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MONOTONE-ITERATIVE TECHNIQUE OF LAKSHMIKANTHAM FOR THE INITIAL VALUE PROBLEM FOR A DIFFERENTIAL EQUATION WITH A STEP FUNCTION

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The initial value problem for a special kind of differential equations with a step function is studied. The monotone-iterative technique of Lakshmikantham for approximate finding of the solutions of the given problem is well grounded.

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1. Introduction. Many evolutionary processes can be described with the help of differential equations. At the same time, the solutions of a small number of linear differential equations can be found as well-known functions. That is why it is necessary to prove some approximate methods for solving different kinds of differential equations. One of the most practically used methods is the monotone-iterative technique of Lakshmikantham [1, 2, 3].

In this note, this method is well grounded for a special type of differential equations. We studied the case when the right part of the equation depends on a piecewise constant function. We note that some qualitative properties of the solutions of differential equations with a piecewise constant function (DEPCF) such as uniqueness, oscillation, and periodicity are investigated in [4]. Research in this direction is motivated by the fact that DEPCF represent a hybrid of continuous and discrete dynamical systems and combine the properties of both differential and difference equations.

2. Preliminary notes and definitions. Let $T > 0$ and $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T$ be fixed numbers.

Definition 2.1. The function $g(t) : [0,T] \to \mathbb{R}$ is called a step function if $g(t) = g_n$ for $t_n \le t < t_{n+1}$ where $g_n = \text{const}$, $n = 0,1,\dots,p$.

Consider the initial value problem (IVP) for the differential equation with a step function

$$x' = f(x(t),x(g(t))) \quad \text{for } t \in [0,T], \qquad x(0) = c_0, \tag{2.1}$$

where $x \in \mathbb{R}$, $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, c_0 is an arbitrary constant, $g(t)$ is a step function.

We denote by $PC^1([0,T],\mathbb{R})$ the set of all functions $u \in C([0,T],\mathbb{R})$ for which the derivative $u'(t)$ exists and is piecewise continuous in $[0,T]$ with points of discontinuity of first kind at the points t_n, $n = 1,2,\dots,p$, $u'(t_n) = u'(t_n+0)$.

Definition 2.2. The function $x(t)$ is a solution of the IVP (2.1) in the interval $[0,T]$ if the following conditions are fulfilled:

(1) $x(t) \in PC^1([0,T],\mathbb{R})$.

(2) The function $x(t)$ turns (2.1) into identities for $t \in [0,T]$.

Definition 2.3. The function $v(t) \in PC^1([0,T],\mathbb{R})$ is called a lower (upper) solution of the IVP (2.1) if

$$v'(t) \le (\ge) f(v(t), v(g(t))), \quad v(0) \le (\ge) c_0. \tag{2.2}$$

Definition 2.4. The function $u(t)$ is called a minimal (maximal) solution of the IVP (2.1) if it is a solution of the IVP (2.1) and, for any other solution $x(t)$ of the IVP (2.1), the inequality $u(t) \le (\ge) x(t)$ holds.

Lemma 2.5. *Let the following conditions be satisfied:*

(1) *the function $g(t):[0,T] \to \mathbb{R}$ is a step one and $0 \le g(t) \le t_n$ for $t \in [t_n, t_{n+1})$, $n = 0,1,\dots,p$;*

(2) *M and N are positive constants such that $(M+N)T \le 1$;*

(3) *the function $p(t) \in PC^1([0,T],\mathbb{R})$ satisfies the inequalities*

$$p'(t) \ge -Mp(t) - Np(g(t)) \quad \text{for } t \in [0,T], \quad p(0) \ge 0. \tag{2.3}$$

Then $p(t) \ge 0$ for $t \in [0,T]$.

Proof

Case 1. Let $p(0) > 0$. Suppose that there exists a point $t \in (0,T]$ such that $p(t) < 0$. And let

$$\xi = \inf\{t \in [0,T] : p(t) \le 0\}. \tag{2.4}$$

Then $\xi \in (0,T]$.

