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ON THE SINE INTEGRAL AND THE CONVOLUTION

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The sine integral $\mathrm{Si}(\lambda x)$ and the cosine integral $\mathrm{Ci}(\lambda x)$ and their associated functions $\mathrm{Si}_+(\lambda x)$, $\mathrm{Si}_-(\lambda x)$, $\mathrm{Ci}_+(\lambda x)$, $\mathrm{Ci}_-(\lambda x)$ are defined as locally summable functions on the real line. Some convolutions of these functions and $\sin(\mu x)$, $\sin_+(\mu x)$, and $\sin_-(\mu x)$ are found.

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The *sine integral* $\mathrm{Si}(x)$ is defined by

$$\int_0^x u^{-1}\sin u\,du, \tag{1}$$

(see Sneddon [6]). This integral is convergent for all x. More generally, for all $\lambda \neq 0$, we define $\mathrm{Si}(\lambda x)$ by

$$\mathrm{Si}(\lambda x) = \int_0^{\lambda x} u^{-1}\sin u\,du = \int_0^x u^{-1}\sin(\lambda u)\,du; \tag{2}$$

and we define $\mathrm{Si}_+(\lambda x)$ and $\mathrm{Si}_-(\lambda x)$ by

$$\mathrm{Si}_+(\lambda x) = H(x)\,\mathrm{Si}(\lambda x), \qquad \mathrm{Si}_-(\lambda x) = H(-x)\,\mathrm{Si}(\lambda x), \tag{3}$$

(see [1]).

It is easily proved that

$$[\mathrm{Si}_+(\lambda x)]' = \sin(\lambda x)x_+^{-1}. \tag{4}$$

We need the following lemma which was proved in [1].

Lemma 1. *If* $\lambda \neq 0$, *then*

$$\int_0^\infty u^{-1}\sin(\lambda u)\,du = \frac{1}{2}\operatorname{sgn}\lambda\cdot\pi. \tag{5}$$

The cosine integral $\mathrm{Ci}(x)$ is defined for $x > 0$ by

$$\mathrm{Ci}(x) = -\int_x^\infty u^{-1}\cos u\,du, \tag{6}$$

(see Sneddon [6]). This integral is divergent for $x \le 0$; but in [3], $\mathrm{Ci}(\lambda x)$ was defined as a locally summable function on the real line by

$$\mathrm{Ci}(\lambda x) = -\int_{\lambda x}^\infty u^{-1}[\cos u - H(1-u)]\,du + H(1-\lambda x)\ln|\lambda x|, \tag{7}$$

where H denotes Heaviside's function. In particular,

$$\mathrm{Ci}(x) = -\int_x^\infty u^{-1}[\cos u - H(1-u)]\,du + H(1-x)\ln|x|. \tag{8}$$

It was proved in [4] that the cosine integral is an even function. We can therefore define $\mathrm{Ci}(\lambda x)$ by

$$\mathrm{Ci}(\lambda x) = -\int_{|\lambda x|}^\infty u^{-1}\cos u\,du = -\int_{|x|}^\infty u^{-1}\cos(\lambda u)\,du, \quad \lambda, x \neq 0, \tag{9}$$

simplifying the definition given in [3].

The locally summable functions $\mathrm{Ci}_+(\lambda x)$ and $\mathrm{Ci}_-(\lambda x)$ are now defined for $\lambda \neq 0$ by

$$\mathrm{Ci}_+(\lambda x) = H(x)\,\mathrm{Ci}(\lambda x), \qquad \mathrm{Ci}_-(\lambda x) = H(-x)\,\mathrm{Ci}(\lambda x). \tag{10}$$

It was proved in [3] that

$$[\mathrm{Ci}_+(\lambda x)]' = \cos(\lambda x)x_+^{-1} - (c - \ln|\lambda|)\delta(x), \tag{11}$$

where

$$c = \int_0^\infty u^{-1}[\cos u - H(1-u)]\,du. \tag{12}$$

We also need the following lemma which was also proved in [4].

Lemma 2. *If $x > 0$, then*

$$\int_0^x u^{-1}[\cos(\lambda u) - 1]\,du = c + \mathrm{Ci}(\lambda x) - \ln|\lambda x|. \tag{13}$$

The classical definition of the convolution of two functions f and g is as follows.

