IJMMS 29:2 (2002) 71–77
PII. S0161171202011705
http://ijmms.hindawi.com

NONCOMPLETE AFFINE STRUCTURES ON LIE ALGEBRAS OF MAXIMAL CLASS

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Received 29 January 2001

Every affine structure on Lie algebra $\mathfrak{g}$ defines a representation of $\mathfrak{g}$ in $\text{aff}(\mathbb{R}^n)$. If $\mathfrak{g}$ is a nilpotent Lie algebra provided with a complete affine structure then the corresponding representation is nilpotent. We describe noncomplete affine structures on the filiform Lie algebra L_n. As a consequence we give a nonnilpotent faithful linear representation of the 3-dimensional Heisenberg algebra.

2000 Mathematics Subject Classification: 53Cxx, 17Bxx.

1. Affine structure on a nilpotent Lie algebra

1.1. Affine structure on nilpotent Lie algebras

Definition 1.1. Let $\mathfrak{g}$ be an n-dimensional Lie algebra over $\mathbb{R}$. An affine structure is given by a bilinear mapping

$$\nabla : \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathfrak{g} \tag{1.1}$$

satisfying

$$\begin{gathered}\nabla(X,Y)-\nabla(Y,X)=[X,Y],\\ \nabla\big(X,\nabla(Y,Z)\big)-\nabla\big(Y,\nabla(X,Z)\big)=\nabla\big([X,Y],Z\big),\end{gathered} \tag{1.2}$$

for all $X,Y,Z\in\mathfrak{g}$.

If $\mathfrak{g}$ is provided with an affine structure, then the corresponding connected Lie group G is an affine manifold such that every left translation is an affine isomorphism of G. In this case, the operator ∇ is nothing but the connection operator of the affine connection on G.

Let $\mathfrak{g}$ be a Lie algebra with an affine structure ∇. Then the mapping

$$f : \mathfrak{g} \longrightarrow \text{End}(\mathfrak{g}), \tag{1.3}$$

defined by

$$f(X)(Y)=\nabla(X,Y), \tag{1.4}$$

is a linear representation (non faithful) of $\mathfrak{g}$ satisfying

$$f(X)(Y)-f(Y)(X)=[X,Y]. \tag{1.5}$$

Remark 1.2. The adjoint representation $\widetilde{f}$ of $\mathfrak{g}$ satisfies

$$\widetilde{f}(X)(Y)-\widetilde{f}(Y)(X)=2[X,Y] \tag{1.6}$$

and cannot correspond to an affine structure.

1.2. Classical examples of affine structures. (i) Let $\mathfrak{g}$ be the n-dimensional abelian Lie algebra. Then the representation

$$f:\mathfrak{g}\longrightarrow \mathrm{End}(\mathfrak{g}),\quad X\longmapsto f(X)=0 \tag{1.7}$$

defines an affine structure.

(ii) Let $\mathfrak{g}$ be a $2p$-dimensional Lie algebra endowed with a symplectic form

$$\theta\in\Lambda^2\mathfrak{g}^* \quad \text{such that } d\theta=0 \tag{1.8}$$

with

$$d\theta(X,Y,Z)=\theta(X,[Y,Z])+\theta(Y,[Z,X])+\theta(Z,[X,Y]). \tag{1.9}$$

For every $X\in\mathfrak{g}$ we can define a unique endomorphism ∇_X by

$$\theta(\mathrm{ad}\,X(Y),Z)=-\theta(Y,\nabla_X(Z)). \tag{1.10}$$

Then $\nabla(X,Y)=\nabla_X(Y)$ is an affine structure on $\mathfrak{g}$.

(iii) Following the work of Benoist [1] and Burde [2, 3, 4], we know that there exists a nilpotent Lie algebra without affine structures.