We consider the following two cases.

Case 1.1. Let $\xi \ne T$. Denote $\lambda = \max_{0 \le t \le \xi} p(t)$, $\lambda > 0$. Then there exists a point $\eta \in [0,\xi)$ such that $p(\eta) = \lambda$. It follows from the mean value theorem that there exists $\xi_0 \in (\eta,\xi)$ for which $p(\xi) - p(\eta) = p'(\xi_0)(\xi - \eta)$. On the other hand, $p(\xi) - p(\eta) \le 0 - \lambda = -\lambda = \lambda_1 < 0$. Then

$$\lambda_1 \ge p'(\xi_0)(\xi - \eta). \tag{2.5}$$

It follows from condition (3) of Lemma 2.5 that $p'(\xi_0) \ge -Mp(\xi_0) - Np(g(\xi_0))$. Since $g(\xi_0) \le \xi_0 < \xi$, the inequalities $p(\xi_0) \le \lambda$, $p(g(\xi_0)) \le \lambda$ hold. Then

$$-Mp(\xi_0) - Np(g(\xi_0)) \ge \lambda_1(M+N). \tag{2.6}$$

It follows from inequalities (2.5) and (2.6) that $\lambda_1 \ge \lambda_1(M+N)(\xi-\eta)$ which is equivalent to $1 \le (M+N)(\xi-\eta)$. Since $(\xi-\eta) < T$, the inequality $1 < (M+N)T$ holds. The last inequality contradicts condition (2) of Lemma 2.5. Therefore, the inequality $p(t) > 0$ holds for $t \in [0,T]$.

Case 1.2. Let $\xi = T$. Then $p(t) > 0$ for $t \in [0,T)$ and $p(T) = 0$, that is, $p(t) \ge 0$ for $t \in [0,T]$.

Case 2. Let $p(0) = 0$. Suppose there exists a point $t \in (0,T]$ such that $p(t) < 0$. And let

$$\zeta = \sup\{t \in [0,T] : p(s) = 0 \text{ for } s \in [0,t]\}. \tag{2.7}$$

We consider the following two cases.

Case 2.1. Let $\zeta = 0$.

Case 2.1.1. There exists a point $\tau > 0$ for which $p(t) > 0$ for $t \in (0,\tau]$. If we consider the point τ instead of the point 0 and follow the proof of Case 1, we get $p(t) \geq 0$ for $t \in [\tau,T]$, that is, $p(t) \geq 0$ for $t \in [0,T]$.

Case 2.1.2. There exists a point $\tau \in (0,t_1)$ such that $p(\tau) < 0$, $p'(\tau) < 0$. According to condition (3) of Lemma 2.5, the inequality $p'(\tau) \geq -Mp(\tau) - Np(g(\tau))$ holds. From condition (1) of Lemma 2.5 and the inequality $\tau < t_1$, it follows that $g(\tau) = g_0 = 0$, that is, $p(g(\tau)) = 0$. Then $p'(\tau) \geq -Mp(\tau) > 0$ which leads to a contradiction. Hence the inequality $p(t) \geq 0$ holds for $t \in [0,T]$.

Case 2.2. Let $\zeta > 0$. If we consider the point ζ instead of the point 0 and follow the proof of Case 2.1, we get $p(t) \geq 0$ for $t \in [\tau,T]$, that is, $p(t) \geq 0$ for $t \in [0,T]$. □

Consider the initial value problem for the linear differential equation with a step function

$$x'(t) = ax(t) + bx(g(t)), \quad x(0) = c_0, \tag{2.8}$$

where a, b, c_0 are constants.

Lemma 2.6. *Let a, b, c_0 be constants and the function $g(t) : [0,T] \to \mathbb{R}$ be a step one such that $0 \leq g_n \leq t_n$ for $t \in [t_n,t_{n+1})$, $n = 0,1,\ldots,p$. Then the initial value problem for the linear differential equation (2.8) has a unique solution for $t \in [0,T]$.*

The proof of Lemma 2.6 is trivial. From Lemma 2.6, the validity of the following result follows.

