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A GALERKIN METHOD OF $O(h^2)$ FOR SINGULAR BOUNDARY VALUE PROBLEMS

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We describe a Galerkin method with special basis functions for a class of singular two-point boundary value problems. The convergence is shown which is of $O(h^2)$ for a certain subclass of the problems.

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1. Introduction. We consider the class of singular two-point boundary value problems:

$$-\frac{1}{p}(pu')' + f(x,u) = 0, \quad 0 < x < 1, \qquad (pu')(0^+) = 0, \quad u(1) = 0. \tag{1.1}$$

We assume that the real-valued function p satisfies

$$p \geq 0, \quad p^{-1} \in L^1_{\text{loc}}(0,1], \qquad p^{-1} \notin L^1_{\text{loc}}([0,\alpha)) \quad \text{for any } \alpha > 0, \tag{1.2}$$

$$\int_x^1 p^{-1} \in L^1_p(0,1), \quad \text{that is,} \quad \int_0^1 \left(\int_x^1 \frac{1}{p(s)} ds\right) p(x)dx < \infty. \tag{1.3}$$

Note that (1.3) is clearly satisfied when p is an increasing function on $(0,1)$. We also assume that $f(x,u)$ is continuous in u such that for any real u, $f(\cdot,u) \in L^\infty_p(0,1)$,

$$q(u,v,x) \equiv \frac{f(x,u) - f(x,v)}{u - v} \geq 0 \quad \text{for } -\infty < u,\ v < \infty,\ u \neq v. \tag{1.4}$$

The singular two-point boundary value problems of the form (1.1) occur frequently in many applied problems, for example, in the study of electrohydrodynamics [9], in the theory of thermal explosions [4], in the separation of variables in partial differential equations [11]; see also [1]. There is a considerable literature on the numerical methods for the singular boundary value problems. Special finite difference methods were considered in Chawla et al. [5]. The Galerkin method for singular problems was considered in Ciarlet et al. [6], Eriksson et al. [7], Jesperson [8]. Ciarlet et al. [6] assumed that $p(x) > 0$ on $(0,1)$, $p \in C^1(0,1)$, and $p^{-1} \in L^1(0,1)$. In this paper, we address the problem with $p^{-1} \notin L^1(0,1)$, and we assume that $p \geq 0$, $p^{-1} \in L^1_{\text{loc}}(0,1)$; see (1.2) and (1.3). We investigate a Galerkin method with the same special patch functions considered by Ciarlet et al. [6] and we show that the method is of $O(h^2)$ when

p is an increasing function on $(0,1)$. The linear case with more general settings was considered in [2] and a nonlinear case was considered in [3]. The special case considered here requires a different approach to establish its order of convergence and to obtain the optimal order of convergence h^2 under an easily checked condition on p; namely that p is increasing on $[0,1]$.

2. Preliminaries. Let $I = (0,1)$ and $H = L_p^2(I)$ denote the weighted Hilbert space with the inner product

$$\langle u,v\rangle_H = \int_I u(x)v(x)p(x)dx. \tag{2.1}$$

Also let V be the Hilbert space consisting of functions $u \in L_p^2(I)$ which are locally absolutely continuous on I, $u(1) = 0$, and $u' \in L_p^2(I)$. The inner product on the space V is defined by

$$\langle u,v\rangle_V = \int_I u'(x)v'(x)p(x)dx. \tag{2.2}$$

The variational formulation of the problem (1.1) now follows:

Find $u \in V$ such that

$$a(u,v) = 0 \quad \forall v \in V, \tag{2.3}$$

where

$$a(u,v) \equiv \langle u,v\rangle_V + \int_0^1 f(x,u(x))v(x)p(x)dx. \tag{2.4}$$

It can be shown [3] that (1.1) and (2.3) have unique absolutely continuous (in $[0,1]$) solutions and that the weak solution of (2.3) coincides with the strong solution of (1.1).

