has

$$d(f^n x,a)\le\frac{c^n}{1-c}\limsup_{\xi\to x}d(\xi,f\xi), \tag{2.17}$$

which proves item (iii). □

Theorem 2.3. *Let (X,d) be a compact metric space such that $X'\ne\emptyset$ and let $f:X\to X$ be a mapping such that $f(X')\subseteq X'$. Suppose that for each $x\in X'$ such that $\liminf_{\xi\to x}d(\xi,f\xi)\ne 0$*

$$\liminf_{\eta\to fx}d(\eta,f\eta)<\liminf_{\xi\to x}d(\xi,f\xi) \tag{2.18}$$

and that the mapping $\varphi(\cdot):=d(\cdot,f\cdot)$ is lower semicontinuous; then f has a fixed point.

Proof. We define $\phi:X'\to[0,+\infty]$ by

$$\phi(x)\overset{\text{def}}{=}\liminf_{\xi\to x}d(\xi,f\xi), \tag{2.19}$$

thus we can observe that such ϕ is lower semicontinuous, in fact for each $a\in(X')'$ one has

$$\liminf_{x\to a}\phi(x)=\liminf_{x\to a}\liminf_{\xi\to x}d(\xi,f\xi)\ge\liminf_{x\to a}d(x,fx)=\phi(a). \tag{2.20}$$

Further, ϕ is defined on the compact set X', in fact it is a closed subset of the compact set X, thus ϕ has a minimum on X'; we call it a, that is,

$$\phi(a)=\min_{x\in X'}\phi(x). \tag{2.21}$$

We now claim that $\phi(a)=0$, in fact suppose by contradiction that this is false, then by the hypotheses we have $fa\in X'$ and

$$\phi(fa)=\liminf_{\eta\to fa}d(\eta,f\eta)<\liminf_{\xi\to a}d(\xi,f\xi)=\phi(a), \tag{2.22}$$

but this contradicts the minimality of $a\in X'$, thus $\phi(a)=0$.

Now for the lower semicontinuity of φ one has

$$d(a,fa)\le\liminf_{\xi\to a}d(\xi,f\xi)=\phi(a)=0, \tag{2.23}$$

thus $fa=a$, and the theorem is proved. □

Mappings satisfying (2.2), (2.10), and (2.18) will be called in the next section, respectively, *liminf contractions, limsup contractions*, and *weak liminf contractions*.

3. Some remarks and examples.

In this section, we give some remarks and examples concerning liminf, limsup, and weak liminf contractions which are not classical ones; we make use of the following notations: $1/2^\infty := 0$ (for ∞ we mean $+\infty$), $P := \{2k \mid k \in \mathbb{Z}\}$, and $D := \{2k+1 \mid k \in \mathbb{Z}\}$.

Remark 3.1. All the contractive conditions we have considered have a local character in the sense that they do not involve two generic points of the underlying space, but a single points and its orbit; under this aspects the results in this paper are related more to [4, Hicks-Rhoades theorem] rather than to Banach-Caccioppoli principle.

Remark 3.2. It is obvious that a Banach-Caccioppoli or an Edelstein contraction f is also, respectively, a liminf and limsup or weak liminf contraction if it satisfies the additional hypothesis $f(X') \subseteq X' \neq \emptyset$, so that our results are sometimes generalizations of the classical ones; the opposite is not true as we will see in the sequel.

Example 3.3. We consider the metric space (X,d) where

$$X \overset{\text{def}}{=} \left\{\left(\frac{1}{2^m}, \frac{1}{2^n}\right) \mid m,n \in \mathbb{Z}\cup\{+\infty\},\ n \geq m\right\} \tag{3.1}$$

and for each $(2^{-m},2^{-n}),(2^{-s},2^{-t}) \in X$

$$d\left[\left(\frac{1}{2^m}, \frac{1}{2^n}\right),\left(\frac{1}{2^s}, \frac{1}{2^t}\right)\right] \overset{\text{def}}{=} \left|\frac{1}{2^m} - \frac{1}{2^s}\right| + \left|\frac{1}{2^n} - \frac{1}{2^t}\right|. \tag{3.2}$$

This metric is equivalent to the Euclidean one in $\mathbb{R}^2$ (restricted to X), and X is a closed subset of $\mathbb{R}^2$ so that (X,d) is actually a complete metric space.