Corollary 2.7. *Let $c_0 = 0$, then the IVP (2.8) has a unique solution $x(t) = 0$ for $t \in [0,T]$.*

Consider the IVP

$$x'(t) = ax(t) + bx(g(t)) + f(t,g(t)), \quad x(0) = c_0, \tag{2.9}$$

where a, b, c_0 are constants, $f : [0,T] \times [0,T] \to \mathbb{R}$.

Theorem 2.8. *Let the function $f \in C([0,T] \times [0,T],\mathbb{R})$ and the function $g(t)$ be a step one such that $0 \leq g(t) \leq t_n$ for $t \in [t_n,t_{n+1})$, $n = 0,1,\ldots,p$. Then the initial value problem for the linear differential equation (2.9) has a unique solution for $t \in [0,T]$.*

Proof

Case 1. Let $a \neq 0$.

Let $t \in [t_0,t_1)$. Consider the IVP

$$x'(t) = ax(t) + bs_0 + f(t,0), \quad x(0) = c_0, \tag{2.10}$$

where $s_0 = x(g_0) = c_0$.

The solution of the IVP (2.10) exists for $t \geq 0$ and satisfies the equality

$$x_0(t) = e^{at}\left(\int_0^t e^{-a\tau} f(\tau,0)\,d\tau + c_0\right) + (e^{at}-1)ba^{-1}c_0. \tag{2.11}$$

Let $t \in [t_1,t_2)$. Consider the IVP

$$x'(t) = ax(t) + bs_1 + f(t,g_1), \qquad x(t_1) = c_1, \tag{2.12}$$

where $s_1 = x(g_1) = x_0(g_1)$, $c_1 = x_0(t_1)$. The solution of the IVP (2.12) exists for $t \geq t_1$ and satisfies the equality

$$x_1(t) = e^{a(t-t_1)}\left(x_0(t_1) + \int_{t_1}^t e^{-a(\tau-t_1)} f(\tau,g_1)\,d\tau\right) + (e^{a(t-t_1)}-1)ba^{-1}x_0(g_1). \tag{2.13}$$

Let $t \in [t_2,t_3)$. Consider the IVP

$$x'(t) = ax(t) + bs_2 + f(t,g_2), \qquad x(t_2) = c_2, \tag{2.14}$$

where $s_2 = x(g_2)$, $c_2 = x_1(t_2)$. Since $g_2 \leq t_2$, then $s_2 = x_k(g_2)$ where

$$k = \begin{cases} 0 & \text{for } g_2 \in [0,t_1], \\ 1 & \text{for } g_2 \in (t_1,t_2]. \end{cases} \tag{2.15}$$

The solution of the IVP (2.14) exists for $t \geq t_1$ and satisfies the equality

$$x_2(t) = e^{a(t-t_2)}\left(x_1(t_2) + \int_{t_2}^t e^{-a(\tau-t_2)} f(\tau,g_2)\,d\tau\right) + (e^{a(t-t_2)}-1)ba^{-1}x_k(g_2). \tag{2.16}$$

With the help of the solution $x_{n-1}(t)$ in the interval $[t_{n-1},t_n)$ and the steps method, we construct a solution $x_n(t)$ of the IVP

$$x'(t) = ax(t) + bs_n + f(t,g_n), \quad x(t_n) = c_n \quad \text{for } t \in [t_n,t_{n+1}), \tag{2.17}$$

where $s_n = x_{n-i}(g_n)$, $c_n = x_{n-1}(t_n)$ and $i \leq n$. The solution of the IVP (2.17) exists for $t \geq t_n$ and satisfies the equality

$$x_n(t) = e^{a(t-t_n)}\left(x_{n-1}(t_n) + \int_{t_n}^t e^{-a(\tau-t_n)} f(\tau,g_n)\,d\tau\right) + (e^{a(t-t_n)}-1)ba^{-1}x_{n-i}(g_n). \tag{2.18}$$

Case 2. Let $a = 0$. Consider the following two cases.