Definition 3. Let f and g be functions. Then the *convolution* $f * g$ is defined by

$$(f * g)(x) = \int_{-\infty}^\infty f(t)g(x-t)\,dt, \tag{14}$$

for all points x for which the integral exists.

It follows easily from the definition that if $f * g$ exists then $g * f$ exists and

$$f * g = g * f, \tag{15}$$

and if $(f * g)'$ and $f * g'$ (or $f' * g$) exists, then

$$(f * g)' = f * g' \,(\text{or } f' * g). \tag{16}$$

Definition 3 can be extended to define the convolution $f * g$ of two distributions f and g in $\mathcal{D}'$ with the following definition, see Gel'fand and Shilov [5].

Definition 4. Let f and g be distributions in $\mathscr{D}'$. Then the *convolution* $f * g$ is defined by the equation

$$\langle (f*g)(x),\phi\rangle = \langle f(y),\langle g(x),\phi(x+y)\rangle\rangle \tag{17}$$

for arbitrary ϕ in $\mathscr{D}$, provided that f and g satisfy either of the following conditions:

(a) either f or g has bounded support,

(b) the supports of f and g are bounded on the same side.

It follows that if the convolution $f * g$ exists by this definition then (15) and (16) are satisfied.

In the following, the locally summable functions $\sin_\pm(\lambda x)$ and $\cos_\pm(\lambda x)$ are defined by

$$\begin{aligned} \sin_+(\lambda x) &= H(x)\sin(\lambda x), & \sin_-(\lambda x) &= H(-x)\sin(\lambda x), \\ \cos_+(\lambda x) &= H(x)\cos(\lambda x), & \cos_-(\lambda x) &= H(-x)\cos(\lambda x). \end{aligned} \tag{18}$$

Theorem 5. *If* $\lambda,\mu \neq 0$, *then the convolution* $\mathrm{Si}_+(\lambda x) * \sin_+(\mu x)$ *exists and*

$$\begin{aligned} \mathrm{Si}_+(\lambda x) * \sin_+(\mu x) = {}& -\frac{1}{2}\mu^{-1}\sin(\mu x)\{\mathrm{Ci}_+[(\lambda-\mu)x]-\mathrm{Ci}_+[(\lambda+\mu)x]\} \\ & +\mu^{-1}\mathrm{Si}_+(\lambda x)-\frac{1}{2}\mu^{-1}\cos(\mu x)\{\mathrm{Si}_+[(\lambda-\mu)x]+\mathrm{Si}_+[(\lambda+\mu)x]\} \\ & -\frac{1}{2}\mu^{-1}\ln\left|\frac{\lambda+\mu}{\lambda-\mu}\right|\sin_+(\mu x) \end{aligned} \tag{19}$$

if $\lambda \neq \pm\mu$; *and*

$$\begin{aligned} \mathrm{Si}_+(\lambda x) * \sin_+(\lambda x) = {}& \lambda^{-1}\mathrm{Si}_+(\lambda x)-\frac{1}{2}\lambda^{-1}\left[\ln\left|\frac{1}{2}\lambda\right|-c\right]\sin_+(\lambda x) \\ & -\frac{1}{2}\lambda^{-1}\cos(\lambda x)\mathrm{Si}_+(2\lambda x) \\ & +\frac{1}{2}\lambda^{-1}\sin(\lambda x)[\mathrm{Ci}_+(2\lambda x)-\ln x_+] \end{aligned} \tag{20}$$

if $\lambda = \pm\mu$.

Proof. It is obvious that $\mathrm{Si}_+(\lambda x) * \sin_+(\mu x) = 0$ if $x<0$. If $x>0$ and $\lambda \neq \pm\mu$, we have

$$\begin{aligned} \mathrm{Si}_+(\lambda x) * \sin_+(\mu x) &= \int_0^x \sin[\mu(x-t)]\int_0^t u^{-1}\sin(\lambda u)\,du\,dt \\ &= \int_0^x u^{-1}\sin(\lambda u)\int_u^x \sin[\mu(x-t)]\,dt\,du \\ &= \mu^{-1}\int_0^x u^{-1}\sin(\lambda u)\{1-\cos[\mu(x-u)]\}\,du && (21) \\ &= \mu^{-1}\mathrm{Si}(\lambda x)-\mu^{-1}I, && (22) \end{aligned}$$

