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A CHARACTERIZATION OF HARMONIC FOLIATIONS BY THE VOLUME PRESERVING PROPERTY OF THE NORMAL GEODESIC FLOW

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We prove that a Riemannian foliation with the flat normal connection on a Riemannian manifold is harmonic if and only if the geodesic flow on the normal bundle preserves the Riemannian volume form of the canonical metric defined by the adapted connection.

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1. Introduction. Let (M, g_M) be a Riemannian manifold. A foliation $\mathcal{F}$ on M is *Riemannian* and g_M *bundle-like* if all the leaves are locally equi-distant to each other. Such a foliation is characterized by the property that a geodesic orthogonal to the foliation at one point is orthogonal everywhere. For a Riemannian foliation, considerable efforts have been made to give global characterizations of the property that it is harmonic, that is, all of its leaves are minimal submanifolds. For examples, a Riemannian foliation is harmonic if and only if either one of the following conditions holds: (1) it is an extremal of the energy functional for special variations (see [2]); (2) it is an extremal of the energy of the foliation under certain variations of the Riemannian metric of the manifold (see [1]). In this paper, we give a dynamical characterization of the harmonicity of a Riemannian foliation which has the flat normal connection in the sense of Oshikiri [4].

Let $\mathcal{F}$ be a Riemannian foliation of dimension p and codimension q on a Riemannian manifold M of dimension n $(p+q=n)$ with bundle-like metric g_M. Throughout, we work in the smooth category and the following notations are used:

- TM is the tangent bundle of M.
- L and $L^\perp$ are the tangent bundle and the normal bundle of $\mathcal{F}$, respectively.
- ΓTM, ΓL, and $\Gamma L^\perp$ are the spaces of sections of TM, L, and $L^\perp$, respectively.
- $\pi : TM \to L^\perp$, $\pi^\perp : TM \to L$, and $P_{\mathcal{F}} : L^\perp \to M$ are the canonical projections.
- ∇^M is the Levi-Civita connection associated with g_M.

Since $\mathcal{F}$ is Riemannian, there exists a unique torsion-free metric connection ∇ on $L^\perp$ which is called *adapted* and given as follows (see [2]): for $Z \in \Gamma L^\perp$,

$$\nabla_X Z = \begin{cases} \pi[X, Z] & \text{for } X \in \Gamma L, \\ \pi(\nabla^M_X Z) & \text{for } X \in \Gamma L^\perp. \end{cases} \tag{1.1}$$

Associated with the above connection there is a bundle map $C_{\mathcal{F}} : TL^\perp \to L^\perp$ called the *connection map associated with* $\mathcal{F}$ given as follows. For $\xi \in T_Z L^\perp$ with $(dP_{\mathcal{F}})(\xi) \neq 0$,

$$C_{\mathcal{F}}(\xi) = \nabla_{\dot\sigma(0)} Z, \tag{1.2}$$

where Z is a curve in $L^\perp$ such that $d/dt|_{t=0} Z = \xi$ and $\sigma(t) = P_{\mathcal{F}}(Z(t))$. This map gives a metric $\tilde g$ on $L^\perp$ defined by

$$\tilde g(\xi,\eta) = g_M\big((dP_{\mathcal{F}})_Z(\xi),(dP_{\mathcal{F}})_Z(\eta)\big) + g_M\big(C_{\mathcal{F}}(\xi),C_{\mathcal{F}}(\eta)\big) \tag{1.3}$$

for $\xi,\eta \in T_Z L^\perp$. We denote the Riemannian volume form on $L^\perp$ associated with $\tilde g$ by $\tilde\mu$.

We define a local flow ϕ_t on $L^\perp$, called the *normal geodesic flow* of $\mathcal{F}$ as follows. For $z \in L^\perp$, let γ be a geodesic with initial velocity z. Since $\mathcal{F}$ is Riemannian, $\dot\gamma(t) \in L^\perp$ for each t in the domain of γ. We put $\phi_t(z) = \dot\gamma(t)$ for $z \in L^\perp$ and t in the domain of γ.

A foliation $\mathcal{F}$ is said to *have the flat normal connection* if the normal bundle $L^\perp$ of $\mathcal{F}$ admits an orthonormal frame field $\{E_{p+1},\dots,E_n\}$ such that $g_M(\nabla^M_Z E_\alpha, E_\beta) = 0$ for all α, $\beta = p+1,\dots,n$ and all $Z \in \Gamma L^\perp$.

