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GENERALIZED BELTRAMI FLOWS AND OTHER CLOSED-FORM SOLUTIONS OF AN UNSTEADY VISCOELASTIC FLUID

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We study flows of an unsteady non-Newtonian fluid by assuming the form of the vorticity a priori. The two forms that have been considered are $\nabla^2\psi = F(t)\psi + G(t)$, which is known as the generalized Beltrami flow and $\nabla^2\psi = f(t)\psi + g(t)y$.

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1. Introduction. At present, numerical solutions to fluid dynamics problems are very attractive due to wide availability of computer codes. But these numerical solutions are insignificant if they cannot be compared with either analytical solutions or experimental results.

Exact solutions of the Navier-Stokes equations are rare since these are nonlinear partial differential equations. Exact solutions are very important not only because they are solutions of some fundamental flows but also because they serve as accuracy checks for experimental, numerical, and asymptotic methods. In an excellent review article, Wang [9] outlines most if not all of the exact solutions to the Navier-Stokes equations.

Over the past decades, non-Newtonian fluids have become more and more important industrially. Polymer solutions and polymer melts provide the most common examples of non-Newtonian fluids. The equations of motion of such fluids are highly nonlinear and one order higher than the Navier-Stokes equations. In spite of the mathematical complexity of these nonlinear equations, there exists a few exact solutions. Kaloni and Huschilt [3], Siddiqui [8], Rajagopal [6, 7], Benharbit and Siddiqui [1], and Labropulu [4, 5] have given a few such exact solutions.

In the present work, following the work of Wang [9, 10], we study generalized Beltrami flows for a non-Newtonian second-grade fluid. These are flows that satisfy $\operatorname{curl}(\omega \times \mathbf{v}) = 0$, $\omega = \operatorname{curl}(\mathbf{v})$, where ω is the vorticity function and $\mathbf{v}$ is the velocity function. For these flows, we assume that $\nabla^2\psi = F(t)\psi + G(t)$ where ψ is the streamfunction. We also obtain solutions when $\nabla^2\psi = f(t)\psi + g(t)y$.

The plan of this paper is as follows: in Section 2, the equations of motion of an unsteady plane incompressible second-grade fluid are given. In Section 3, solutions to generalized Beltrami flows are found. In Section 4, solutions are obtained under the assumption that $\nabla^2\psi = f(t)\psi + g(t)y$.

2. Flow equations. The flow of a viscous incompressible non-Newtonian second-grade fluid, neglecting thermal effects and body forces, is governed by

$$\operatorname{div} \underset{\sim}{V}^* = 0, \qquad \rho \dot{\underset{\sim}{V}}^* = \operatorname{div} \underset{\approx}{T} \tag{2.1}$$

when the constitutive equation for the Cauchy stress tensor $\underset{\approx}{T}$ which describes second-grade fluids given by Coleman and Noll [2] is

$$\begin{gathered} \underset{\approx}{T} = -p^{*}\underset{\approx}{I} + \mu\underset{\approx 1}{A} + \alpha_1\underset{\approx 2}{A} + \alpha_2\underset{\approx 1}{A}^2, \\ \underset{\approx 1}{A} = \left(\operatorname{grad}\underset{\sim}{V}^{*}\right) + \left(\operatorname{grad}\underset{\sim}{V}^{*}\right)^T, \qquad \underset{\approx 2}{A} = \underset{\approx 1}{\dot{A}} + \left(\operatorname{grad}\underset{\sim}{V}^{*}\right)\underset{\approx 1}{A} + \underset{\approx 1}{A}\left(\operatorname{grad}\underset{\sim}{V}^{*}\right). \end{gathered} \tag{2.2}$$

Here $\underset{\sim}{V}^{*}$ is the velocity vector field, p^{*} is the fluid pressure function, ρ is the constant fluid density, μ is the constant coefficient of viscosity, and α_1, α_2 are the normal stress moduli.

