IJMMS 30:1 (2002) 9–14
PII. S016117120201253X
http://ijmms.hindawi.com

THE EQUIVALENCE BETWEEN THE HAMILTONIAN AND LAGRANGIAN FORMULATIONS FOR THE PARAMETRIZATION-INVARIANT THEORIES

S. I. MUSLIH

Received 27 March 2001 and in revised form 13 August 2001

The link between the treatment of singular Lagrangians as field systems and the canonical Hamiltonian approach is studied. It is shown that the singular Lagrangians as field systems are always in exact agreement with the canonical approach for the parametrization invariant theories.

2000 Mathematics Subject Classification: 37K10, 37K05, 17B80.

1. Introduction. [2, 3, 4, 5], the Hamilton-Jacobi formulation of constrained systems has been studied. This formulation leads us to obtain the set of Hamilton-Jacobi partial differential equations (HJPDE) as follows:

$$H'_\alpha\left(t_\beta, q_a, \frac{\partial S}{\partial q_a}, \frac{\partial S}{\partial t_\alpha}\right) = 0, \quad \alpha, \beta = 0, n-r+1, \ldots, n;\ a = 1, \ldots, n-r, \tag{1.1}$$

where

$$H'_\alpha = H_\alpha(t_\beta, q_a, p_a) + p_\alpha, \tag{1.2}$$

and H_0 is defined as

$$H_0 = p_a w_a + p_\mu \dot{q}_\mu|_{p_\nu = -H_\nu} - L(t, q_i, \dot{q}_\nu, \dot{q}_a = w_a), \quad \mu, \nu = n-r+1, \ldots, n. \tag{1.3}$$

The equations of motion are obtained as total differential equations in many variables as follows:

$$\begin{gathered} dq_a = \frac{\partial H'_\alpha}{\partial p_a} dt_\alpha, \qquad dp_a = -\frac{\partial H'_\alpha}{\partial q_a} dt_\alpha, \qquad dp_\beta = -\frac{\partial H'_\alpha}{\partial t_\beta} dt_\alpha, \\ dz = \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a}\right) dt_\alpha; \quad \alpha, \beta = 0, n-r+1, \ldots, n,\ a = 1, \ldots, n-r, \end{gathered} \tag{1.4}$$

where $z = S(t_\alpha; q_a)$. The set of equations (1.4) is integrable (see [4, 5]) if

$$dH'_0 = 0, \qquad dH'_\mu = 0, \quad \mu = n-r+1, \ldots, n. \tag{1.5}$$

If conditions (1.5) are not satisfied identically, we consider them as new constraints and again test the consistency conditions. Hence, the canonical formulation leads to obtain the set of canonical phase space coordinates q_a and p_a as functions of t_α; besides, the canonical action integral is obtained in terms of the canonical coordinates. The Hamiltonians H'_α are considered as the infinitesimal generators of canonical transformations given by parameters t_α, respectively.

In [1], the singular Lagrangians are treated as field systems. The Euler-Lagrange equations of singular systems are proposed in the form

$$\frac{\partial}{\partial t_\alpha}\left[\frac{\partial L'}{\partial(\partial_\alpha q_a)}\right]-\frac{\partial L'}{\partial q_a}=0, \quad \partial_\alpha q_a=\frac{\partial q_a}{\partial t_\alpha}, \tag{1.6}$$

with constraints

$$dG_0=-\frac{\partial L'}{\partial t}dt, \tag{1.7}$$

$$dG_\mu=-\frac{\partial L'}{\partial q_\mu}dt, \tag{1.8}$$

where

$$\begin{gathered} L'(t_\alpha,\partial_\alpha q_a,\dot{q}_\mu,q_a)=L(q_a,q_\alpha,\dot{q}_a=(\partial_\alpha q_a)\dot{t}_\alpha), \quad \dot{q}_\mu=\frac{dq_\mu}{dt}, \\ G_\alpha=H_\alpha\Big(q_a,t_\beta,p_a=\frac{\partial L}{\partial \dot{q}_a}\Big). \end{gathered} \tag{1.9}$$

In order to have a consistent theory, we should consider the variations of the constraints (1.7) and (1.8).

In this paper, we study the link between the treatment of singular Lagrangians as field systems and the canonical formalism for the parametrization invariant theories.

2. Parametrization-invariant theories as singular systems. In [4], the canonical method treatment of the parametrization-invariant theories is studied and will be briefly reviewed here.