where

$$\begin{aligned}
I &= \int_0^x u^{-1}\sin(\lambda u)\cos[\mu(x-u)]\,du \\
&= \cos(\mu x)\int_0^x u^{-1}\sin(\lambda u)\cos(\mu u)\,du + \sin(\mu x)\int_0^x u^{-1}\sin(\lambda u)\sin(\mu u)\,du \\
&= \frac{1}{2}\cos(\mu x)\int_0^x u^{-1}\{\sin[(\lambda-\mu)u]+\sin[(\lambda+\mu)u]\}\,du \\
&\quad + \frac{1}{2}\sin(\mu x)\int_0^x u^{-1}\{\cos[(\lambda-\mu)u]-\cos[(\lambda+\mu)u]\}\,du \\
&= \frac{1}{2}\cos(\mu x)\{\mathrm{Si}[(\lambda-\mu)x]+\mathrm{Si}[(\lambda+\mu)x]\} \\
&\quad + \frac{1}{2}\sin(\mu x)\left\{\mathrm{Ci}[(\lambda-\mu)x]-\mathrm{Ci}[(\lambda+\mu)x]+\ln\left|\frac{\lambda+\mu}{\lambda-\mu}\right|\right\}
\end{aligned} \tag{23}$$

on using Lemma 2; and (19) follows from (22) and (23).

If $\lambda = \pm\mu$, (21) is replaced by

$$\begin{aligned}
\mathrm{Si}_+(\lambda x) * \sin_+(\lambda x) &= \lambda^{-1}\int_0^x u^{-1}\sin(\lambda u)\{1-\cos[\lambda(x-u)]\}\,du \\
&= \lambda^{-1}\mathrm{Si}(\lambda x) - \lambda^{-1}J,
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
J &= \int_0^x u^{-1}\sin(\lambda u)\cos[\lambda(x-u)]\,du \\
&= \frac{1}{2}\cos(\lambda x)\int_0^x u^{-1}\sin(2\lambda u)\,du - \frac{1}{2}\sin(\lambda x)\int_0^x u^{-1}[\cos(2\lambda u)-1]\,du \\
&= \frac{1}{2}\cos(\lambda x)\,\mathrm{Si}(2\lambda x) + \frac{1}{2}[\ln|2\lambda x| - c]\sin(\lambda x) - \frac{1}{2}\sin(\lambda x)\,\mathrm{Ci}(2\lambda x)
\end{aligned} \tag{25}$$

on using Lemma 2; and (20) follows from (24) and (25). □

Corollary 6. *If $\lambda,\mu \neq 0$, then the convolution $\mathrm{Si}_+(\lambda x) * \cos_+(\mu x)$ exists and*

$$\begin{aligned}
\mathrm{Si}_+(\lambda x) * \cos_+(\mu x) = {} & \frac{1}{2}\mu^{-1}\sin(\mu x)\{\mathrm{Si}_+[(\lambda-\mu)x]+\mathrm{Si}_+[(\lambda+\mu)x]\} \\
& - \frac{1}{2}\mu^{-1}\cos(\mu x)\{\mathrm{Ci}_+[(\lambda-\mu)x]-\mathrm{Ci}_+[(\lambda+\mu)x]\} \\
& - \frac{1}{2}\mu^{-1}\ln\left|\frac{\lambda+\mu}{\lambda-\mu}\right|\cos_+(\mu x)
\end{aligned} \tag{26}$$

if $\lambda \neq \pm\mu$; and

$$\begin{aligned}
\mathrm{Si}_+(\lambda x) * \cos_+(\lambda x) = {} & \frac{1}{2}\lambda^{-1}\sin(\lambda x)\,\mathrm{Si}_+(2\lambda x) - \frac{1}{2}\lambda^{-1}\left[\ln\left|\frac{1}{2}\lambda\right| - c\right]\cos_+(\lambda x) \\
& + \frac{1}{2}\lambda^{-1}\cos(\lambda x)[\mathrm{Ci}_+(2\lambda x) - \ln x_+]
\end{aligned} \tag{27}$$

if $\lambda = \pm\mu$.