Now consider the mapping $f : X \to X$ defined by

$$f\left(\frac{1}{2^m}, \frac{1}{2^n}\right) \overset{\text{def}}{=} \begin{cases} (0,0) & \text{if } m = n = \infty, \\ \left(\dfrac{1}{2^{m+3}}, 0\right) & \text{if } m \in P,\ n = \infty, \\ \left(\dfrac{1}{2^{m-1}}, 0\right) & \text{if } m \in D,\ n = \infty, \\ \left(\dfrac{1}{2^{m+3}}, \dfrac{1}{2^{n+3}}\right) & \text{if } m \in P,\ n \neq \infty, \\ \left(\dfrac{1}{2^{m-2}}, \dfrac{1}{2^{n+2}}\right) & \text{if } m \in D,\ n \neq \infty. \end{cases} \tag{3.3}$$

It is easy to see that $X' = \{(2^{-m},0) \mid m \in \mathbb{Z}\cup\{+\infty\}\}$ and that $f(X') \subseteq X'$.

Let now $x = (2^{-2k},0)$ with $k \in \mathbb{Z}$, then $fx = (2^{-(2k+3)},0)$ and

$$\begin{aligned} \liminf_{\eta\to fx} d(\eta,f\eta) &= \lim_{n\to+\infty}\left[\left|\frac{1}{2^{2k+3}} - \frac{1}{2^{2k+1}}\right| + \left|\frac{1}{2^n} - \frac{1}{2^{n+2}}\right|\right] = \frac{3}{2^{2k+3}}, \\ \liminf_{\xi\to x} d(\xi,f\xi) &= \lim_{n\to+\infty}\left[\left|\frac{1}{2^{2k}} - \frac{1}{2^{2k+3}}\right| + \left|\frac{1}{2^n} - \frac{1}{2^{n+3}}\right|\right] = \frac{7}{2^{2k+3}}, \end{aligned} \tag{3.4}$$

thus

$$\liminf_{\eta\to fx} d(\eta,f\eta) \le \frac{3}{7}\liminf_{\xi\to x} d(\xi,f\xi), \tag{3.5}$$

$$d(x,fx) = \left|\frac{1}{2^{2k}} - \frac{1}{2^{2k+3}}\right| = \frac{7}{2^{2k+3}} = \liminf_{\xi\to x} d(\xi,f\xi), \tag{3.6}$$

while in the case $x = (2^{-(2k+1)},0)$ $(k \in \mathbb{Z})$ we have $fx = (2^{-2k},0)$ and

$$\begin{aligned}\liminf_{\eta\to fx} d(\eta,f\eta) &= \lim_{n\to+\infty}\left[\left|\frac{1}{2^{2k}} - \frac{1}{2^{2k+3}}\right| + \left|\frac{1}{2^n} - \frac{1}{2^{n+3}}\right|\right] = \frac{7}{2^{2k+3}},\\ \liminf_{\xi\to x} d(\xi,f\xi) &= \lim_{n\to+\infty}\left[\left|\frac{1}{2^{2k+1}} - \frac{1}{2^{2k-1}}\right| + \left|\frac{1}{2^n} - \frac{1}{2^{n+2}}\right|\right] = \frac{3}{2^{2k+1}} = \frac{12}{2^{2k+3}},\end{aligned} \tag{3.7}$$

thus

$$\liminf_{\eta\to fx} d(\eta,f\eta) \le \frac{7}{12}\liminf_{\xi\to x} d(\xi,f\xi), \tag{3.8}$$

$$d(x,fx) = \left|\frac{1}{2^{2k+1}} - \frac{1}{2^{2k}}\right| = \frac{1}{2^{2k+1}} < \frac{3}{2^{2k+1}} = \liminf_{\xi\to x} d(\xi,f\xi). \tag{3.9}$$