Considering the flow to be plane, we take $\underset{\sim}{V}^{*} = (u^{*}(x^{*},y^{*},t^{*}), v^{*}(x^{*},y^{*},t^{*}))$ and $p^{*} = p^{*}(x^{*},y^{*},t^{*})$ so that our flow equations (2.1) and (2.2) take the form

$$\frac{\partial u^{*}}{\partial x^{*}} + \frac{\partial v^{*}}{\partial y^{*}} = 0, \tag{2.3}$$

$$\begin{aligned} &\frac{\partial u^{*}}{\partial t^{*}} + u^{*}\frac{\partial u^{*}}{\partial x^{*}} + v^{*}\frac{\partial u^{*}}{\partial y^{*}} + \frac{1}{\rho}\frac{\partial p^{*}}{\partial x^{*}} \\ &= \nu\nabla^{*2}u^{*} + \frac{\alpha_1}{\rho}\left\{\frac{\partial}{\partial t^{*}}(\nabla^{*2}u^{*})\right. \\ &\qquad + \frac{\partial}{\partial x^{*}}\left[2u^{*}\frac{\partial^2 u^{*}}{\partial x^{*2}} + 2v^{*}\frac{\partial^2 u^{*}}{\partial x^{*}\partial y^{*}} + 4\left(\frac{\partial u^{*}}{\partial x^{*}}\right)^2 + 2\frac{\partial v^{*}}{\partial x^{*}}\left(\frac{\partial v^{*}}{\partial x^{*}} + \frac{\partial u^{*}}{\partial y^{*}}\right)\right] \\ &\qquad \left. + \frac{\partial}{\partial y^{*}}\left[\left(u^{*}\frac{\partial}{\partial x^{*}} + v^{*}\frac{\partial}{\partial y^{*}}\right)\left(\frac{\partial v^{*}}{\partial x^{*}} + \frac{\partial u^{*}}{\partial y^{*}}\right) + 2\frac{\partial u^{*}}{\partial x^{*}}\frac{\partial u^{*}}{\partial y^{*}} + 2\frac{\partial v^{*}}{\partial x^{*}}\frac{\partial v^{*}}{\partial y^{*}}\right]\right\} \\ &\quad + \frac{\alpha_2}{\rho}\frac{\partial}{\partial x^{*}}\left[4\left(\frac{\partial u^{*}}{\partial x^{*}}\right)^2 + \left(\frac{\partial v^{*}}{\partial x^{*}} + \frac{\partial u^{*}}{\partial y^{*}}\right)^2\right], \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\frac{\partial v^{*}}{\partial t^{*}} + u^{*}\frac{\partial v^{*}}{\partial x^{*}} + v^{*}\frac{\partial v^{*}}{\partial y^{*}} + \frac{1}{\rho}\frac{\partial p^{*}}{\partial y^{*}} \\ &= \nu\nabla^{*2}v^{*} + \frac{\alpha_1}{\rho}\left\{\frac{\partial}{\partial t^{*}}(\nabla^{*2}v^{*})\right. \\ &\qquad + \frac{\partial}{\partial x^{*}}\left[2\frac{\partial v^{*}}{\partial x^{*}}\frac{\partial v^{*}}{\partial y^{*}} + \left(u^{*}\frac{\partial}{\partial x^{*}} + v^{*}\frac{\partial}{\partial y^{*}}\right)\left(\frac{\partial v^{*}}{\partial x^{*}} + \frac{\partial u^{*}}{\partial y^{*}}\right) + 2\frac{\partial v^{*}}{\partial x^{*}}\frac{\partial v^{*}}{\partial y^{*}}\right] \\ &\qquad \left. + \frac{\partial}{\partial y^{*}}\left[2u^{*}\frac{\partial^2 v^{*}}{\partial x^{*}\partial y^{*}} + 4\left(\frac{\partial v^{*}}{\partial y^{*}}\right)^2 + 2v^{*}\frac{\partial^2 v^{*}}{\partial x^{*2}} + 2\frac{\partial u^{*}}{\partial y^{*}}\left(\frac{\partial v^{*}}{\partial x^{*}} + \frac{\partial u^{*}}{\partial y^{*}}\right)\right]\right\} \\ &\quad + \frac{\alpha_2}{\rho}\frac{\partial}{\partial y^{*}}\left[4\left(\frac{\partial v^{*}}{\partial y^{*}}\right)^2 + \left(\frac{\partial v^{*}}{\partial x^{*}} + \frac{\partial u^{*}}{\partial y^{*}}\right)^2\right], \end{aligned} \tag{2.5}$$