Proof. It follows from (4), (11), (16), and (19) that

$$
\begin{aligned}
\left[\operatorname{Si}_+(\lambda x) * \sin_+(\mu x)\right]' &= \mu \operatorname{Si}_+(\lambda x) * \cos_+(\mu x) \\
&= \mu^{-1}\sin(\lambda x)x_+^{-1} - \frac{1}{2}\ln\left|\frac{\lambda+\mu}{\lambda-\mu}\right|\cos_+(\mu x) \\
&\quad - \frac{1}{2}\cos(\mu x)\{\operatorname{Ci}_+[(\lambda-\mu)x] - \operatorname{Ci}_+[(\lambda+\mu)x]\} \\
&\quad - \frac{1}{2}\mu^{-1}\sin(\mu x)\{\cos[(\lambda-\mu)x] - \cos[(\lambda+\mu)x]\}x_+^{-1} \\
&\quad - \frac{1}{2}\mu^{-1}\ln\left|\frac{\lambda+\mu}{\lambda-\mu}\right|\sin(\mu x)\delta(x) \\
&\quad + \frac{1}{2}\sin(\mu x)\{\operatorname{Si}_+[(\lambda-\mu)x] + \operatorname{Si}_+[(\lambda+\mu)x]\} \\
&\quad - \frac{1}{2}\mu^{-1}\cos(\mu x)\{\sin[(\lambda+\mu)x] + \sin[(\lambda-\mu)x]\}x_+^{-1} \\
&= -\frac{1}{2}\ln\left|\frac{\lambda+\mu}{\lambda-\mu}\right|\cos_+(\mu x) \\
&\quad - \frac{1}{2}\cos(\mu x)\{\operatorname{Ci}_+[(\lambda+\mu)x] - \operatorname{Ci}_+[(\lambda-\mu)x]\} \\
&\quad + \frac{1}{2}\sin(\mu x)\{\operatorname{Si}_+[(\lambda+\mu)x] + \operatorname{Si}_+[(\lambda-\mu)x]\}
\end{aligned}
\tag{28}
$$

and (26) follows.

If $\lambda = \pm\mu$, it follows from (4), (11), (16), and (20) that

$$
\begin{aligned}
\lambda \operatorname{Si}_+(\lambda x) * \cos_+(\lambda x) &= \lambda^{-1}\sin(\lambda x)x_+^{-1} - \frac{1}{2}\left[\ln\left|\frac{1}{2}\lambda\right| - c\right]\cos_+(\lambda x) \\
&\quad + \frac{1}{2}\sin(\lambda x)\operatorname{Si}_+(2\lambda x) - \frac{1}{2}\lambda^{-1}\cos(\lambda x)\sin(2\lambda x)x_+^{-1} \\
&\quad + \frac{1}{2}\cos(\lambda x)[\operatorname{Ci}_+(2\lambda x) - \ln x_+] \\
&\quad + \frac{1}{2}\lambda^{-1}\sin(\lambda x)[\cos(2\lambda x) - 1]x_+^{-1} \\
&= -\frac{1}{2}\left[\ln\left|\frac{1}{2}\lambda\right| - c\right]\cos_+(\lambda x) + \frac{1}{2}\sin(\lambda x)\operatorname{Si}_+(2\lambda x) \\
&\quad + \frac{1}{2}\cos(\lambda x)[\operatorname{Ci}_+(2\lambda x) - \ln x_+]
\end{aligned}
\tag{29}
$$

and (27) follows. □

Corollary 7. *If $\lambda, \mu \neq 0$, then the convolutions $\operatorname{Si}_-(\lambda x) * \sin_-(\mu x)$ and $\operatorname{Si}_-(\lambda x) * \cos_-(\mu x)$ exist and*

$$
\begin{aligned}
\operatorname{Si}_-(\lambda x) * \sin_-(\mu x) &= \frac{1}{2}\mu^{-1}\sin(\mu x)\{\operatorname{Ci}_-[(\lambda-\mu)x] - \operatorname{Ci}_-[(\lambda+\mu)x]\} \\
&\quad - \mu^{-1}\operatorname{Si}_-(\lambda x) + \frac{1}{2}\mu^{-1}\cos(\mu x)\{\operatorname{Si}_-[(\lambda-\mu)x] + \operatorname{Si}_-[(\lambda+\mu)x]\} \\
&\quad + \frac{1}{2}\mu^{-1}\ln\left|\frac{\lambda+\mu}{\lambda-\mu}\right|\sin_-(\mu x),
\end{aligned}
\tag{30}
$$