Case 2.1. Let $b \neq 0$. Using the steps method, we construct the functions $x_n(t)$, $t \in [t_n,t_{n+1})$, $n = 0,1,2,\ldots,p$ as solutions of the IVP

$$x'(t) = bs_n + f(t,g_n), \qquad x(t_n) = c_n, \tag{2.19}$$

where $s_n = x_{n-i}(g_n)$, $c_n = x_{n-1}(t_n)$ for $0 < n \leq p$ and $i \leq n$.

Therefore,

$$x_n(t) = \int_{t_n}^t f(\tau,g_n)\,d\tau + bx_{n-i}(g_n)(t-t_n) + x_{n-1}(t_n). \tag{2.20}$$

Case 2.2. Let $b = 0$. Using the steps method, we construct the functions $x_n(t)$, $t \in [t_n, t_{n+1})$, $n = 0, 1, 2, \ldots, p$ as solutions of the IVP

$$x'(t) = f(t, g_n), \qquad x(t_n) = c_n, \tag{2.21}$$

where $c_n = x_{n-1}(t_n)$.

Therefore,

$$x_n(t) = \int_{t_n}^{t} f(\tau, g_n)\, d\tau + x_{n-1}(t_n). \tag{2.22}$$

Define the function

$$x(t) = \begin{cases} x_0(t) & \text{for } t \in [0, t_1), \\ x_1(t) & \text{for } t \in [t_1, t_2), \\ \vdots & \\ x_p(t) & \text{for } t \in [t_p, t_{p+1}]. \end{cases} \tag{2.23}$$

The function $x(t)$ is a solution of the IVP (2.9) in $[0, T]$. Suppose there exist two different solutions $x(t)$ and $y(t)$ of the IVP (2.9). Define the function $q(t) = x(t) - y(t)$, $t \in [0, T]$. The function $q(t)$ satisfies the IVP (2.8), where $c_0 = 0$. By the Corollary 2.7, it follows that $q(t) = 0$ for $t \in [0, T]$. Therefore the IVP (2.9) has a unique solution. □

3. Main results.

We will apply the monotone-iterative technique to find an approximate solution of the initial value problem for a nonlinear differential equation with a step function.

Theorem 3.1. *Let the following conditions be fulfilled:*

(1) *the function $g(t) \in ([0, T], \mathbb{R})$ is a step one such that $0 \le g(t) \le t_n$ for $t \in [t_n, t_{n+1})$, $n = 0, 1, \ldots, p$;*

(2) *M and N are positive constants such that $(M + N)T \le 1$;*

(3) *the function $f \in C(\mathbb{R}^2, \mathbb{R})$ and for $x_1 \ge x_2$, $y_1 \ge y_2$, the inequality*

$$f(x_1, y_1) - f(x_2, y_2) \ge -M(x_1 - x_2) - N(y_1 - y_2) \tag{3.1}$$

holds;

(4) *the functions $v_0(t)$ and $w_0(t)$ are lower and upper solutions of the IVP (2.1) and $v_0(t) \le w_0(t)$ for $t \in [0, T]$.*

Then there exist two sequences of functions $\{v_n(t)\}_0^\infty$ and $\{w_n(t)\}_0^\infty$ such that

(a) *the sequences are increasing and decreasing, respectively;*

(b) *the functions $v_n(t)$, $w_n(t)$ are lower and upper solutions of the IVP (2.1);*

(c) *the sequences are uniformly convergent in the interval $[0, T]$;*

(d) *the limits $v(t) = \lim_{n\to\infty} v_n(t)$, $w(t) = \lim_{n\to\infty} w_n(t)$ are minimal and maximal solutions of the IVP (2.1), respectively.*

Proof. Let the function $\eta(t) \in C([0,T],\mathbb{R})$, $v_0(t) \le \eta(t) \le w_0(t)$, be fixed. Consider the initial value problem for the linear differential equation with a step function

$$x'(t) = f(\eta(t),\eta(g(t))) - M(x(t)-\eta(t)) - N(x(g(t)) - \eta(g(t))),\quad x(0) = c_0. \tag{3.2}$$