3. The Galerkin approximation and convergence results. Let $\pi : 0 = x_0 < x_1 < \cdots < x_{N+1} = 1$ be a mesh on the interval $[0,1]$ and, for $i = 1,2,\ldots,N$, define the patch functions

$$r_i(x) = \begin{cases} r_i^-(x) & \text{if } x_{i-1} \le x \le x_i, \\ r_i^+(x) & \text{if } x_i \le x \le x_{i+1}, \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

where

$$\begin{aligned} r_1^-(x) &= 1, \\ r_i^-(x) &= \frac{\int_{x_{i-1}}^{x}(1/p(s))ds}{\int_{x_{i-1}}^{x_i}(1/p(s))ds}, \quad i = 2,3,\ldots,N, \\ r_i^+(x) &= \frac{\int_{x}^{x_{i+1}}(1/p(s))ds}{\int_{x_i}^{x_{i+1}}(1/p(s))ds}, \quad i = 1,2,\ldots,N. \end{aligned} \tag{3.2}$$

Define the discrete subspace V_N of V by

$$V_N = \operatorname{span}\{r_i\}_{i=1}^{N}. \tag{3.3}$$

The discrete version of the weak problem (2.3) reads:

Find $u^G \in V_N$ such that

$$a(u^G, v_N) = 0 \quad \forall v_N \in V_N. \tag{3.4}$$

Note that (3.4) has a unique solution $u^G \in AC[0,1]$. It follows from (2.3) and (3.4) that

$$\langle u - u^G, v_N \rangle_V + \int_0^1 \frac{f(x,u) - f(x,u^G)}{u - u^G}(u - u^G) v_N p = 0. \tag{3.5}$$

Let $\tilde{q}(x)$ be the unique function (because u and u^G are unique) defined by

$$\tilde{q}(x) \equiv \begin{cases} \dfrac{f(x,u(x)) - f(x,u^G(x))}{u(x) - u^G(x)}, & u(x) \neq u^G(x) \\ 0, & u(x) = u^G(x). \end{cases} \tag{3.6}$$

We assume that f is such that

$$C_{\tilde{q}} := \int_0^1 \tilde{q}(x) \int_x^1 \frac{ds}{p(s)} p(x)\,dx < \infty. \tag{3.7}$$

This is the case for example if f satisfies a Lipschitz condition in its second argument (see (1.3)). We can now state our results on the convergence of the Galerkin solution u^G to the weak solution u of (2.3).

Theorem 3.1. *The following relation holds:*

$$||u^G - u||_\infty \le (1 + 4C_{\tilde{q}})||f(\cdot,u(\cdot))||_\infty \ell(\pi_N), \tag{3.8}$$

where $\ell(\pi_N)$ is given by

$$\ell(\pi_N) = \max_{0 \le i \le N} \int_{x_i}^{x_{i+1}} \left(\int_s^{x_{i+1}} \frac{1}{p(t)} dt \right) p(s)\,ds. \tag{3.9}$$

Corollary 3.2. *If p is increasing then the method is $O(h^2)$ where*

$$h = \max_{0 \le i \le N} (x_{i+1} - x_i). \tag{3.10}$$

Remark 3.3. The absolute continuity of the solution u and the continuity of f imply that $\|f(\cdot,u(\cdot))\|_\infty < \infty$ in the above expression for the error.

4. Proof of the results. Let

$$u^G(x) = \sum_{i=1}^{N} \alpha_i r_i(x) \tag{4.1}$$

be the Galerkin approximation and u^I be the V_N-interpolant of the solution u given by

$$u^I(x) = \sum_{i=1}^{N} u_i r_i(x), \tag{4.2}$$

where $u_i = u(x_i)$ and r_i is given by (3.1), $i = 1,\dots,N$. We note here that u^I is the orthogonal projection of u with respect to the inner product $\langle\cdot,\cdot\rangle_V$:

$$\langle u - u^I, v_N\rangle_V = 0 \tag{4.3}$$

for all $v_N \in V_N$. The following relation is also easily checked (using (3.5) and (4.3))

$$\langle u^G - u^I, v_N\rangle_V = \langle \tilde{q}(u - u^G), v_N\rangle_p, \tag{4.4}$$

for all $v_N \in V_N$. We have the following lemma.

Lemma 4.1. *The following relation holds:*

$$\|u - u^I\|_\infty \le \|f(\cdot,u(\cdot))\|_\infty \ell(\pi_N). \tag{4.5}$$

Proof. For any $x \in [x_i, x_{i+1}]$, $i = 0,1,\dots,N$

$$u(x) - u^I(x) \le \int_{x_i}^{x_{i+1}} |g(s)| \left(\int_s^{x_{i+1}} \frac{dt}{p(t)}\right) p(s)\,ds, \tag{4.6}$$

where $g(s) = -f(s,u(s))$. To see this we consider two cases: $i = 0$ and $i \ge 1$.