$$\begin{aligned}\operatorname{Si}_-(\lambda x) * \cos_-(\mu x) = &-\frac{1}{2}\mu^{-1}\sin(\mu x)\{\operatorname{Si}_-[(\lambda-\mu)x]+\operatorname{Si}_-[(\lambda+\mu)x]\}\\ &+\frac{1}{2}\mu^{-1}\cos(\mu x)\{\operatorname{Ci}_-[(\lambda-\mu)x]-\operatorname{Ci}_-[(\lambda+\mu)x]\}\\ &+\frac{1}{2}\mu^{-1}\ln\left|\frac{\lambda+\mu}{\lambda-\mu}\right|\sin_-(\mu x)\end{aligned} \tag{31}$$

if $\lambda \neq \pm\mu$; *and*

$$\begin{aligned}\operatorname{Si}_-(\lambda x) * \sin_-(\lambda x) = &-\lambda^{-1}\operatorname{Si}_-(\lambda x)+\frac{1}{2}\lambda^{-1}\left[\ln\left|\frac{1}{2}\lambda\right|-c\right]\sin_-(\lambda x)\\ &+\frac{1}{2}\lambda^{-1}\cos(\lambda x)\operatorname{Si}_-(2\lambda x)\\ &-\frac{1}{2}\lambda^{-1}\sin(\lambda x)[\operatorname{Ci}_-(2\lambda x)-\ln x_-],\end{aligned} \tag{32}$$

$$\begin{aligned}\operatorname{Si}_-(\lambda x) * \cos_-(\lambda x) = &-\frac{1}{2}\lambda^{-1}\sin(\lambda x)\operatorname{Si}_-(2\lambda x)+\frac{1}{2}\lambda^{-1}\left[\ln\left|\frac{1}{2}\lambda\right|-c\right]\cos_-(\lambda x)\\ &-\frac{1}{2}\lambda^{-1}\cos(\lambda x)[\operatorname{Ci}_-(2\lambda x)-\ln x_-]\end{aligned} \tag{33}$$

if $\lambda = \pm\mu$.

Proof. Equations (30) and (31) follow on replacing x by $-x$ in (19) and (26), respectively. Equations (32) and (33) follow on replacing x by $-x$ in (20) and (27), respectively.

Definition 4 of the convolution is rather restrictive and so a neutrix convolution was introduced in [2]. In order to define the neutrix convolution we, first of all, let τ be a function in $\mathscr{D}$ satisfying the following properties:

(i) $\tau(x) = \tau(-x)$,
(ii) $0 \leq \tau(x) \leq 1$,
(iii) $\tau(x) = 1$ for $|x| \leq 1/2$,
(iv) $\tau(x) = 0$ for $|x| \geq 1$.

The function τ_ν is now defined by

$$\tau_\nu(x) = \begin{cases} 1, & |x| \leq \nu,\\ \tau(\nu^\nu x - \nu^{\nu+1}), & x > \nu,\\ \tau(\nu^\nu x + \nu^{\nu+1}), & x < -\nu,\end{cases} \tag{34}$$

for $\nu > 0$. □

The following definition of the neutrix convolution was given in [2].

Definition 8. Let f and g be distributions in $\mathscr{D}'$ and let $f_\nu = f\tau_\nu$ for $\nu > 0$. Then the *neutrix convolution* $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_\nu * g\}$, provided that the limit h exists in the sense that

$$\underset{\nu\to\infty}{\text{N}-\lim}\langle f_\nu * g, \varphi\rangle = \langle h, \varphi\rangle, \tag{35}$$

for all φ in $\mathcal{D}$, where N is the neutrix (see van der Corput [7]), having domain N' the positive reals, range N'' the real numbers and with negligible functions finite linear sums of the functions

$$\nu^{\lambda}\ln^{r-1}\nu, \quad \ln^{r}\nu, \quad (\lambda \neq 0, r = 1,2,\dots) \tag{36}$$

and all functions which converge to zero in the usual sense as ν tends to infinity.