By Theorem 2.8, the IVP (3.2) has a unique solution $x(t)$ for $t \in [0,T]$. Define the mapping A by the equality $A\eta(t) = x(t)$ where $x(t)$ is the unique solution of the IVP (3.2). We prove that the operator A satisfies the following properties:

(i) $v_0(t) \le Av_0(t)$, $w_0(t) \ge Aw_0(t)$;

(ii) for any function $u_1(t),u_2(t) \in PC^1([0,T],\mathbb{R})$ such that $v_0(t) \le u_1(t) \le u_2(t) \le w_0(t)$, the inequality $Au_1(t) \le Au_2(t)$ holds.

Indeed, let $Av_0(t) = v_1(t)$. The function $v_1(t)$ is continuous and it is the solution of the IVP (3.2) for $\eta(t) = v_0(t)$. Set $p(t) = v_1(t) - v_0(t)$. Then $p'(t) = v_1'(t) - v_0'(t) \ge v_1'(t) - f(v_0(t),v_0(g(t))) = -Mp(t) - Np(g(t))$ and $p(0) = v_1(0) - v_0(0) \ge 0$.

By Lemma 2.5 the function $p(t)$ is nonnegative in $[0,T]$, that is, $Av_0(t) \ge v_0(t)$. Let $Aw_0(t) = w_1(t)$. The function $w_1(t)$ is continuous and it is a solution of (3.2) for $\eta(t) = w_0(t)$. Set $p(t) = w_0(t) - w_1(t)$. Then

$$\begin{aligned} p'(t) &\ge f(w_0(t),w_0(g(t))) - f(w_0(t),w_0(g(t))) \\ &\quad + M(w_1(t) - w_0(t)) + N(w_1(g(t)) - w_0(g(t))) \\ &= -Mp(t) - Np(g(t)), \quad p(0) \ge 0. \end{aligned} \tag{3.3}$$

By Lemma 2.5 the function $p(t)$ is nonnegative in $[0,T]$, that is, $Aw_0(t) \le w_1(t)$. Therefore, property (i) is satisfied.

Let $u_1,u_2 \in PC^1([0,T],\mathbb{R})$ and $v_0(t) \le u_1(t) \le u_2(t) \le w_0(t)$. If $x_1(t) = Au_1(t)$ and $x_2(t) = Au_2(t)$, then the function $p(t) = x_2(t) - x_1(t)$ satisfies the equality

$$\begin{aligned} p'(t) &= f(u_2(t),u_2(g(t))) - M(x_2(t) - u_2(t)) - N(x_2(g(t)) - u_2(g(t))) \\ &\quad - f(u_1(t),u_1(g(t))) + M(x_1(t) - u_1(t)) + N(x_1(g(t)) - u_1(g(t))). \end{aligned} \tag{3.4}$$

Due to Theorem 3.1(3), we get $p'(t) \ge -M(x_2(t)-x_1(t)) - N(x_2(g(t)) - x_1(g(t))) = -Mp(t) - Np(g(t))$ and $p(0) = 0$. By Lemma 2.5 the inequality $p(t) \ge 0$ holds, that is, $Au_1(t) \le Au_2(t)$. Therefore, property (ii) is satisfied.

Define the sequences $\{v_n(t)\}_0^\infty$ and $\{w_n(t)\}_0^\infty$ with the help of the equalities $v_n(t) = Av_{n-1}(t)$, $w_n(t) = Aw_{n-1}(t)$, $n \ge 1$. By the proof of Theorem 2.8 we get

$$\begin{aligned} v_n^{(0)}(t) &= e^{-Mt}\left(c_0 + \int_0^t e^{M\tau}\varphi_{n-1}^{(0)}(\tau,0)\,d\tau\right) \\ &\quad + (e^{-Mt} - 1)NM^{-1}c_0 \quad \text{for } t \in [0,t_1), \end{aligned} \tag{3.5}$$