where $\nu = \mu/\rho$ is the kinematic viscosity. The star on a variable indicates its dimensional form. We non-dimensionalize the above equations according to

$$\begin{gathered} x = \frac{U_0}{\nu}x^{*}, \qquad y = \frac{U_0}{\nu}y^{*}, \qquad t = \frac{U_0^2}{\nu}t^{*}, \\ u = \frac{1}{U_0}u^{*}, \qquad v = \frac{1}{U_0}v^{*}, \qquad p = \frac{1}{\rho U_0^2}p^{*}, \end{gathered} \tag{2.6}$$

where U_0 is some characteristic velocity. The flow equations in non-dimensional form are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.7}$$

$$\begin{aligned}
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial p}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}\\
&\quad=\nabla^2 u+W_e\left\{\frac{\partial}{\partial t}(\nabla^2 u)+\frac{\partial}{\partial x}\left[2u\frac{\partial^2 u}{\partial x^2}+2v\frac{\partial^2 u}{\partial x\partial y}+4\left(\frac{\partial u}{\partial x}\right)^2+2\frac{\partial v}{\partial x}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]\right.\\
&\qquad\left.+\frac{\partial}{\partial y}\left[\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+2\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right]\right\}\\
&\qquad+\beta\frac{\partial}{\partial x}\left[4\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right],
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
&\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{\partial p}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial y}\\
&\quad=\nabla^2 v+W_e\left\{\frac{\partial}{\partial t}(\nabla^2 v)+\frac{\partial}{\partial x}\left[\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+2\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right]\right.\\
&\qquad\left.+\frac{\partial}{\partial y}\left[2u\frac{\partial^2 v}{\partial x\partial y}+2v\frac{\partial^2 v}{\partial y^2}+4\left(\frac{\partial v}{\partial y}\right)^2+2\frac{\partial u}{\partial y}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]\right\}\\
&\qquad+\beta\frac{\partial}{\partial y}\left[4\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right],
\end{aligned} \tag{2.9}$$

where $W_e=\alpha_1 U_0^2/\rho\nu^2$ is the Weissenberg number and $\beta=\alpha_2 U_0^2/\rho\nu^2$.

Continuity equation (2.7) implies the existence of a streamfunction $\psi(x,y,t)$ such that

$$u=\frac{\partial\psi}{\partial y},\qquad v=-\frac{\partial\psi}{\partial x}. \tag{2.10}$$

Substitution of (2.10) in (2.8) and (2.9) and elimination of pressure from the resulting equations using $p_{xy}=p_{yx}$ yields

$$\frac{\partial}{\partial t}(\nabla^2\psi)-W_e\frac{\partial}{\partial t}(\nabla^4\psi)-\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,y)}+W_e\frac{\partial(\psi,\nabla^4\psi)}{\partial(x,y)}-\nabla^4\psi=0. \tag{2.11}$$

Having obtained a solution of (2.11), the velocity components are given by (2.10) and the pressure can be found by integrating equations (2.8) and (2.9).

3. Generalized Beltrami flows.

We assume that

$$\nabla^2\psi=F(t)\psi+G(t). \tag{3.1}$$

Using (3.1) in (2.11), we obtain

$$F(1-W_eF)\frac{\partial\psi}{\partial t}+(F'-2W_eFF'-F^2)\psi=FG-G'+W_e(FG'+F'G), \tag{3.2}$$

where the prime denotes differentiation with respect to time. Thus, the streamfunction $\psi(x, y, t)$ satisfies a system of two linear partial differential equations (3.1) and (3.2).