In short for each $x \in X'$ one has

$$\liminf_{\eta\to fx} d(\eta,f\eta) \le \frac{7}{12}\liminf_{\xi\to x} d(\xi,f\xi) < +\infty \tag{3.10}$$

and, by (3.6) and (3.9), the mapping $y \mapsto d(y,fy)$ is lower semicontinuous, so that all the hypotheses of Theorem 2.1 are satisfied (in fact f has $(0,0)$ as fixed point), but f is not a contraction in the sense of Hicks and Rhoades (see [4]), in fact for $x = (2^{-(2k+1)},0)$ $(k \in \mathbb{Z})$ we have $(fx = (2^{-2k},0)$, $f^2x = (2^{-(2k+3)},0))$:

$$d(fx,f^2x) = \left|\frac{1}{2^{2k}} - \frac{1}{2^{2k+3}}\right| = \frac{7}{2^{2k+3}} > \frac{1}{2^{2k+1}} = \left|\frac{1}{2^{2k+1}} - \frac{1}{2^{2k}}\right| = d(x,fx). \tag{3.11}$$

The mapping of this example is actually both a liminf and a limsup contraction, but starting from it we give two other examples: in the first one (Example 3.4) the mapping we give is a liminf but not limsup contraction, while the opposite is true in Example 3.5 (all the details are left to the interested reader).

Example 3.4. Let (X,d) be as above, we consider the following mapping:

$$f\left(\frac{1}{2^m},\frac{1}{2^n}\right) \overset{\text{def}}{=} \begin{cases} (0,0) & \text{if } m = n = \infty,\\ \left(\dfrac{1}{2^{m+3}},0\right) & \text{if } m \in P,\ n = \infty,\\ \left(\dfrac{1}{2^{m-1}},0\right) & \text{if } m \in D,\ n = \infty,\\ \left(\dfrac{1}{2^{m+3}},\dfrac{1}{2^{n+3}}\right) & \text{if } m \in P,\ n \ne \infty,\\ \left(\dfrac{1}{2^{m-2}},\dfrac{1}{2^{n+2}}\right) & \text{if } m \in D,\ n \in P,\\ \left(\dfrac{1}{2^{m-3}},\dfrac{1}{2^{n+3}}\right) & \text{if } m,n \in D.\end{cases} \tag{3.12}$$

It is easy to see that f is a liminf but not limsup contraction, in fact for $x = (2^{-2k},0)$ $(k \in \mathbb{Z})$ one has

$$\begin{aligned}
\liminf_{\eta\to fx} d(\eta,f\eta) &= \frac{3}{2^{2k+3}} = \frac{3}{7}\cdot\frac{7}{2^{2k+3}} = \frac{3}{7}\liminf_{\xi\to x} d(\xi,f\xi),\\
\limsup_{\eta\to fx} d(\eta,f\eta) &= \frac{7}{2^{2k+3}} = \limsup_{\xi\to x} d(\xi,f\xi).
\end{aligned}\tag{3.13}$$

Example 3.5. Let (X,d) be as in Examples 3.4 and 3.5 and let f be the following mapping:

$$f\left(\frac{1}{2^m},\frac{1}{2^n}\right) \overset{\text{def}}{=} \begin{cases} (0,0) & \text{if } m = n = \infty,\\ \left(\frac{1}{2^{m+3}},0\right) & \text{if } m \in P,\ n = \infty,\\ \left(\frac{1}{2^{m-1}},0\right) & \text{if } m \in D,\ n = \infty,\\ \left(\frac{1}{2^{m+3}},\frac{1}{2^{n+3}}\right) & \text{if } m \in P,\ n \neq \infty,\\ \left(\frac{1}{2^{m-2}},\frac{1}{2^{n+2}}\right) & \text{if } m,n \in D,\\ \left(\frac{1}{2^{m-1}},\frac{1}{2^{n+1}}\right) & \text{if } m \in D,\ n \in P. \end{cases}\tag{3.14}$$