The purpose of this paper is to prove the following theorem.

Theorem 1.1. *Let $\mathcal{F}$ be a Riemannian foliation on a Riemannian manifold which has a flat normal connection and $\tilde\mu$ the Riemannian volume form on $L^\perp$ corresponding to $\tilde g$. Then $\mathcal{F}$ is harmonic if and only if (ϕ_t) preserves $\tilde\mu$.*

2. The proof.

Let ζ be a vector field on $L^\perp$ generated by the geodesic flow. It suffices to show that $\mathcal{F}$ is harmonic if and only if $(\Theta_\zeta\tilde\mu)(z) = 0$ at any given point $z \in L^\perp$, where Θ_ζ denotes the Lie derivative. Let $\{e_1,\dots,e_n\}$ be an orthonormal basis of the tangent space of M at the point $m = P_{\mathcal{F}}(z)$ such that $e_i \in L_m$ for $i = 1,\dots,p$ and $e_\alpha \in L_m^\perp$ for $\alpha = p+1,\dots,n$. In a neighborhood of m, we may choose a frame $\{E_\alpha : \alpha = p+1,\dots,n\}$ of $L^\perp$, called an *adapted frame*, satisfying the following properties: $E_\alpha(m) = e_\alpha$, $\alpha = p+1,\dots,n$, $\nabla_{e_\alpha} E_\beta = \pi(\nabla^M_{e_\alpha} E_\beta) = 0$ and $\nabla_X E_\alpha = \pi([X,E_\alpha]) = 0$ for any smooth section X of L on U (see [3]). Since $\mathcal{F}$ has the flat normal connection, we may choose E_α so that $\nabla_{E_\alpha} E_\beta = 0$ for α, $\beta = p+1,\dots,n$. Completing this frame by an orthonormal frame $\{E_i : i = 1,\dots,p\}$ of L with $E_i(m) = e_i$, we get a local orthonormal frame $\{E_1,\dots,E_n\}$ of TM on a neighborhood U of m with $E_A(m) = e_A$ for $A = 1,\dots,n$. Let E_A^H for $A = 1,\dots,n$ be the *horizontal lift* of E_A to $TL^\perp$, that is, the unique vector field on a neighborhood of z in $L^\perp$ such that $dP_{\mathcal{F}}(E_A^H) = E_A$ and $C_{\mathcal{F}}(E_A^H) = 0$, and E_α^V for $\alpha = p+1,\dots,n$ the *vertical lift* of E_α on $TL^\perp$, that is, the vector field on a neighborhood of z such that $dP(E_\alpha^V) = 0$ and $C_{\mathcal{F}}(E_\alpha^V) = E_\alpha$. We put $E_A^H(z) = e_A^H$ and $E_\alpha^V(z) = e_\alpha^V$. Now we compute

$$\begin{aligned}
&\big[(\Theta_\zeta\tilde\mu)(z)\big](e_1^H,\dots,e_n^H,e_{p+1}^V,\dots,e_n^V) \\
&\quad = -\sum_{i=1}^{p} \tilde\mu(e_1^H,\dots,[\zeta,E_i^H](z),\dots,e_p^H,e_{p+1}^H,\dots,e_n^H,e_{p+1}^V,\dots,e_n^V) \\
&\qquad - \sum_{\alpha=p+1}^{n} \tilde\mu(e_1^H,\dots,e_p^H,e_{p+1}^H,\dots,[\zeta,E_\alpha^H](z),\dots,e_n^H,e_{p+1}^V,\dots,e_n^V) \\
&\qquad - \sum_{\alpha=p+1}^{n} \tilde\mu(e_1^H,\dots,e_n^H,e_{p+1}^V,\dots,[\zeta,E_\alpha^V](z),\dots,e_n^V).
\end{aligned} \tag{2.1}$$

But,

$$\begin{aligned}
&\tilde{\mu}(e_1^H,\dots,[\zeta,E_i^H](z),\dots,e_p^H,e_{p+1}^H,\dots,e_n^H,e_{p+1}^V,\dots,e_n^V) \\
&\qquad = \tilde{g}([\zeta,E_i^H](z),e_i^H) = g_M((dP_{\mathcal{F}})[\zeta,E_i^H](m),e_i), \\
&\tilde{\mu}(e_1^H,\dots,e_p^H,e_{p+1}^H,\dots,[\zeta,E_\alpha^H](z),\dots,e_n^H,e_{p+1}^V,\dots,e_n^V) \\
&\qquad = g_M((dP_{\mathcal{F}})([\zeta,E_\alpha^H](z)),e_\alpha),
\end{aligned} \tag{2.2}$$

where $m = P_{\mathcal{F}}(z)$ and α is the second fundamental form of $\mathcal{F}$ (see [2]).