1.3. Faithful representations associated to an affine structure. Let ∇ be an affine structure on an n-dimensional Lie algebra $\mathfrak{g}$. We consider the $(n+1)$-dimensional linear representation given by

$$\rho:\mathfrak{g}\longrightarrow \mathrm{End}\,(\mathfrak{g}\bigoplus\mathbb{R}) \tag{1.11}$$

given by

$$\rho(X):(Y,t)\longmapsto (\nabla(X,Y)+tX,0). \tag{1.12}$$

It is easy to verify that ρ is a faithful representation of dimension $n+1$.

We can note that this representation gives also an affine representation of $\mathfrak{g}$

$$\psi:\mathfrak{g}\longrightarrow \mathrm{aff}\,(\mathbb{R}^n),\quad X\longmapsto \begin{pmatrix} A(X) & X\\ 0 & 0\end{pmatrix}, \tag{1.13}$$

where $A(X)$ is the matrix of the endomorphisms $\nabla_X:Y\to\nabla(X,Y)$ in a given basis.

Definition 1.3. We say that the representation ρ is nilpotent if the endomorphisms $\rho(X)$ are nilpotent for every X in $\mathfrak{g}$.

Proposition 1.4. *Suppose that $\mathfrak{g}$ is a complex non-abelian indecomposable nilpotent Lie algebra and let ρ be a faithful representation of $\mathfrak{g}$. Then there exists a faithful nilpotent representation of the same dimension.*

Proof. Consider the $\mathfrak{g}$-module M associated to ρ. Then, as $\mathfrak{g}$ is nilpotent, M can be decomposed as

$$M = \bigoplus_{i=1}^{k} M_{\lambda_i}, \tag{1.14}$$

where M_{λ_i} is a $\mathfrak{g}$-submodule, and the λ_i are linear forms on $\mathfrak{g}$. For all $X \in \mathfrak{g}$, the restriction of $\rho(X)$ to M_i is in the following form:

$$\begin{pmatrix} \lambda_i(X) & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \lambda_i(X) \end{pmatrix}. \tag{1.15}$$

Let $\mathbb{K}_{\lambda_i}$ be the one-dimensional $\mathfrak{g}$-module defined by

$$\mu : X \in \mathfrak{g} \longrightarrow \mu(X) \in \operatorname{End}\mathbb{K} \tag{1.16}$$

with

$$\mu(X)(a) = \rho(X)(a) = \lambda_i(X)a. \tag{1.17}$$

The tensor product $M_{\lambda_i} \otimes \mathbb{K}_{-\lambda_i}$ is the $\mathfrak{g}$-module associated to

$$X \cdot (Y \otimes a) = \rho(X)(Y) \otimes a - Y \otimes \lambda_i(X)a. \tag{1.18}$$

Then $\widetilde{M} = \bigoplus(M_{\lambda_i} \otimes K_{-\lambda_i})$ is a nilpotent $\mathfrak{g}$-module. We prove that $\widetilde{M}$ is faithful. Recall that a representation ρ of $\mathfrak{g}$ is faithful if and only if $\rho(Z) \neq 0$ for every $Z \neq 0 \in Z(\mathfrak{g})$. Consider $X \neq 0 \in Z(\mathfrak{g})$. If $\widetilde{\rho}(X) = 0$, then $\rho(X)$ is a diagonal endomorphism. By hypothesis $\mathfrak{g} \neq Z(\mathfrak{g})$ and there is $i \geq 1$ such that $X \in \mathscr{C}^i(\mathfrak{g})$, we have

$$X = \sum_j a_j [Y_j, Z_j] \tag{1.19}$$

with $Y_j \in \mathscr{C}^{i-1}(\mathfrak{g})$ and $Z_j \in \mathfrak{g}$. The endomorphisms $\rho(Y_j)\rho(Z_j) - \rho(Z_j)\rho(Y_j)$ are nilpotent and the eigenvalues of $\rho(X)$ are 0. Thus $\rho(X) = 0$ and ρ is not faithful. Then $\widetilde{\rho}(X) \neq 0$ and $\widetilde{\rho}$ is a faithful representation.