For $i = 0$, that is, for $x \in [0, x_1]$ we have

$$\begin{aligned} u(x) - u^I(x) &= u(x) - u(x_1) \\ &= \int_x^{x_1} \frac{1}{p(s)} \int_0^s g(t)p(t)\,dt \\ &= \int_x^{x_1} \frac{ds}{p(s)} \int_0^x g(s)p(s)\,ds + \int_x^{x_1} g(s)p(s) \int_s^{x_1} \frac{dt}{p(t)}\,ds \\ &\le \int_0^x |g(s)|\,p(s) \int_s^{x_1} \frac{dt}{p(t)}\,ds + \int_x^{x_1} |g(s)|\,p(s) \int_s^{x_1} \frac{dt}{p(t)}\,ds \\ &= \int_0^{x_1} |g(s)| \int_s^{x_1} \frac{dt}{p(t)}\,p(s)\,ds. \end{aligned} \tag{4.7}$$

It can be shown, using the fact $\sum_{i=1}^{N} r_i(x) = 1$ and integrating by parts, that for $x \in [x_i, x_{i+1}]$, $i = 1,\dots,N$,

$$\begin{aligned}
&u(x) - u^I(x)\\
&\quad = r_i^+(x)\int_{x_i}^{x}\left(\int_{x_i}^{s}\frac{dt}{p(t)}\right)g(s)p(s)ds + r_{i+1}^-(x)\int_{x}^{x_{i+1}}\left(\int_{s}^{x_{i+1}}\frac{dt}{p(t)}\right)g(s)p(s)ds\\
&\quad = \frac{\int_{x}^{x_{i+1}} ds/p(s)}{\int_{x_i}^{x_{i+1}} ds/p(s)}\int_{x_i}^{x}\left(\int_{x_i}^{s} dt/p(t)\right)g(s)p(s)ds\\
&\qquad + \frac{\int_{x_i}^{x} ds/p(s)}{\int_{x_i}^{x_{i+1}} ds/p(s)}\int_{x}^{x_{i+1}}\int_{s}^{x_{i+1}}\frac{dt}{p(t)}g(s)p(s)ds\\
&\quad \le \left(\int_{x}^{x_{i+1}}\frac{ds}{p(s)}\right)\int_{x_i}^{x}|g(s)|p(s)ds + \int_{x}^{x_{i+1}}\int_{s}^{x_{i+1}}\frac{dt}{p(t)}|g(s)|p(s)ds\\
&\quad \le \int_{x_i}^{x}|g(s)|p(s)\int_{s}^{x_{i+1}}\frac{dt}{p(t)}ds + \int_{x}^{x_{i+1}}\int_{s}^{x_{i+1}}\frac{dt}{p(t)}|g(s)|p(s)ds\\
&\quad = \int_{x_i}^{x_{i+1}}|g(s)|\int_{s}^{x_{i+1}}\frac{dt}{p(t)}p(s)ds
\end{aligned} \tag{4.8}$$

The result thus follows. □

Proof of Theorem 3.1. In (4.4) taking $v_N = r_i$ for $i = 1,\dots,N$, we obtain

$$\langle u^G - u^I, r_i\rangle_V = \langle \tilde{q}(u - u^G), r_i\rangle_p, \tag{4.9}$$

which can be written as

$$\sum_{j=1}^{N}\left[\langle r_j, r_i\rangle_V + \langle \tilde{q} r_j, r_i\rangle_p\right](\alpha_j - u_j) = \langle \tilde{q}(u - u^I), r_i\rangle_p. \tag{4.10}$$

This gives the system

$$(\mathbf{A} + \mathbf{Q})\mathbf{e} = \mathbf{d}, \tag{4.11}$$

where $\mathbf{A} = (a_{ij}) = (\langle r_i, r_j\rangle_V)$ is a symmetric and tridiagonal matrix given by

$$\begin{aligned}
a_{11} &= \frac{1}{\int_{x_1}^{x_2}(1/p(s))ds},\\
a_{ii} &= \frac{1}{\int_{x_{i-1}}^{x_i}(1/p(s))ds} + \frac{1}{\int_{x_i}^{x_{i+1}}(1/p(s))ds}, \quad i = 2,\dots,N,\\
a_{i,i+1} &= -\frac{1}{\int_{x_i}^{x_{i+1}}(1/p(s))ds}, \quad i = 1,\dots,N-1,
\end{aligned} \tag{4.12}$$