Consider a system with the action integral as

$$S(q_i)=\int dt\mathscr{L}(q_i,\dot{q}_i,t), \quad i=1,\dots,n, \tag{2.1}$$

where $\mathscr{L}$ is a regular Lagrangian with Hessian n. Parametrize the time $t\to\tau(t)$, with $\dot{\tau}=d\tau/dt>0$. The velocities $\dot{q}_i$ may be expressed as

$$\dot{q}_i=q'_i\dot{\tau}, \tag{2.2}$$

where q'_i are defined as

$$q'_i=\frac{dq_i}{d\tau}. \tag{2.3}$$

Denote $t=q_0$ and $q_\mu=(q_0,q_i)$, $\mu=0,1,\dots,n$, then the action integral (2.1) may be written as

$$S(q_\mu)=\int d\tau\dot{t}\mathscr{L}\Big(q_\mu,\frac{q'_i}{\dot{t}}\Big), \tag{2.4}$$

which is parametrization invariant since L is homogeneous of first degree in the velocities q'_μ with L given as

$$L(q_\mu, \dot{q}_\mu) = \dot{t}\mathcal{L}\left(q_\mu, \frac{q'_i}{\dot{t}}\right). \tag{2.5}$$

The Lagrangian L is now singular since its Hessian is n.

The canonical method in [2, 3, 4, 5] leads us to obtain the set of Hamilton-Jacobi partial differential equations as follows:

$$\begin{aligned} H'_0 &= p_\tau - L(q_0, q_i, \dot{q}_0, \dot{q}_i = w_i) \\ &\qquad + p_i^\tau q'_i + p_t \dot{q}_0|_{p_t=-H_t} = 0, \quad p_\tau = \frac{\partial S}{\partial \tau}, \\ H'_t &= p_t + H_t = 0, \quad p_t = \frac{\partial S}{\partial t}, \end{aligned} \tag{2.6}$$

where H_t is defined as

$$H_t = -\mathcal{L}(q_i, w_i) + p_i^\tau w_i. \tag{2.7}$$

Here, p_i^τ and p_t are the generalized momenta conjugated to the generalized coordinates q_i and t, respectively.

The equations of motion are obtained as total differential equations in many variables as follows:

$$dq^i = \frac{\partial H'_0}{\partial p_i} d\tau + \frac{\partial H'_t}{\partial p_i} dq^0 = \frac{\partial H'_t}{\partial p_i} dq^0, \tag{2.8}$$

$$dp^i = -\frac{\partial H'_0}{\partial q_i} d\tau - \frac{\partial H'_t}{\partial q_i} dq^0 = -\frac{\partial H'_t}{\partial q_i} dq^0, \tag{2.9}$$

$$dp_t = -\frac{\partial H'_0}{\partial q_0} d\tau - \frac{\partial H'_t}{\partial q_0} dq^0 = 0. \tag{2.10}$$

Since

$$dH'_t = dp_t + H_t \tag{2.11}$$

vanishes identically, this system is integrable and the canonical phase space coordinates q_i and p_i are obtained in terms of the time $(q_0 = t)$.

Now, we look at the Lagrangian (2.5) as a field system. Since the rank of the Hessian matrix is n, this Lagrangian can be treated as a field system in the form

$$q_i = q_i(\tau, t), \tag{2.12}$$

thus, the expression

$$q'_i = \frac{\partial q_i}{\partial \tau} + \frac{\partial q_i}{\partial t}\dot{t}, \tag{2.13}$$

can be substituted in (2.5) to obtain the modified Lagrangian L':

$$L' = \dot{t}\mathcal{L}\left(q_\mu, \frac{1}{\dot{t}}\left(\frac{\partial q_i}{\partial \tau} + \frac{\partial q_i}{\partial t}\dot{t}\right)\right). \tag{2.14}$$

Making use of (1.6), we have

$$\frac{\partial L'}{\partial q_i} - \frac{\partial}{\partial t}\left(\frac{\partial L'}{\partial(\partial q_i/\partial t)}\right) - \frac{\partial}{\partial \tau}\left(\frac{\partial L'}{\partial(\partial q_i/\partial \tau)}\right) = 0. \tag{2.15}$$

Calculations show that (2.15) leads to a well-known Lagrangian equation as

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial(dq_i/dt)}\right) = 0. \tag{2.16}$$

Using (2.7), we have

$$H_t = -\mathcal{L} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_i. \tag{2.17}$$

In order to have a consistent theory, we should consider the total variation of H_t. In fact,

$$dH_t = -\frac{\partial \mathcal{L}}{\partial t}dt. \tag{2.18}$$

Making use of (1.8), we find that

$$dH_t = -\frac{\partial L'}{\partial t}d\tau. \tag{2.19}$$

Besides, the quantity H_0 is identically satisfied and does not lead to constraints.

We notice that (2.8) and (2.9) are equivalent to (2.15) and (2.16).

3. Classical fields as constrained systems. In the following sections, we study the Hamiltonian and Lagrangian formulations for classical field systems and demonstrate the equivalence between these two formulations for the reparametrization-invariant fields.