2. Affine structures on Lie algebra of maximal class

2.1. Definition

Definition 2.1. An n-dimensional nilpotent Lie algebra $\mathfrak{g}$ is called of maximal class if the smallest k such that $\mathscr{C}^k\mathfrak{g} = \{0\}$ is equal to $n-1$.

In this case the descending sequence is

$$\mathfrak{g} \supset \mathscr{C}^1\mathfrak{g} \supset \cdots \supset \mathscr{C}^{n-2}\mathfrak{g} \supset \{0\} = \mathscr{C}^{n-1}\mathfrak{g} \tag{2.1}$$

and we have

$$\begin{gathered} \dim \mathscr{C}^1\mathfrak{g} = n-2, \\ \dim \mathscr{C}^i\mathfrak{g} = n-i-1, \quad \text{for } i = 1, \ldots, n-1. \end{gathered} \tag{2.2}$$

Example 2.2. The n-dimensional nilpotent Lie algebra L_n defined by

$$[X_1, X_i] = X_{i+1} \quad \text{for } i \in \{2, \dots, n-1\} \tag{2.3}$$

is of maximal class.

We can note that any Lie algebra of maximal class is a linear deformation of L_n [5]. □

2.2. On non-nilpotent affine structure. Let $\mathfrak{g}$ be an n-dimensional Lie algebra of maximal class provided with an affine structure ∇. Let ρ be the $(n+1)$-dimensional faithful representation associated to ∇ and we note that $M = \mathfrak{g} \bigoplus \mathbb{C}$ is the corresponding complex $\mathfrak{g}$-module. As $\mathfrak{g}$ is of maximal class, its decomposition has one of the following forms

$$M = M_0, \quad M \text{ is irreducible}, \tag{2.4}$$

or

$$M = M_0 \bigoplus M_\lambda, \quad \lambda \neq 0. \tag{2.5}$$

For a general faithful representation, we call characteristic the ordered sequence of the dimensions of the irreducible submodules. In the case of maximal class we have $c(\rho) = (n+1)$ or $(n, 1)$ or $(n-1, 1, 1)$ or $(n-1, 2)$. In fact, the maximal class of $\mathfrak{g}$ implies that there exists an irreducible submodule of dimension greater than or equal to $n-1$. More generally, if the characteristic sequence of a nilpotent Lie algebra is equal to $(c_1, \dots, c_p, 1)$ (see [5]) then for every faithful representation ρ we have $c(\rho) = (d_1, \dots, d_q)$ with $d_1 \geq c_1$.

Theorem 2.3. *Let $\mathfrak{g}$ be the Lie algebra of the maximal class L_n. Then there are faithful $\mathfrak{g}$-modules which are not nilpotent.*

Proof. Consider the following representation given by the matrices $\rho(X_i)$ where $\{X_1, \dots, X_n\}$ is a basis of $\mathfrak{g}$

$$\rho(X_1) = \begin{pmatrix}
a & a & 0 & \cdots & \cdots & & & 0 & 1 \\
a & a & 0 & & & & & \vdots & 0 \\
0 & 0 & 0 & & & & & 0 & 0 \\
\vdots & \ddots & \frac{1}{2} & \ddots & & & & \vdots & 0 \\
\vdots & & \ddots & \ddots & \ddots & & & \vdots & 0 \\
\vdots & & & \ddots & \frac{i-3}{i-2} & \ddots & & \vdots & 0 \\
0 & 0 & & & \ddots & \ddots & \ddots & \vdots & 0 \\
\alpha & \beta & 0 & \cdots & \cdots & 0 & \frac{n-3}{n-2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$\rho(X_2) = \begin{pmatrix} a & a & 0 & \cdots & \cdots & & \cdots & 0 & 0 \\ a & a & 0 & & & & & \vdots & 1 \\ -1 & 1 & 0 & & & & & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \ddots & & & & \vdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots & 0 \\ \vdots & & & \ddots & \frac{1}{i-2} & \ddots & & \vdots & 0 \\ 0 & 0 & & & \ddots & \ddots & \ddots & \vdots & 0 \\ \beta & \alpha & 0 & \cdots & \cdots & \cdots & \frac{1}{n-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.6}$$