Now f is a limsup but not liminf contraction, in fact for $x = (2^{-(2k+1)},0)$ $(k \in \mathbb{Z})$ one has

$$\begin{aligned}
\limsup_{\eta\to fx} d(\eta,f\eta) &= \frac{7}{2^{2k+3}} = \frac{7}{12}\cdot\frac{12}{2^{2k+3}} = \frac{7}{12}\limsup_{\xi\to x} d(\xi,f\xi),\\
\liminf_{\eta\to fx} d(\eta,f\eta) &= \frac{7}{2^{2k+3}} > \frac{1}{2^{2k+1}} = \liminf_{\xi\to x} d(\xi,f\xi).
\end{aligned}\tag{3.15}$$

The final example shows that a weak liminf contraction may not be an Edelstein one (even in this case all the details are left to the interested reader).

Example 3.6. Let X be the following set:

$$X \overset{\text{def}}{=} \left\{\left(\frac{1}{m},\frac{1}{n}\right) \mid m,n \in (\mathbb{Z}^+ - \{0\}) \cup \{\infty\}\ n \geq m\right\},\tag{3.16}$$

and let $d : X^2 \to \mathbb{R}^+$ be a mapping such that for each $(x,y),(u,v) \in X$ one has

$$d[(x,y),(u,v)] \overset{\text{def}}{=} \begin{cases} \max\{x,u\} + |y-v| & \text{if } x \neq u,\\ |y-v| & \text{if } x = u, \end{cases}\tag{3.17}$$

with the definitions above it is easy to see that (X,d) becomes a compact metric space and that $X' = \{(1/m,0) \mid m \in (\mathbb{Z}^+ - \{0\}) \cup \{\infty\}\}$. We define the mapping $f : X \to X$ in the following manner (in this example D and P denote, respectively, the set of positive odd numbers and the set of positive even numbers):

$$f\left(\frac{1}{m},\frac{1}{n}\right) \stackrel{\text{def}}{=} \begin{cases} (0,0) & \text{if } m = n = \infty, \\ \left(\frac{1}{m-1},0\right) & \text{if } m \in D-\{1\},\ n = \infty, \\ \left(\frac{1}{3},0\right) & \text{if } m = 1,\ n = \infty, \\ \left(\frac{1}{m+3},0\right) & \text{if } m \in P,\ n = \infty, \\ \left(\frac{1}{m-2},\frac{1}{n+2}\right) & \text{if } m \in D-\{1\},\ n \in \mathbb{Z}^+-\{0\}, \\ \left(1,\frac{1}{n+1}\right) & \text{if } m = 1,\ n \in \mathbb{Z}^+-\{0\}, \\ \left(\frac{1}{m+3},\frac{1}{n+3}\right) & \text{if } m \in P,\ n \in \mathbb{Z}^+-\{0\}. \end{cases} \tag{3.18}$$

This mapping is now a weak liminf contraction, that is, it satisfies (2.18) and all the hypotheses of Theorem 2.3 but it does not satisfy Edelstein condition (1.2); further it is not a liminf contraction in the sense of Theorem 2.1, in fact one has

$$\sup_{x \in X'-\{(0,0)\}} \frac{\liminf_{\eta \to fx} d(\eta, f\eta)}{\liminf_{\xi \to x} d(\xi, f\xi)} = 1, \tag{3.19}$$

so that for every $c \in [0,1[$, (2.2) may not be satisfied.

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