$\mathbf{Q} = (q_{ij}) = (\langle \tilde{q} r_j, r_i \rangle_p)$, $\mathbf{e} = (e_i) = (\alpha_i - u_i)$, and $\mathbf{d} = (d_i)$ is given by

$$
\begin{aligned}
d_1 &= \int_{x_0}^{x_1} h(s)p(s)\,ds + \frac{\int_{x_1}^{x_2} h(s)p(s)\int_s^{x_2}(dt/p(t))\,ds}{\int_{x_1}^{x_2} dt/p(t)} \\
d_i &= \frac{\int_{x_{i-1}}^{x_i} h(s)p(s)\int_{x_{i-1}}^{s}(dt/p(t))\,ds}{\int_{x_{i-1}}^{x_i} dt/p(t)} + \frac{\int_{x_i}^{x_{i+1}} h(s)p(s)\int_s^{x_{i+1}}(dt/p(t))\,ds}{\int_{x_i}^{x_{i+1}} dt/p(t)}, \quad i > 1,
\end{aligned}
\tag{4.13}
$$

where $h(s)$ stands for $\tilde{q}(s)(u(s) - u^I(s))$. Now $\mathbf{A}$ is an $\mathbf{M}$-matrix, $q_{ij} \ge 0$ (see (1.4)), $q_{ij} < -a_{ij} (i \ne j)$ for sufficiently small mesh size and therefore, $\mathbf{A}+\mathbf{Q}$ is an $\mathbf{M}$-matrix with $(\mathbf{A}+\mathbf{Q})^{-1} \le \mathbf{A}^{-1}$ (see Ortega [10]). Thus $|\mathbf{e}| \le \mathbf{A}^{-1}|\mathbf{d}|$. The inverse of the matrix $\mathbf{A}$, denoted by $\mathbf{B} = (b_{ij})$, can be explicitly written as

$$
b_{ij} = \begin{cases} \displaystyle\int_{x_j}^{1} \frac{ds}{p(s)} & \text{if } i \le j, \\ \displaystyle\int_{x_i}^{1} \frac{ds}{p(s)} & \text{if } i \ge j. \end{cases} \tag{4.14}
$$

Therefore,

$$
\begin{aligned}
|e_i| &\le \sum_{j=1}^{N} b_{ij}|d_j| \\
&= \sum_{j=1}^{i} \int_{x_i}^{1} \frac{ds}{p(s)}|d_j| + \sum_{j=i+1}^{N} \int_{x_j}^{1} \frac{ds}{p(s)}|d_j| \\
&\le \sum_{j=1}^{N} \int_{x_j}^{1} \frac{ds}{p(s)}|d_j|.
\end{aligned}
\tag{4.15}
$$

We see that

$$
\begin{aligned}
\int_{x_1}^{1} \frac{ds}{p(s)}|d_1| &\le \int_{x_1}^{1} \frac{ds}{p(s)} \int_{x_0}^{x_1} |h(s)|p(s)\,ds + \int_{x_1}^{1} \frac{ds}{p(s)} \frac{\int_{x_1}^{x_2} |h(s)|p(s)\int_s^{x_2}(dt/p(t))\,ds}{\int_{x_1}^{x_2} dt/p(t)} \\
&= \int_{x_1}^{1} \frac{ds}{p(s)} \int_{x_0}^{x_1} |h(s)|p(s)\,ds + \int_{x_1}^{x_2} \frac{ds}{p(s)} \frac{\int_{x_1}^{x_2} |h(s)|p(s)\int_s^{x_2}(dt/p(t))\,ds}{\int_{x_1}^{x_2} dt/p(t)} \\
&\quad + \int_{x_2}^{1} \frac{ds}{p(s)} \frac{\int_{x_1}^{x_2} |h(s)|p(s)\int_s^{x_2}(dt/p(t))\,ds}{\int_{x_1}^{x_2} dt/p(t)} \\
&\le \int_{x_1}^{1} \frac{ds}{p(s)} \int_{x_0}^{x_1} |h(s)|p(s)\,ds + \int_{x_1}^{x_2} |h(s)|p(s) \int_s^{x_2} \frac{dt}{p(t)}\,ds \\
&\quad + \int_{x_2}^{1} \frac{ds}{p(s)} \int_{x_1}^{x_2} |h(s)|p(s)\,ds \\
&= \int_{x_1}^{1} \frac{ds}{p(s)} \int_{x_0}^{x_1} |h(s)|p(s)\,ds + \int_{x_1}^{x_2} |h(s)|p(s) \int_s^{1} \frac{dt}{p(t)}\,ds \\
&\le \int_{x_0}^{x_1} |h(s)|p(s) \int_s^{1} \frac{dt}{p(t)}\,ds + \int_{x_1}^{x_2} |h(s)|p(s) \int_s^{1} \frac{dt}{p(t)}\,ds.
\end{aligned}
\tag{4.16}
$$