If $1 - W_e F(t) = 0$ then $F(t) = 1/W_e$ and $\psi = -W_e G(t)$ which corresponds to an irrotational flow. We exclude this case from further consideration. In the following, we assume that $1 - W_e F(t) \neq 0$.

We consider the following two cases:

(1) $F(t) = 0$.

(2) $F(t) \neq 0$.

3.1. Solutions when $F(t) = 0$. In this case, (3.2) implies that

$$G(t) = \text{constant} = a_0. \tag{3.3}$$

Thus, the streamfunction $\psi(x, y)$ satisfies

$$\nabla^2 \psi = a_0 \tag{3.4}$$

with the general solution given by

$$\psi = \frac{1}{2}(a_0 - a_1)x^2 + a_2 x + \frac{1}{2}a_1 y^2 + a_3 y + a_4, \tag{3.5}$$

where a_1 to a_4 are arbitrary constants. This is a steady-state solution with constant vorticity. In the above expression for the streamfunction, we can add any irrotational solution. This solution is independent of the Weissenberg number W_e, thus the same as the Newtonian case studied by Wang [10] who gave some useful nontrivial solutions as follows.

(a) Source or vortex in shear flow

$$\begin{aligned} \psi &= ay + by^2 + c\tan^{-1}\frac{y}{x}, \\ \psi &= ay + by^2 + c\ln(x^2 + y^2). \end{aligned} \tag{3.6}$$

(b) Shear flow over convection cells

$$\psi = ay^2 + be^{-\lambda y}\cos\lambda x, \quad \lambda > 0. \tag{3.7}$$

Figure 3.1 depicts the streamlines corresponding to this flow.

(c) Elliptic vortex of Kirchhoff

$$\psi = ax^2 + by^2, \quad a, b > 0. \tag{3.8}$$

(d) Oblique impingement of two jets

$$\psi = ay^2 + bxy. \tag{3.9}$$

Other useful nontrivial solutions are given by

(e)

$$\psi = x + x^2 - y^2 + \ln(x^2 + y^2). \tag{3.10}$$

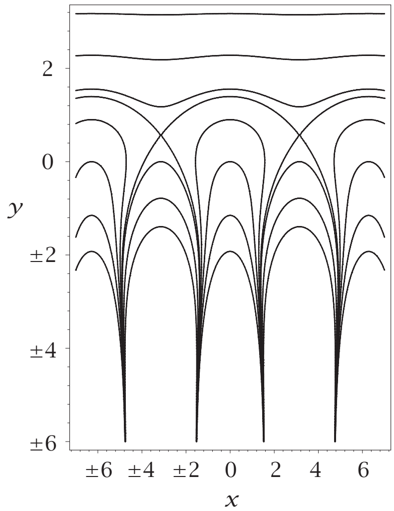

FIGURE 3.1. Streamline pattern for $-y^2 + 2e^{-y}\cos x = \text{constant}$.

The streamlines are shown in Figure 3.2. This corresponds to impingement of two jets.

(f)

$$\psi = 4x^2 + 6y^2 - 3y - \ln(x^2 + y^2). \tag{3.11}$$

Figure 3.3 shows the streamline pattern.

(g)

$$\psi = x^2 - 3x + 2xy + \tan^{-1}\left(\frac{y}{x}\right). \tag{3.12}$$

Figure 3.4 depicts these streamlines.

(h)

$$\psi = x^2 + y^2 - x + 10y + 5xy. \tag{3.13}$$

The streamline pattern is shown in Figure 3.5.

(i)

$$\psi = x^2 + 3y^2 + 3x^2y - y^3. \tag{3.14}$$

(j)

$$\psi = x^2 + y^2 - 5x + xy^3 - x^3y. \tag{3.15}$$

3.2. Solutions when $F(t) \neq 0$. Dividing (3.2) by $F - W_eF^2 \neq 0$, we get

$$\frac{\partial \psi}{\partial t} + \left(\frac{F' - 2W_eFF'}{F - W_eF^2} - \frac{F}{1 - W_eF}\right)\psi = \frac{1}{F - W_eF^2}\left[FG - G' + W_e(FG' + F'G)\right] \tag{3.16}$$