A classical relativistic field $\phi_i = \phi_i(\vec{x}, t)$ in four space-time dimensions may be described as the action functional

$$S(\phi_i) = \int dt \int d^3x \{\mathcal{L}(\phi_i, \partial_\mu \phi_i)\}, \quad \mu = 0, 1, 2, 3;\ i = 1, 2, \ldots, n, \tag{3.1}$$

which leads to the Euler-Lagrange equations of motion as

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\right] = 0. \tag{3.2}$$

We can go over from the Lagrangian description to the Hamiltonian description by using the definition

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}, \tag{3.3}$$

then the canonical Hamiltonian is defined as

$$H_0 = \int d^3x (\pi_i \dot{\phi}_i - \mathcal{L}). \tag{3.4}$$

The equations of motion are obtained

$$\dot{\pi}_i = -\frac{\partial H_0}{\partial \phi_i}, \qquad \dot{\phi}_i = \frac{\partial H_0}{\partial \pi_i}. \tag{3.5}$$

4. Reparametrization-invariant fields. In analogy with the finite dimensional systems, we introduce the reparametrization-invariant action for the field system:

$$S = \int d\tau \int \mathcal{L}_R d^3x, \tag{4.1}$$

where

$$\mathcal{L}_R = \dot{t}\mathcal{L}(\phi_i, \partial_\mu \phi_i). \tag{4.2}$$

Following the canonical method [2, 3, 4, 5], we obtain the set of [HJPDE],

$$\begin{aligned} H'_0 &= \pi_\tau + \pi_i^{(\tau)} \frac{d\phi_i}{d\tau} + \pi_t \frac{dt}{d\tau} - \mathcal{L}_R = 0, \quad \pi_\tau = \frac{\partial S}{\partial \tau}, \\ H'_t &= \pi_t + H_t = 0, \quad \pi_t = \frac{\partial S}{\partial t}, \end{aligned} \tag{4.3}$$

where H_t is defined as

$$H_t = -\mathcal{L}(\phi_i, \partial_\mu \phi_i) + \pi_i^{(\tau)} \frac{d\phi_i}{dt}, \tag{4.4}$$

and $\pi_i^{(\tau)}$, π_t are the generalized momenta conjugated to the generalized coordinates ϕ_i and t, respectively.

The equations of motion are obtained as follows:

$$d\phi_i = \frac{\partial H'_0}{\partial \pi_i} d\tau + \frac{\partial H'_t}{\partial \pi_i} dt = \frac{\partial H'_t}{\partial \pi_i} dt, \tag{4.5}$$

$$d\pi_i = -\frac{\partial H'_0}{\partial \phi_i} d\tau - \frac{\partial H'_t}{\partial \phi_i} dt = -\frac{\partial H'_t}{\partial \phi_i} dt, \tag{4.6}$$

$$d\pi_t = -\frac{\partial H'_0}{\partial t} d\tau - \frac{\partial H'_t}{\partial t} dt = 0. \tag{4.7}$$

Now the Euler-Lagrangian equation for the field system reads as

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial (\partial \phi_i / \partial x_\mu)} \right) = 0. \tag{4.8}$$

Again as for the finite-dimensional systems, (4.5) and (4.6) are equivalent to (4.8) for field systems.

5. Conclusion. As it is mentioned in the introduction, if the rank of the Hessian matrix for discrete systems is $(n - r)$, $0 < r < n$, then the systems can be treated as field systems [1]. The treatment of Lagrangians as field systems is always in exact agreement with the Hamilton-Jacobi treatment for reparametrization-invariant theories. The equations of motion (2.8) and (2.9) are equivalent to the equations of motion (2.15) and (2.16). Besides, the variations of constraints (2.18) and (2.19) are identically satisfied and no further constraints arise.

In analogy with the finite-dimensional systems, it is observed that the Lagrangian and the Hamilton-Jacobi treatments for the reparametrization-invariant fields are in exact agreement.

References

[1] N. I. Farahat and Y. Güler, *Singular Lagrangians as field systems*, Phys. Rev. A (3) **51** (1995), no. 1, 68–72.

[2] Y. Güler, *Canonical formulation of singular systems*, Nuovo Cimento B (11) **107** (1992), no. 12, 1389–1395.

[3] ———, *Integration of singular systems*, Nuovo Cimento B (11) **107** (1992), no. 10, 1143–1149.

[4] S. I. Muslih and Y. Güler, *On the integrability conditions of constrained systems*, Nuovo Cimento B (11) **110** (1995), no. 3, 307–315.

[5] ———, *Is gauge fixing of constrained systems necessary?* Nuovo Cimento B **113** (1998), 277–289.

S. I. Muslih: Department of Physics, Al-Azhar University, Gaza, Palestine
E-mail address: sami_muslih@hotmail.com