

## TWO PROBLEMS ON VARIETIES OF GROUPS GENERATED BY WREATH PRODUCTS

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*To Professor B. I. Plotkin on his 75th birthday*

We outline results on varieties of groups generated by Cartesian and direct wreath products of abelian groups and pose two problems related to our recent results in that direction. A few related topics are also considered.

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**1. Introduction.** The aim of this paper is to pose two problems concerning varieties of groups generated (1) by (Cartesian or direct) wreath products  $NWrH$  of abelian groups  $N$  and  $H$  and, more generally, (2) by (Cartesian or direct) wreath products  $\mathfrak{X}Wr\mathfrak{Y}$  of sets of abelian groups  $\mathfrak{X}$  and  $\mathfrak{Y}$  (see background information below). Recall that for the sets  $\mathfrak{X}$  and  $\mathfrak{Y}$  of (not necessarily abelian) groups, their wreath product  $\mathfrak{X}Wr\mathfrak{Y}$  is defined as the set

$$\mathfrak{X}Wr\mathfrak{Y} = \{NWrH \mid N \in \mathfrak{X}, H \in \mathfrak{Y}\}. \quad (1.1)$$

**PROBLEM 1.1** (see [14, Problem 6.5]). Let  $N$  and  $H$  be arbitrary (nilpotent, metabelian, soluble) groups. Find a criterion by which the following equation holds:

$$\text{var}(NWrH) = \text{var}(N) \cdot \text{var}(H). \quad (1.2)$$

The second problem is the much more general analog of the first one.

**PROBLEM 1.2.** Let  $\mathfrak{X}$  and  $\mathfrak{Y}$  be arbitrary sets of (nilpotent, metabelian, soluble) groups. Find a criterion by which the following equation holds:

$$\text{var}(\mathfrak{X}Wr\mathfrak{Y}) = \text{var}(\mathfrak{X}) \cdot \text{var}(\mathfrak{Y}). \quad (1.3)$$

(A restricted version of this problem, for the sets of *abelian* groups only was posed in [14]. Since that problem is solved in [12] (see the second criterion in Section 3 of this paper), we present that problem in this general form, that is, for the case of arbitrary sets of groups.)

Notice that in our problems we did not specify whether the wreath product considered is *Cartesian* or *direct*. The point is that for arbitrary groups (or group sets) their Cartesian wreath product and their direct wreath product always generate the same variety of groups [15]. Thus, in order to avoid immaterial repetitions we consider Cartesian wreath products only and do not formulate each problem, theorem, or criterion for both wreath products.

Section 2 contains the background information for our problems, in particular, the theorems of Higman and Houghton who considered the problem of some abelian groups. Section 3 presents our recent results generalizing the theorems of Higman and Houghton for the case of arbitrary abelian groups and for the case of arbitrary sets of abelian groups. Section 4 gives an outline of techniques and arguments used in the proofs. Section 5 contains the first examples that are not exclusively related to abelian groups. These examples illustrate our problems and also show that the criteria mentioned in Sections 2 and 3 do not work for the case of non-abelian groups. Section 6 is an illustration of our construction: we consider the subvariety lattice of the variety  $\mathfrak{A}_p^2$  and find the subvarieties that can be generated by wreath products of abelian groups.