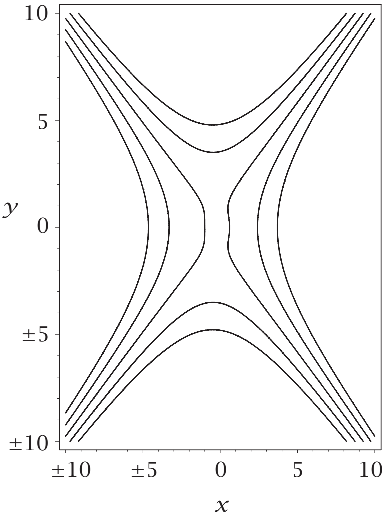

FIGURE 3.2. Streamline pattern for $x^2 - y^2 + x + \ln(x^2 + y^2) = \text{constant}$.

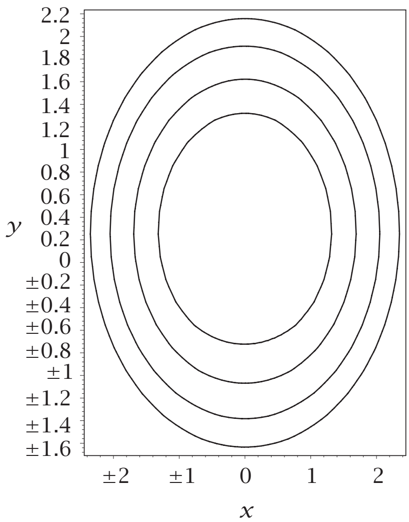

FIGURE 3.3. Streamline pattern for $4x^2 + 6y^2 - 3y - \ln(x^2 + y^2) = \text{constant}$.

which upon one integration gives

$$\psi = -\frac{G}{F} + \frac{1}{F - W_e F^2} \exp\left[\int \frac{F}{1 - W_e F} dt\right] h(x, y), \tag{3.17}$$

where $h(x, y)$ is an unknown function to be determined.

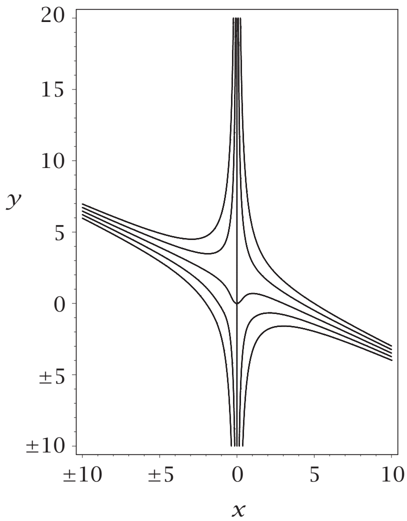

FIGURE 3.4. Streamline pattern for $x^2 - 3x + 2xy + \tan^{-1}(y/x)x = \text{constant}$.

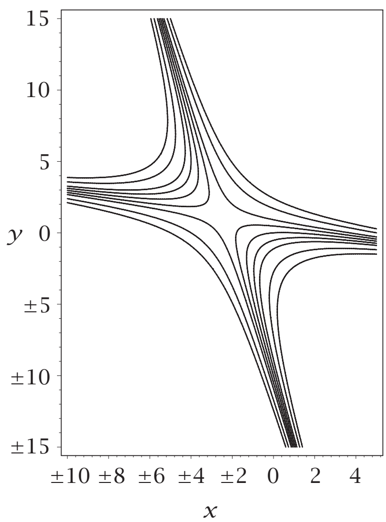

FIGURE 3.5. Streamline pattern for $x^2 + y^2 - x + 10y + 5xy = \text{constant}$.