Note that in this definition, the convolution $f_\nu * g$ is defined in Gel'fand's and Shilov's sense, the distribution f_ν having bounded support.

It is easily seen that any results proved with the original definition hold with the new definition. The following theorem (proved in [2]) therefore holds, showing that the neutrix convolution is a generalization of the convolution.

Theorem 9. *Let f and g be distributions in $\mathcal{D}'$ satisfying either condition (a) or condition (b) of Definition 4 (Gel'fand's and Shilov's [5]). Then the neutrix convolution $f \circledast g$ exists and*

$$f \circledast g = f * g. \tag{37}$$

The next theorem was also proved in [2].

Theorem 10. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix convolution $f \circledast g$ exists. Then the neutrix convolution $f \circledast g'$ exists and*

$$(f \circledast g)' = f \circledast g'. \tag{38}$$

Note, however, that the neutrix convolution $(f \circledast g)'$ is not necessarily equal to $f' \circledast g$.

We now increase the set of negligible functions given here to include finite linear sums of the functions

$$\cos(\lambda\nu), \quad \sin(\lambda\nu), \quad (\lambda \neq 0). \tag{39}$$

Theorem 11. *If $\lambda, \mu \neq 0$, then the neutrix convolution $\mathrm{Si}_+(\lambda x) \circledast \sin(\mu x)$ exists and*

$$\begin{aligned} \mathrm{Si}_+(\lambda x) \circledast \sin(\mu x) = {} & -\frac{1}{4}\mu^{-1}[\operatorname{sgn}(\lambda+\mu)+\operatorname{sgn}(\lambda-\mu)]\pi\cos(\mu x) \\ & -\frac{1}{2}\mu^{-1}\ln\left|\frac{\lambda+\mu}{\lambda-\mu}\right|\sin(\mu x) \end{aligned} \tag{40}$$

if $\lambda \neq \pm\mu$; and

$$\begin{aligned} \mathrm{Si}_+(\lambda x) \circledast \sin(\lambda x) = {} & -\frac{1}{4}\lambda^{-1}\operatorname{sgn}\lambda \cdot \pi\cos(\lambda x) \\ & +\frac{1}{2}\lambda^{-1}[c-\ln|2\lambda|]\sin(\lambda x) \end{aligned} \tag{41}$$

if $\lambda = \pm\mu$.

Proof. We put $[\mathrm{Si}_+(\lambda x)]_\nu = \mathrm{Si}_+(\lambda x)\tau_\nu(x)$. Then the convolution $[\mathrm{Si}_+(\lambda x)]_\nu * \sin(\mu x)$ exists by Definition 3 and

$$\begin{aligned}[\mathrm{Si}_+(\lambda x)]_\nu * \sin(\mu x) &= \int_0^\nu \mathrm{Si}_+(\lambda t)\sin[\mu(x-t)]\,dt \\ &\quad + \int_\nu^{\nu+\nu^{-\nu}} \mathrm{Si}_+(\lambda t)\sin[\mu(x-t)]\tau_\nu(t)\,dt \\ &= I_1 + I_2,\end{aligned} \tag{42}$$

where it is easily seen that

$$\lim_{n\to\infty} I_2 = 0. \tag{43}$$

Further,

$$\begin{aligned} I_1 &= \int_0^\nu u^{-1}\sin(\lambda u)\int_u^\nu \sin[\mu(x-t)]\,dt\,du \\ &= \mu^{-1}\int_0^\nu u^{-1}\sin(\lambda u)\{\cos[\mu(x-\nu)] - \cos[\mu(x-u)]\}\,du \\ &= \mu^{-1}\cos[\mu(x-\nu)]\,\mathrm{Si}(\lambda\nu) \\ &\quad - \frac{1}{2}\mu^{-1}\cos(\mu x)\int_0^\nu u^{-1}\{\sin[(\lambda+\mu)u] + \sin[(\lambda-\mu)u]\}\,du \\ &\quad + \frac{1}{2}\mu^{-1}\sin(\mu x)\int_0^\nu u^{-1}\{\cos[(\lambda+\mu)u] - \cos[(\lambda-\mu)u]\}\,du \\ &= \mu^{-1}\cos[\mu(x-\nu)]\,\mathrm{Si}(\lambda\nu) - \frac{1}{2}\mu^{-1}\cos(\mu x)\{\mathrm{Si}[(\lambda+\mu)\nu] + \mathrm{Si}[(\lambda-\mu)\nu]\} \\ &\quad + \frac{1}{2}\mu^{-1}\sin(\mu x)\left\{\mathrm{Ci}[(\lambda+\mu)\nu] - \mathrm{Ci}[(\lambda-\mu)\nu] - \ln\left|\frac{\lambda+\mu}{\lambda-\mu}\right|\right\} \end{aligned} \tag{44}$$