## 2. Background information and the history of the problem

**2.1. Varieties of groups, their products, and wreath products.** A *variety of groups* is a nonempty class of groups closed under homomorphic images, subgroups, and Cartesian products. By Birkhoff's theorem [3, 15] each variety  $\mathfrak{V}$  is defined by a suitable word set  $V \subseteq F_\infty$  (here  $F_\infty$  denotes the absolutely free group of infinite rank), that is,  $\mathfrak{V}$  consists of all the groups  $G$  on which  $v(g_1, \dots, g_k) = 1$  holds for each word  $v(x_1, \dots, x_n) \in V$  and for each set of elements  $g_1, \dots, g_k \in G$ . Then  $v(x_1, \dots, x_n) \equiv 1$  is said to be an *identity* of the group  $G$ . The set  $\mathfrak{A}$  of all abelian groups, for example, is a variety defined by a single identity  $[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2 \equiv 1$ . Another example related to the topic of this paper is the variety  $\mathfrak{A}_n$  of all abelian groups of exponents dividing  $n$ , where  $n$  is a positive integer, obtained by adding a new identity  $x^n \equiv 1$  to the previous one. Every variety of abelian groups should be equal either to  $\mathfrak{A}$  or to  $\mathfrak{A}_n$  for a suitable  $n$ ;  $\mathfrak{N}_c$  is the variety of all nilpotent groups of class at most  $c$ ;  $\mathfrak{B}_e$  is the Burnside variety of all groups exponents of which divide the positive integer  $e$ ;  $\mathfrak{E}$  is the trivial variety consisting only of the trivial group  $\{1\}$ . The variety  $\mathfrak{V}$  is generated by the group  $G$  (by the groups set  $\mathfrak{X}$ ) if  $\mathfrak{V}$  is the minimal variety containing the group  $G$  (the set  $\mathfrak{X}$ ) or, equivalently, if  $\mathfrak{V}$  consists of all factors of all Cartesian powers of  $G$  (of factors of Cartesian products of groups of the set  $\mathfrak{X}$ ). This fact is denoted by  $\mathfrak{V} = \text{var}(G)$  (by  $\mathfrak{V} = \text{var}(\mathfrak{X})$ ). For further information on varieties of groups see [15].

We omit the definition and the basic properties of Cartesian and direct wreath products (denoted by  $\text{Wr}$  and  $\text{wr}$ , respectively) and refer to [8, 11, 15, 16] for detailed information.

Wreath products are particularly useful tools (probably the most useful ones) while studying the product varieties of groups. Recall that the product variety  $\mathfrak{U} \cdot \mathfrak{V}$  is defined to consist of all extensions of groups of  $\mathfrak{U}$  by means of groups of  $\mathfrak{V}$ , that is,  $\mathfrak{U} \cdot \mathfrak{V}$  consists of groups  $G$  which have a normal subgroup  $N$  such that  $N \in \mathfrak{U}$  and  $G/N \in \mathfrak{V}$ . And, by Kaloujnine and Krassner's theorem [10], every extension  $G$  of mentioned type can be isomorphically embedded into the Cartesian wreath product  $N\text{Wr}H$ , where  $H \cong G/N$ .

This together with the fact that the abelian groups (at least finitely generated abelian groups) in some sense are the simplest class of groups, explain the very early interest to the wreath products of abelian groups as means of study for metabelian groups and for metabelian varieties of groups (a group  $G$  or a variety  $\mathfrak{V}$  are said to be metabelian

if they have the identity

$$[[x_1, x_2], [x_3, x_4]] \equiv 1; \quad (2.1)$$

thus, a metabelian group is simply an extension of an abelian group by an abelian group).

**2.2. The theorems of Higman and Houghton.** The initial result in the mentioned direction belongs to Higman (see [7, Lemma 4.5 and Example 4.9] and [15, 24.65, 54.41]), who proved that, *when  $C_p$  and  $C_n$  are finite cycles of orders  $p$  and  $n$  (where  $p$  is a prime), then the wreath product  $C_p \text{Wr} C_n$  generates the product variety  $\mathfrak{A}_p \cdot \mathfrak{A}_n$  (clearly,  $\text{var}(C_n) = \mathfrak{A}_n$ ).*

Houghton's result covered the case of arbitrary finite cycles  $A = C_m$  and  $B = C_n$ . Namely, the equality

$$\text{var}(C_m \text{Wr} C_n) = \text{var}(C_m) \cdot \text{var}(C_n) = \mathfrak{A}_m \cdot \mathfrak{A}_n \quad (2.2)$$

holds if and only if  $m$  and  $n$  are coprime [15].

Also there are several other known cases in the literature. For instance, it is a well-known example that, if  $H = C_p \oplus C_p \oplus \dots$  is an infinite direct power of the cycle  $C_p$ , then  $C_p \text{Wr} H$  generates  $\mathfrak{A}_p \cdot \mathfrak{A}_p$ . On the other hand, the group

$$C_p \text{Wr} \underbrace{C_p \oplus \dots \oplus C_p}_s \quad (2.3)$$

does not generate  $\mathfrak{A}_p \cdot \mathfrak{A}_p$  for any positive integer  $s$  [15]. That is, the Houghton's theorem does *not* have obvious generalization for the case of arbitrary abelian groups. Since  $H = C_p \oplus C_p \oplus \dots$  is a discriminating group (see the definition in Section 4.3), the mentioned example is a consequence of the following much more general result: for an arbitrary group  $N$  and an arbitrary discriminating group  $H$  the wreath product  $N \text{Wr} H$  discriminates and, thus, generates the variety  $\text{var}(N) \cdot \text{var}(H)$  [1, 2].