Employing (3.17) in (3.1), we obtain

$$\nabla^2 h = Fh \tag{3.18}$$

which implies that

$$\frac{\nabla^2 h}{h} = F(t) = \text{constant} = A \neq 0. \tag{3.19}$$

Thus, the streamfunction is given by

$$\psi = -\frac{G(t)}{A} + \frac{1}{A(1-W_eA)}\exp\left[\frac{At}{1-W_eA}\right]h(x,y), \tag{3.20}$$

where $G(t)$ is any function of time t and $h(x,y)$ satisfies the following equation:

$$\nabla^2 h = Ah. \tag{3.21}$$

If we assume that $h(x,y) = X(x)Y(y)$, then (3.21) gives

$$\begin{aligned} X''(x) - \lambda X(x) &= 0, \\ Y''(y) + (\lambda - A)Y(y) &= 0, \end{aligned} \tag{3.22}$$

where λ is the separation constant.

Thus, the function $h(x,y)$ is given by

$$h(x,y) = \begin{cases} (a_0 + a_1 x)\left(c_0 e^{\sqrt{A}y} + c_1 e^{-\sqrt{A}y}\right), & \text{if } \lambda = 0,\ A > 0; \\ (a_0 + a_1 x)\left(c_2 \cos\sqrt{-A}y + c_3 \sin\sqrt{-A}y\right), & \text{if } \lambda = 0,\ A < 0; \\ \left(a_2 e^{kx} + a_3 e^{-kx}\right)\left(c_4 e^{\sqrt{A-k^2}y} + c_5 e^{-\sqrt{A-k^2}y}\right), & \text{if } \lambda = k^2,\ A - k^2 > 0; \\ \left(a_2 e^{kx} + a_3 e^{-kx}\right)\left(c_6 + c_7 y\right), & \text{if } \lambda = k^2,\ A = k^2; \\ \left(a_2 e^{kx} + a_3 e^{-kx}\right)\left(c_8 \cos\sqrt{k^2 - A}y + c_9 \sin\sqrt{k^2 - A}y\right), & \text{if } \lambda = k^2,\ A - k^2 < 0; \\ \left(a_4 \cos kx + a_5 \sin kx\right)\left(c_{10} e^{\sqrt{A+k^2}y} + c_{11} e^{-\sqrt{A+k^2}y}\right), & \text{if } \lambda = -k^2,\ A + k^2 > 0; \\ \left(a_4 \cos kx + a_5 \sin kx\right)\left(c_{12} + c_{13} y\right), & \text{if } \lambda = -k^2,\ A = -k^2; \\ \left(a_4 \cos kx + a_5 \sin kx\right)\left(c_{14} \cos\sqrt{-A-k^2}y + c_{15} \sin\sqrt{-A-k^2}y\right), & \text{if } \lambda = -k^2,\ A + k^2 < 0, \end{cases} \tag{3.23}$$

where a_0 to a_5 and c_0 to c_{15} are arbitrary constants of integration.

Assuming that $h(x,y) = X(x) + Y(y)$, then (3.21) gives

$$X''(x) - AX(x) = \lambda, \qquad Y''(y) - AY(y) = -\lambda, \tag{3.24}$$

where λ is the separation constant.

Thus, the function $h(x,y)$ is given by

$$h(x,y) = \begin{cases} b_0 e^{\sqrt{A}x} + b_1 e^{-\sqrt{A}x} + b_2 e^{\sqrt{A}y} + b_3 e^{-\sqrt{A}y} - \dfrac{\lambda}{A}, & \text{if } A > 0; \\ b_4 \cos\sqrt{-A}x + b_5 \sin\sqrt{-A}x + b_6 \cos\sqrt{-A}y + b_7 \sin\sqrt{-A}y + \dfrac{\lambda}{A}, & \text{if } A < 0, \end{cases} \tag{3.25}$$

where b_0 to b_7 are constants of integration.

4. Solutions for $\nabla^2\psi = f(t)\psi + g(t)y$. We assume that

$$\nabla^2\psi = f(t)\psi + g(t)y. \tag{4.1}$$

Using (4.1) in (2.11), we obtain

$$f(1-W_e f)\frac{\partial\psi}{\partial t} - g(1-W_e f)\frac{\partial\psi}{\partial x} + (f' - 2W_e f f' - f^2)\psi = [fg - g' + W_e(fg' + f'g)]y, \tag{4.2}$$

where the prime denotes differentiation with respect to time. Thus, the streamfunction $\psi(x,y,t)$ satisfies a system of two linear partial differential equations (4.1) and (4.2).