on using Lemma 2. It follows that

$$\mathrm{N}\text{-}\lim_{\nu\to\infty} I_1 = -\frac{1}{4}\mu^{-1}[\mathrm{sgn}(\lambda+\mu) + \mathrm{sgn}(\lambda-\mu)]\pi\cos(\mu x) - \frac{1}{2}\mu^{-1}\ln\left|\frac{\lambda+\mu}{\lambda-\mu}\right|\sin(\mu x) \tag{45}$$

on using Lemma 1. Equation (40) now follows from (42), (43), and (45) .

If $\lambda = \pm\mu$, we have

$$\begin{aligned}[\mathrm{Si}_+(\lambda x)]_\nu * \sin(\lambda x) &= \int_0^\nu \mathrm{Si}_+(\lambda t)\sin[\lambda(x-t)]\,dt \\ &\quad + \int_\nu^{\nu+\nu^{-\nu}} \mathrm{Si}_+(\lambda t)\sin[\lambda(x-t)]\tau_\nu(t)\,dt \\ &= J_1 + J_2,\end{aligned} \tag{46}$$

where it is easily seen that

$$\lim_{n\to\infty} J_2 = 0. \tag{47}$$

Further,

$$\begin{aligned}
J_1 &= \int_0^{\nu} u^{-1}\sin(\lambda u)\int_u^{\nu}\sin[\lambda(x-t)]\,dt\,du \\
&= \lambda^{-1}\int_0^{\nu} u^{-1}\sin(\lambda u)\{\cos[\lambda(x-\nu)]-\cos[\lambda(x-u)]\}\,du \\
&= \lambda^{-1}\cos[\lambda(x-\nu)]\,\mathrm{Si}(\lambda\nu) \\
&\quad -\frac{1}{2}\lambda^{-1}\cos(\lambda x)\int_0^{\nu} u^{-1}\sin(2\lambda u)\,du \\
&\quad +\frac{1}{2}\lambda^{-1}\sin(\lambda x)\int_0^{\nu} u^{-1}[\cos(2\lambda u)-1]\,du \\
&= \lambda^{-1}\cos[\lambda(x-\nu)]\,\mathrm{Si}(\lambda\nu)-\frac{1}{2}\lambda^{-1}\cos(\lambda x)\,\mathrm{Si}(2\lambda\nu) \\
&\quad +\frac{1}{2}\lambda^{-1}\sin(\lambda x)[c+\mathrm{Ci}(2\lambda\nu)-\ln|2\lambda\nu|]
\end{aligned} \tag{48}$$

on using Lemma 2. It follows that

$$\mathrm{N}\text{-}\lim_{\nu\to\infty} J_1 = -\frac{1}{4}\lambda^{-1}\,\mathrm{sgn}\,\lambda\cdot\pi\cos(\lambda x)+\frac{1}{2}\lambda^{-1}[c-\ln|2\lambda|]\sin(\lambda x) \tag{49}$$

on using Lemma 1. Equation (41) now follows from (46), (47), and (49). □

Corollary 12. *If* $\lambda,\mu \neq 0$, *then the neutrix convolution* $\mathrm{Si}_+(\lambda x) \circledast \cos(\mu x)$ *exists and*

$$\begin{aligned}
\mathrm{Si}_+(\lambda x)\circledast\cos(\mu x) &= \frac{1}{4}\pi\mu^{-1}[\mathrm{sgn}(\lambda+\mu)+\mathrm{sgn}(\lambda-\mu)]\sin(\mu x) \\
&\quad -\frac{1}{2}\mu^{-1}\ln\left|\frac{\lambda+\mu}{\lambda-\mu}\right|\cos(\mu x)
\end{aligned} \tag{50}$$

if $\lambda \neq \pm\mu$; *and*

$$\mathrm{Si}_+(\lambda x)\circledast\cos(\lambda x) = \frac{1}{4}\lambda^{-1}\,\mathrm{sgn}\,\lambda\cdot\pi\sin(\lambda x)+\frac{1}{2}\lambda^{-1}[c-\ln|2\lambda|]\cos(\lambda x) \tag{51}$$

if $\lambda = \pm\mu$.