Also for $j = 2,\dots,N$, by a similar approach, we have

$$\begin{aligned}\int_{x_j}^{1} \frac{ds}{p(s)} |d_j| &\le \int_{x_j}^{1} \frac{ds}{p(s)} \int_{x_{j-1}}^{x_j} |h(s)| p(s) ds \\ &\quad + \int_{x_j}^{1} \frac{ds}{p(s)} \frac{\int_{x_i}^{x_{i+1}} |h(s)| p(s) \int_s^{x_{i+1}} (dt/p(t)) ds}{\int_{x_i}^{x_{i+1}} dt/p(t)} \\ &\le \int_{x_{j-1}}^{x_j} |h(s)| p(s) \int_s^1 \frac{dt}{p(t)} ds + \int_{x_j}^{x_{j+1}} |h(s)| p(s) \int_s^1 \frac{dt}{p(t)} ds.\end{aligned} \tag{4.17}$$

Substituting these two inequalities in (4.15) we obtain

$$\begin{aligned}|e_i| &\le \int_{x_0}^{x_N} |h(s)| p(s) \int_s^1 \frac{dt}{p(t)} ds + \int_{x_1}^{x_{N+1}} |h(s)| p(s) \int_s^1 \frac{dt}{p(t)} ds \\ &\le 2\int_0^1 |h(s)| p(s) \int_s^1 \frac{dt}{p(t)} ds \\ &= 2\int_0^1 |\tilde{q}(s)(u(s) - u^I(s))| p(s) \int_s^1 \frac{dt}{p(t)} ds.\end{aligned} \tag{4.18}$$

Thus using (3.7), we have

$$\max_{1\le i\le N} |\alpha_i - u_i| \le 2C_{\tilde{q}} ||u - u^I||_\infty. \tag{4.19}$$

It can be shown that

$$||u^G - u^I||_\infty \le 2 \max_{1\le i\le N} |\alpha_i - u_i|. \tag{4.20}$$

Therefore,

$$\begin{aligned}||u - u^G||_\infty &\le ||u - u^I||_\infty + ||u^G - u^I||_\infty \\ &\le ||u - u^I||_\infty + 2\max_{1\le i\le N} |u_i - \alpha_i| \\ &\le (1 + 4C_{\tilde{q}}) ||u - u^I||_\infty.\end{aligned} \tag{4.21}$$

The result thus follows from Lemma 4.1. □

5. Example. In this section we give examples which are solved by the Galerkin method just described above with equal mesh size h. We then compare the results with the actual solutions.

Example 5.1. We consider the boundary value problem

$$-\frac{1}{x}(xu')' + e^u = 0, \quad 0 < x < 1, \qquad u'(0) = u(1) = 0. \tag{5.1}$$

The exact solution is known: $u(x) = 2\ln((1+\beta)/(1+\beta x^2))$, $\beta = -5+2\sqrt{6}$. It is seen that $\|u^G - u\|_\infty = 0.188845 \times 10^{-2}$ for $h = 0.1$ and $\|u^G - u\|_\infty = 0.189 \times 10^{-4}$ for $h = 0.01$. According to the Corollary 3.2 the method is $O(h^2)$ which is reflected in these results.

Example 5.2. We consider the equation

$$-\frac{1}{x^\alpha}(x^\alpha u')' + \frac{\beta^2 x^{2\beta-2}}{5(4+x^\beta)}e^u = \frac{\beta(\alpha+\beta-1)x^{\beta-2}}{4+x^\beta}$$
$$(x^\alpha u')(0^+) = 0, \qquad u(1) = 0. \tag{5.2}$$

The exact solution is $u = \ln 5 - \ln(4+x^\beta)$. The following results were obtained:

Table 5.1

α	β	h	$\|u^G - u\|_\infty$
0.5	2	0.02	1.0299×10^{-4}
0.5	2	0.01	2.6147×10^{-5}
1.0	2	0.02	9.9647×10^{-5}
1.0	2	0.01	2.4913×10^{-5}
2.0	6	0.02	3.4133×10^{-4}
2.0	6	0.01	8.6170×10^{-5}

Remark 5.3. Our method does not differentiate between $0 < \alpha < 1$ and $\alpha \geq 1$ as is the case in many articles in the literature.

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