and for $j \geq 3$ the endomorphisms $\rho(X_j)$ satisfy

$$\begin{aligned} \rho(X_j)(e_1) &= -\frac{1}{j-1}e_{j+1}, \\ \rho(X_j)(e_2) &= \frac{1}{j-1}e_{j+1}, \\ \rho(X_j)(e_3) &= \frac{1}{j(j-1)}e_{j+2}, \\ &\vdots \\ \rho(X_j)(e_{i-j+1}) &= \frac{(j-2)!(i-j-1)!}{(i-2)!}e_i, \quad i = j-2,\dots,n, \\ \rho(X_j)(e_{i-j+1}) &= 0, \quad i = n+1,\dots,n+j-1, \\ \rho(X_j)(e_{n+1}) &= e_j, \end{aligned} \tag{2.7}$$

where $\{e_1,\dots,e_n,e_{n+1}\}$ is the basis given by $e_i = (X_i,0)$ and $e_{n+1} = (0,1)$. We easily verify that these matrices describe a nonnilpotent faithful representation. □

2.3. Noncomplete affine structure on L_n. The previous representation is associated to an affine structure on the Lie algebra L_n given by

$$\nabla(X_i,Y) = \rho(X_i)(Y,0), \tag{2.8}$$

where L_n is identified to the n-dimensional first factor of the $(n+1)$-dimensional faithful module. This affine structure is complete if and only if the endomorphisms $R_X \in \mathrm{End}(\mathfrak{g})$ defined by

$$R_X(Y) = \nabla(Y,X) \tag{2.9}$$

are nilpotent for all $X \in \mathfrak{g}$ (see [6]). But the matrix of R_{X_1} has the form

$$\begin{pmatrix} a & a & 0 & \cdots & 0 & \cdots & 0 & 0 \\ a & a & & & \vdots & & \vdots & 0 \\ 0 & -1 & & & \vdots & & \vdots & 0 \\ 0 & 0 & -\frac{1}{2} & \cdots & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & 0 & \ddots & & \cdots & \vdots & 0 \\ 0 & 0 & \vdots & \ddots & -\frac{1}{j-1} & & \vdots & 0 \\ \alpha & \beta & \vdots & \cdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{n-2} & 0 \end{pmatrix}. \tag{2.10}$$

Its trace is $2a$ and for $a \neq 0$ it is not nilpotent. We have proved the following proposition.

Proposition 2.4. *There exist affine structures on the Lie algebra of maximal class L_n which are noncomplete.*

Remark 2.5. The most simple example is on $\dim 3$ and concerns the Heisenberg algebra. We find a nonnilpotent faithful representation associated to the noncomplete affine structure given by

$$\nabla_{X_1} = \begin{pmatrix} a & a & 0 \\ a & a & 0 \\ \alpha & \beta & 0 \end{pmatrix}, \qquad \nabla_{X_2} = \begin{pmatrix} a & a & 0 \\ a & a & 0 \\ \beta - 1 & \alpha + 1 & 0 \end{pmatrix}, \qquad \nabla_{X_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.11}$$

where X_1, X_2, and X_3 are a basis of H_3 satisfying $[X_1, X_2] = X_3$ and ∇_{X_i} the endomorphisms of $\mathfrak{g}$ given by

$$\nabla_{X_i}(X_j) = \nabla(X_i, X_j). \tag{2.12}$$

The affine representation is written as

$$\begin{pmatrix} a(x_1 + x_2) & a(x_1 + x_2) & 0 & x_1 \\ a(x_1 + x_2) & a(x_1 + x_2) & 0 & x_2 \\ \alpha x_1 + (\beta - 1)x_2 & \beta x_1 + (\alpha + 1)x_2 & 0 & x_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.13}$$

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