Let W_i be any vector field on M satisfying $W_i(\varphi_t^i m) = \tilde{\varphi}_t^i z$ for the local flows (φ_t^i) of E_i and $(\tilde{\varphi}_t^i)$ of E_i^H. From $dP_{\mathcal{F}} \circ E_i^H = E_i \circ P_{\mathcal{F}}$, we have $P_{\mathcal{F}} \circ \tilde{\varphi}_t^i = \varphi_t^i \circ P_{\mathcal{F}}$ for any t. Therefore,

$$\begin{aligned}
dP_{\mathcal{F}}([\zeta,E_i^H](z)) &= \frac{d}{dt}|_{t=0}(dP_{\mathcal{F}} \circ d\tilde{\varphi}_{-t}^i)(\zeta(\tilde{\varphi}_t^i(z))) \\
&= \frac{d}{dt}|_{t=0}(d\varphi_{-t}^i \circ dP_{\mathcal{F}})(\zeta(\tilde{\varphi}_t^i(z))) \\
&= \frac{d}{dt}|_{t=0}(d\varphi_{-t}^i \circ \tilde{\varphi}_t^i)(z) \\
&= \frac{d}{dt}|_{t=0}(d\varphi_{-t}^i)(W_i(\varphi_t^i(m))) \\
&= [W_i,E_i](m).
\end{aligned} \tag{2.3}$$

Hence we have

$$\begin{aligned}
g_M(dP_{\mathcal{F}}([\zeta,E_i^H](z)),E_i(z)) &= g_M([W_i,E_i],E_i)(m) \\
&= g_M(W_i,\nabla_{E_i}^M E_i)(m) \\
&= g_M(W_i(m),\alpha(E_i,E_i)(m)) \\
&= g_M(z,\alpha(E_i(m),E_i(m))).
\end{aligned} \tag{2.4}$$

Thus, we have

$$\begin{aligned}
&-\sum_{i=1}^{p}\tilde{\mu}(e_1^H,\dots,[\zeta,E_i^H](z),\dots,e_p^H,e_{p+1}^H,\dots,e_n^H,e_{p+1}^V,\dots,e_n^V) \\
&\qquad = -g_M\left(z,\sum_{i=1}^{p}\alpha(E_i(m),E_i(m))\right) \\
&\qquad = -g_M(z,\tau(m)),
\end{aligned} \tag{2.5}$$

where $\tau(m)$ is the mean curvature vector of $\mathcal{F}$ at m (see [2]).

On the other hand, we have

$$g_M((dP_{\mathcal{F}}[\zeta,E_\alpha^H])(m),e_\alpha) = g_M([W_\alpha,E_\alpha](m),e_\alpha), \tag{2.6}$$

where W_α is any vector field on M satisfying $W_\alpha(\varphi_t^\alpha m) = \tilde{\varphi}_t^\alpha z$ for the local flows φ_t^α of E_α and $\tilde{\varphi}_t^\alpha$ of E_α^H, $\alpha = p+1,\dots,n$. Since $W_\alpha(\varphi_t^\alpha m)$ is an integral curve of E_α^H, we have $\pi(\nabla_{E_\alpha}^M W_\alpha) = C_{\mathcal{F}}(E_\alpha^H) = 0$. Moreover, by the choice of $\{E_\alpha\}$, we have $\pi(\nabla_{W_\alpha}^M E_\alpha)(m) = 0$. Therefore,

$$g_M\big((dP_{\mathcal{F}}[\zeta, E_\alpha^H])(m), e_\alpha\big) = g_M\big((\nabla_{W_\alpha}^M E_\alpha)(m) - (\nabla_{E_\alpha}^M W_\alpha)(m), e_\alpha\big) = 0. \tag{2.7}$$