$$\begin{aligned} v_n^{(m)}(t) &= e^{-M(t-t_m)}\left(v_n^{(m-1)}(t_m) + \int_{t_m}^t e^{M\left(\tau - t_m\right)}\varphi_{n-1}^{(m)}(\tau,g_m)\,d\tau\right) \\ &\quad + (e^{-M(t-t_m)} - 1)NM^{-1}v_n^{(m-i)}(g_m) \\ &\quad \text{for } t \in [t_m,t_{m+1}),\ m = 1,2,\dots,p, \quad i \le m, \end{aligned} \tag{3.6}$$

$$\begin{aligned} w_n^{(0)}(t) &= e^{-Mt}\left(c_0+\int_0^t e^{M\tau}\psi_{n-1}^{(0)}(\tau,0)d\tau\right) \\ &\quad + (e^{-Mt}-1)NM^{-1}c_0 \quad \text{for } t\in[0,t_1), \end{aligned} \tag{3.7}$$

$$\begin{aligned} w_n^{(m)}(t) &= e^{-M(t-t_m)}\left(w_n^{(m-1)}(t_m)+\int_{t_m}^t e^{M(\tau-t_m)}\psi_{n-1}^{(m)}(\tau,g_m)d\tau\right) \\ &\quad + (e^{-M(t-t_m)}-1)NM^{-1}w_n^{(m-i)}(g_m) \\ &\quad \text{for } t\in[t_m,t_{m+1}), \quad m=1,2,\dots,p, \quad i\le m, \end{aligned} \tag{3.8}$$

$$v_n(t)=\begin{cases} v_n^{(0)}(t) & \text{for } t\in[0,t_1), \\ v_n^{(1)}(t) & \text{for } t\in[t_1,t_2), \\ \vdots & \\ v_n^{(p)}(t) & \text{for } t\in[t_p,t_{p+1}], \end{cases} \qquad w_n(t)=\begin{cases} w_n^{(0)}(t) & \text{for } t\in[0,t_1), \\ w_n^{(1)}(t) & \text{for } t\in[t_1,t_2), \\ \vdots & \\ w_n^{(p)}(t) & \text{for } t\in[t_p,t_{p+1}], \end{cases} \tag{3.9}$$

where, for $m=0,1,\dots,p$,

$$\begin{aligned} \varphi_{n-1}^{(m)}(t,g_m) &= Mv_{n-1}^{(m)}(t)+Nv_{n-1}^{(m)}(g_m)+f\left(v_{n-1}^{(m)}(t),v_{n-1}^{(m)}(g_m)\right), \\ \psi_{n-1}^{(m)}(t,g_m) &= Mw_{n-1}^{(m)}(t)+Nw_{n-1}^{(m)}(g_m)+f\left(w_{n-1}^{(m)}(t),w_{n-1}^{(m)}(g_m)\right). \end{aligned} \tag{3.10}$$

By properties (i) and (ii) of the operator A, the following inequalities hold:

$$v_0(t)\le v_1(t)\le\cdots\le v_n(t)\le w_n(t)\le\cdots\le w_1(t)\le w_0(t) \quad \text{for } t\in[0,T]. \tag{3.11}$$

The sequences $\{v_n(t)\}_0^\infty$ and $\{w_n(t)\}_0^\infty$ are equicontinuous and uniformly bounded in the intervals $[t_m,t_{m+1})$, $m=0,1,\dots,p$. Therefore, they are uniformly convergent on $[t_m,t_{m+1})$. We denote $\lim_{n\to\infty}v_n^{(m)}(t)=v^{(m)}(t)$ and $\lim_{n\to\infty}w_n^{(m)}(t)=w^{(m)}(t)$.