If $1 - W_e f(t) = 0$ then $f(t) = 1/W_e$ and $\psi = -W_e g(t)y$ which corresponds to an irrotational flow. We exclude this case from further consideration. In the following, we assume that $1 - W_e f(t) \neq 0$.

We have to consider the following three cases.

(1) $f(t) = 0$, $g(t) \neq 0$.

(2) $g(t) = 0$, $f(t) \neq 0$.

(3) $f(t) \neq 0$, $g(t) \neq 0$.

4.1. Solutions when $f(t) = 0$, $g(t) \neq 0$. In this case, (4.2) implies that

$$\frac{\partial\psi}{\partial x} = \frac{g'}{g}y \tag{4.3}$$

which upon one integration with respect to x gives

$$\psi = \frac{g'}{g}xy + f(y), \tag{4.4}$$

where $f(y)$ is an unknown function of y to be determined. Using (4.4) in (4.1), we get

$$\frac{1}{y}\frac{d^2 f}{dy^2} = g(t) = 6b_0 = \text{constant}. \tag{4.5}$$

Integrating $(1/y)(d^2f/dy^2) = 6b_0$ twice with respect to y, we obtain

$$f(y) = b_0 y^3 + b_1 y + b_2, \tag{4.6}$$

where b_0, b_1, and b_2 are arbitrary constants. Thus, the streamfunction is given by

$$\psi(x,y) = b_0 y^3 + b_1 y + b_2. \tag{4.7}$$

This is a steady-state solution.

4.2. Solutions when $g(t) = 0$, $f(t) \neq 0$. If $g(t) = 0$, then (4.2) becomes

$$\frac{\partial\psi}{\partial t} + \frac{f' - 2W_e f f' - f^2}{f(1-W_e f)}\psi = 0 \tag{4.8}$$

which upon one integration with respect to time t gives

$$\psi = \frac{1}{f - W_e f^2} \exp\left[\int \frac{f}{1 - W_e f} dt\right] h(x, y), \tag{4.9}$$

where $h(x, y)$ is a function to be determined.

Employing equation (4.9) into (4.1), we obtain

$$\frac{\nabla^2 h}{h} = f(t) = \text{constant} = A. \tag{4.10}$$

Hence the streamfunction $\psi(x, y, t)$ is given by

$$\psi = \frac{1}{A - W_e A^2} \exp\left[\frac{At}{1 - W_e A}\right] h(x, y), \tag{4.11}$$

where $h(x, y)$ satisfies

$$\nabla^2 h = Ah. \tag{4.12}$$

Solutions for this equation are given by (3.23) and (3.25) above.

4.3. Solutions when $f(t) \neq 0$, $g(t) \neq 0$. Dividing (4.2) by $f - W_e f^2 \neq 0$, we get

$$\frac{\partial \psi}{\partial t} - \frac{g}{f}\frac{\partial \psi}{\partial x} + \left(\frac{f' - 2W_e f f'}{f - W_e f^2} - \frac{f}{1 - W_e f}\right)\psi = \frac{1}{f - W_e f^2}\left[fg - g' + W_e(fg' + f'g)\right]y. \tag{4.13}$$

Introducing new variables $\xi = x + \int (g/f) dt$ and t, we find that

$$\frac{\partial(\xi, t)}{\partial(x, t)} = 1 \neq 0. \tag{4.14}$$

Transforming (4.13) into new independent variables ξ, t, we have

$$\frac{\partial \psi}{\partial t} + \left(\frac{f' - 2W_e f f'}{f - W_e f^2} - \frac{f}{1 - W_e f}\right)\psi = \frac{fg - g' + W_e(fg' + f'g)}{f - W_e f^2} y. \tag{4.15}$$

The general solution of this equation is given by

$$\psi = -\frac{gy}{f} + \frac{1}{f - W_e f^2} \exp\left[\int \frac{f}{1 - W_e f} dt\right] H(\xi, y), \tag{4.16}$$

where $H(\xi, y)$ is a function to be determined.