Thus, to complete the proof, it suffices to show that

$$\tilde{\mu}\big(e_1^H,\dots,e_n^H, e_{p+1}^V,\dots,[\zeta, E_\alpha^V](z),\dots,e_n^V\big) = 0, \tag{2.8}$$

that is,

$$g_M\big(C_{\mathcal{F}}([\zeta, E_\alpha^V](z)), e_\alpha\big) = 0. \tag{2.9}$$

For this purpose, we introduce a local coordinate system around a point $z \in L^\perp$ as follows: let $(x^A)_{A=1,\dots,n} : U \to \mathbb{R}^n$ be a distinguished chart on a neighborhood U of $m \in M$. To $z \in P_{\mathcal{F}}^{-1}(U)$ with $P_{\mathcal{F}}(z) = m$, we assign $(x^1(m),\dots,x^n(m), z^{p+1}(m),\dots,z^n(m))$ as its coordinates, where $z = \sum_{\alpha=p+1}^n z^\alpha(m) E_\alpha(m)$. Let γ be a geodesic orthogonal to the leaves of $\mathcal{F}$ and $(x^A(t) : A = 1,\dots,n)$ its local coordinates.

Write

$$\dot{\gamma}(t) = \sum_{\alpha=p+1}^{n} z^\alpha(t) E_\alpha(\gamma(t)). \tag{2.10}$$

By the choice of $\{E_\alpha\}$, we get

$$\frac{d}{dt} z^\alpha = 0 \tag{2.11}$$

for $\alpha = p+1,\dots,n$. Moreover, if we express E_α as $E_\alpha = \sum_{A=1}^n f_\alpha^A (\partial/\partial x^A)$, where f_α^A is a smooth function on U, we have

$$\sum_{A=1}^{n}\left(\frac{d}{dt}x^A\right)\frac{\partial}{\partial x^A} = \dot{\gamma} = \sum_{\alpha=p+1}^{n} z^\alpha E_\alpha = \sum_{\alpha=p+1}^{n}\sum_{A=1}^{n} z^\alpha f_\alpha^A \frac{\partial}{\partial x^A}. \tag{2.12}$$

Equations (2.10) and (2.11) imply that $(x^A(t), z^\alpha(t))$ satisfy

$$\frac{d}{dt}x^A = \sum_{\alpha=p+1}^{n} z^\alpha f_\alpha^A, \qquad \frac{d}{dt} z^\alpha = 0. \tag{2.13}$$

It follows that ζ can be locally expressed as

$$\zeta = \sum_{\alpha,A} z^\alpha f_\alpha^A \frac{\partial}{\partial x^A}. \tag{2.14}$$

A simple computation using the above expression of ζ gives

$$[\zeta, E_\alpha^V] = -\sum_A \left(f_\alpha^A + \sum_\beta z^\beta E_\alpha(f_\beta^A)\right)\frac{\partial}{\partial x^A}. \tag{2.15}$$

It is easy to show that for a vector field $\xi = \sum_A \xi^A(\partial/\partial x^A) + \sum_\alpha \tilde{\xi}^\alpha(\partial/\partial z^\alpha)$, $C_{\mathcal{F}}(\xi)$ is given by

$$C_{\mathcal{F}}(\xi) = \sum_\alpha \left(\tilde{\xi}^\alpha + \sum_{\beta,A} \Gamma^\alpha_{\beta A} z^\beta \xi^A \right) E_\alpha, \tag{2.16}$$

where $z = \sum_\alpha z^\alpha E_\alpha$ and $\nabla_{\partial_A} E_\alpha = \sum_{\gamma=p+1}^n \Gamma^\gamma_{\alpha A} E_\gamma$. Therefore,

$$C_{\mathcal{F}}([\zeta, E^V_\alpha]) = -\sum_{\delta,\sigma,A} \left\{ f^A_\alpha + \sum_\beta z^\beta E_\alpha(f^A_\beta) \right\} \Gamma^\delta_{\sigma A} z^\sigma E_\delta. \tag{2.17}$$

But $\Gamma^\delta_{\sigma A} = 0$ on U for $A = 1, \dots, n$ and $\delta, \sigma = p+1, \dots, n$ by the choice of the frame $\{E_A\}$. Hence $C_{\mathcal{F}}([\zeta, E^V_\alpha]) = 0$ and the proof is complete. □

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