Taking the limit as $n\to\infty$ into equalities (3.6) and (3.8), we obtain that the functions $v^{(m)}(t)$ and $w^{(m)}(t)$ are solutions of the integral equations,

$$\begin{aligned} v^{(m)}(t) &= e^{-M(t-t_m)}\left(v^{(m-1)}(t_m)+\int_{t_m}^t e^{M(\tau-t_m)}\varphi^{(m)}(\tau,g_m)d\tau\right) \\ &\quad + (e^{-M(t-t_m)}-1)NM^{-1}v^{(m-i)}(g_m) \quad \text{for } t\in[t_m,t_{m+1}),\ i\le m, \\ w^{(m)}(t) &= e^{-M(t-t_m)}\left(w^{(m-1)}(t_m)+\int_{t_m}^t e^{M(\tau-t_m)}\psi^{(m)}(\tau,g_m)d\tau\right) \\ &\quad + (e^{-M(t-t_m)}-1)NM^{-1}w^{(m-i)}(g_m) \quad \text{for } t\in[t_m,t_{m+1}),\ i\le m, \end{aligned} \tag{3.12}$$

where, for $m=0,1,\dots,p$,

$$\begin{aligned} \varphi^{(m)}(t,g_m) &= Mv^{(m)}(t)+Nv^{(m)}(g_m)+f\left(v^{(m)}(t),v^{(m)}(g_m)\right), \\ \psi^{(m)}(t,g_m) &= Mw^{(m)}(t)+Nw^{(m)}(g_m)+f\left(w^{(m)}(t),w^{(m)}(g_m)\right). \end{aligned} \tag{3.13}$$

Define the functions

$$v(t)=\begin{cases} v^{(0)}(t) & \text{for } t\in[0,t_1),\\ v^{(1)}(t) & \text{for } t\in[t_1,t_2),\\ \vdots & \\ v^{(p)}(t) & \text{for } t\in[t_p,t_{p+1}],\end{cases}\qquad w(t)=\begin{cases} w^{(0)}(t) & \text{for } t\in[0,t_1),\\ w^{(1)}(t) & \text{for } t\in[t_1,t_2),\\ \vdots & \\ w^{(p)}(t) & \text{for } t\in[t_p,t_{p+1}].\end{cases} \tag{3.14}$$

From equalities (3.12) it follows that the functions $v(t)$ and $w(t)$ are solutions of the IVP (2.1). We prove that $v(t)$ and $w(t)$ are, respectively, minimal and maximal solutions of the IVP (2.1). Let $x(t)$ be an arbitrary solution of the IVP (2.1) in $[0,T]$ such that $v_0(t)\le x(t)\le w_0(t)$.

Assume that $v_n(t)\le x(t)\le w_n(t)$ in $[0,T]$ for some n. Set $p(t)=x(t)-v_{n+1}(t)$. Then we get

$$\begin{aligned} p'(t)&=f(x(t),x(g(t)))-f(v_n(t),v_n(g(t)))+M(v_{n+1}(t)-v_n(t))\\ &\qquad+N(v_{n+1}(g(t))-v_n(g(t)))\\ &\ge -Mp(t)-Np(g(t))\quad \text{for } t\in[0,T],\\ p(0)&=0. \end{aligned} \tag{3.15}$$

By Lemma 2.5 the inequality $p(t)\ge 0$ holds, that is, $x(t)\ge v_{n+1}(t)$ for $t\in[0,T]$. By arguments analogous to those above, we get $x(t)\le w_{n+1}(t)$ for $t\in[0,T]$.

By induction we obtain that $v_n(t)\le x(t)\le w_n(t)$ for any $n\in N\cup\{0\}$. After passing to the limit for $n\to\infty$ we get $v(t)\le x(t)\le w(t)$, that is, $v(t)$ is a minimal solution of the IVP (2.1) and $w(t)$ is a maximal solution of the IVP (2.1). □

Remark 3.2. If the conditions of Theorem 3.1 are fulfilled and the IVP (2.1) has a unique solution $x(t)\in[0,T]$, then there exist two sequences $\{v_n(t)\}_0^\infty$ and $\{w_n(t)\}_0^\infty$ that are uniformly convergent to the unique solution $x(t)$ in the interval $[0,T]$.

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