Employing (4.16) into (4.1), we obtain

$$\frac{\partial^2 H}{\partial \xi^2} + \frac{\partial^2 H}{\partial y^2} = fH. \tag{4.17}$$

Since (ξ, t) are independent variables, then we must have

$$f(t) = \text{constant} = B \neq 0. \tag{4.18}$$

Hence, the streamfunction is given by

$$\psi = -\frac{g(t)y}{B} + \frac{1}{B - W_e B^2} \exp\left[\frac{Bt}{1 - W_e B}\right] H(\xi, y), \tag{4.19}$$

where $g(t)$ is any function of time t and $H(\xi,y)$ must satisfy the following equation:

$$\frac{\partial^2 H}{\partial \xi^2}+\frac{\partial^2 H}{\partial y^2}=BH. \tag{4.20}$$

Assuming that $H(\xi,y)=X(\xi)Y(y)$, then (4.20) gives

$$X''(\xi)-\lambda X(\xi)=0, \qquad Y''(y)+(\lambda-B)Y(y)=0, \tag{4.21}$$

where λ is the separation constant.

Thus, the function $H(\xi,y)$ is given by

$$H(\xi,y)$$
$$=\begin{cases}(a_0+a_1\xi)\left(b_0e^{\sqrt{B}y}+b_1e^{-\sqrt{B}y}\right), & \text{if } \lambda=0,\ B>0;\\ (a_0+a_1\xi)\left(b_2\cos\sqrt{-B}y+b_3\sin\sqrt{-B}y\right), & \text{if } \lambda=0,\ B<0;\\ (a_2e^{k\xi}+a_3e^{-k\xi})\left(b_4e^{\sqrt{B-k^2}y}+b_5e^{-\sqrt{B-k^2}y}\right), & \text{if } \lambda=k^2,\ B-k^2>0;\\ (a_2e^{k\xi}+a_3e^{-k\xi})(b_6+b_7y), & \text{if } \lambda=k^2,\ B=k^2;\\ (a_2e^{k\xi}+a_3e^{-k\xi})\left(b_8\cos\sqrt{k^2-B}y+b_9\sin\sqrt{k^2-B}y\right), & \text{if } \lambda=k^2,\ B-k^2<0;\\ (a_4\cos k\xi+a_5\sin k\xi)\left(b_{10}e^{\sqrt{B+k^2}y}+b_{11}e^{-\sqrt{B+k^2}y}\right), & \text{if } \lambda=-k^2,\ B+k^2>0;\\ (a_4\cos k\xi+a_5\sin k\xi)(b_{12}+b_{13}y), & \text{if } \lambda=-k^2,\ B=-k^2;\\ (a_4\cos k\xi+a_5\sin k\xi)\left(b_{14}\cos\sqrt{-B-k^2}y+b_{15}\sin\sqrt{-B-k^2}y\right), & \text{if } \lambda=-k^2,\ B+k^2<0,\end{cases} \tag{4.22}$$

where a_0 to a_5 and b_0 to b_{15} are arbitrary constants of integration.

Assuming that $H(\xi,y)=X(\xi)+Y(y)$, then (4.20) gives

$$X''(\xi)-BX(\xi)=\lambda, \qquad Y''(y)-BY(y)=-\lambda, \tag{4.23}$$

where λ is the separation constant.

Thus, the function $H(\xi,y)$ is given by

$$H(\xi,y)=\begin{cases}c_0e^{\sqrt{B}\xi}+c_1e^{-\sqrt{B}\xi}+c_2e^{\sqrt{B}y}+c_3e^{-\sqrt{B}y}-\dfrac{\lambda}{B}, & \text{if } B>0;\\ c_4\cos\sqrt{-B}\xi+c_5\sin\sqrt{-B}\xi+c_6\cos\sqrt{-B}y+c_7\sin\sqrt{-B}y+\dfrac{\lambda}{B}, & \text{if } B<0,\end{cases} \tag{4.24}$$

where c_0 to c_7 are constants of integration.

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