Proof. It follows from (38) and (40) that

$$\begin{aligned}
[\mathrm{Si}_+(\lambda x)\circledast\sin(\mu x)]' &= \mu\,\mathrm{Si}_+(\lambda x)\circledast\cos(\mu x) \\
&= \frac{1}{4}\pi[\mathrm{sgn}(\lambda+\mu)+\mathrm{sgn}(\lambda-\mu)]\sin(\mu x) \\
&\quad -\frac{1}{2}\ln\left|\frac{\lambda+\mu}{\lambda-\mu}\right|\cos(\mu x)
\end{aligned} \tag{52}$$

and (50) follows.

If $\lambda = \pm\mu$, it follows from (38) and (41) that

$$\lambda\,\mathrm{Si}_+(\lambda x) \circledast \cos(\lambda x) = \frac{1}{4}\,\mathrm{sgn}\,\lambda \cdot \pi \sin(\lambda x) + \frac{1}{2}[c - \ln|2\lambda|]\cos(\lambda x) \tag{53}$$

and (51) follows. □

Corollary 13. *If $\lambda, \mu \neq 0$, then the neutrix convolutions $\mathrm{Si}_-(\lambda x) \circledast \sin(\mu x)$ and $\mathrm{Si}_-(\lambda x) \circledast \cos(\mu x)$ exist and*

$$\begin{aligned}\mathrm{Si}_-(\lambda x) \circledast \sin(\mu x) = &-\frac{1}{4}\pi\mu^{-1}[\mathrm{sgn}(\lambda+\mu) + \mathrm{sgn}(\lambda-\mu)]\cos(\mu x)\\ &+\frac{1}{2}\mu^{-1}\ln\left|\frac{\lambda+\mu}{\lambda-\mu}\right|\sin(\mu x),\end{aligned} \tag{54}$$

$$\begin{aligned}\mathrm{Si}_-(\lambda x) \circledast \cos(\mu x) = &\frac{1}{4}\pi\mu^{-1}[\mathrm{sgn}(\lambda+\mu) + \mathrm{sgn}(\lambda-\mu)]\sin(\mu x)\\ &+\frac{1}{2}\mu^{-1}\ln\left|\frac{\lambda+\mu}{\lambda-\mu}\right|\cos(\mu x)\end{aligned} \tag{55}$$

if $\lambda \neq \pm\mu$; and

$$\begin{aligned}\mathrm{Si}_-(\lambda x) \circledast \sin(\lambda x) &= \frac{1}{4}\lambda^{-1}\,\mathrm{sgn}\,\lambda \cdot \pi\cos(\lambda x) - \frac{1}{2}\lambda^{-1}[c - \ln|2\lambda|]\sin(\lambda x),\\ \mathrm{Si}_-(\lambda x) \circledast \cos(\lambda x) &= \frac{1}{4}\lambda^{-1}\,\mathrm{sgn}\,\lambda \cdot \pi\sin(\lambda x) - \frac{1}{2}\lambda^{-1}[c - \ln|2\lambda|]\cos(\lambda x)\end{aligned} \tag{56}$$

if $\lambda = \pm\mu$.

Proof. Equations (54) and (55) follow on replacing x by $-x$ in (40) and (50), respectively; and (56) follow on replacing x by $-x$ in (41) and (51), respectively. □

The final neutrix convolutions follow easily from the above results:

$$\mathrm{Si}(\lambda x) \circledast \sin(\mu x) = 0, \qquad \mathrm{Si}(\lambda x) \circledast \cos(\mu x) = 0 \tag{57}$$

if $\lambda \neq \pm\mu$; and

$$\mathrm{Si}(\lambda x) \circledast \sin(\lambda x) = 0, \qquad \mathrm{Si}(\lambda x) \circledast \cos(\lambda x) = 0 \tag{58}$$

if $\lambda = \pm\mu$.

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