**2.3. Generalization for the case of arbitrary abelian groups.** The results listed lead to the following more general questions.

**QUESTION 2.1.** Let  $N$  and  $H$  be arbitrary abelian groups. Find a criterion by which the following equation holds:

$$\text{var}(N \text{Wr} H) = \text{var}(N) \cdot \text{var}(H). \quad (2.4)$$

It is easy to notice that the variety generated by  $N \text{Wr} H$  is always contained in  $\text{var}(N) \cdot \text{var}(H)$  but is not in general equal to the latter. So the actual problem is to find a criterion under which  $\text{var}(N) \cdot \text{var}(H)$  contains a group not belonging to  $\text{var}(N \text{Wr} H)$ .

The next question is a generalization of the first one.

**QUESTION 2.2.** Let  $\mathfrak{X}$  and  $\mathfrak{Y}$  be arbitrary sets of abelian groups. Find a criterion by which the following equation holds:

$$\text{var}(\mathfrak{X} \text{Wr} \mathfrak{Y}) = \text{var}(\mathfrak{X}) \cdot \text{var}(\mathfrak{Y}). \quad (2.5)$$

The answers to these questions are given in [12, 14] (see Section 3).

### 3. The general criteria for Questions 2.1 and 2.2

**3.1. The general criterion for the case of wreath products of arbitrary abelian groups.** The answer to Question 2.1 is given by [14, Theorem 6.1]. The mentioned theorem, however, is based on special notation. Thus, we reformulate that result in a form which depends on traditional notions only.

**THEOREM 3.1.** *For arbitrary abelian groups  $N$  and  $H$ ,*

$$\text{var}(N\text{Wr}H) = \text{var}(N) \cdot \text{var}(H) \quad (3.1)$$

*holds if and only if*

- (1) *at least one of the groups  $N$  and  $H$  is of infinite exponent; or*
- (2) *if  $\exp N = m$  and  $\exp H = n$  are finite and for every prime number  $p$  dividing both  $m$  and  $n$  the  $p$ -primary component  $H(p)$  of  $H$  is a direct sum  $H(p) = \sum_{i \in I} C_{p^{k_i}}$ , of cycles  $C_{p^{k_i}}$  with orders  $p^{k_i}$ ,  $i \in I$ , such that infinitely many of these summands are of order  $p^{k'}$ , where  $p^{k'}$  is the highest power of  $p$  dividing  $n$ .*

(Recall that, by Prüfer's theorem, every abelian group of finite exponent is really a direct product of finite cycles.)

This theorem shows that (3.1) can be falsified only if both  $N$  and  $H$  are of finite exponents and there is a prime divisor  $p$  of  $\exp N$  and  $\exp H$  such that the  $p$ -primary component  $H(p)$  of  $H$  contains only finitely many direct summands  $C_{p^{k'}}$  (where, as above,  $p^{k'}$  is the highest power of  $p$  dividing  $n$ ).

In particular, when  $N$  and  $H$  are finite groups, our condition simply means that  $\exp N$  and  $\exp H$  are coprime.

**3.2. Examples.** By Houghton's theorem, the wreath product

$$C_{p^k} \text{Wr} C_{p^s} \quad (k > 0, s > 0) \quad (3.2)$$

does not generate the variety  $\mathfrak{A}_{p^k} \cdot \mathfrak{A}_{p^s}$  (this also immediately follows from the fact that (3.2) is a nilpotent groups while  $\mathfrak{A}_{p^k} \cdot \mathfrak{A}_{p^s}$  is not a nilpotent variety). According to the criterion cited above, the even wreath product  $N\text{Wr}H$  with

$$\begin{aligned} H = & C_{p^k} \oplus \cdots \oplus C_{p^k} \oplus \cdots \\ & C_{p^{k-1}} \oplus \cdots \oplus C_{p^{k-1}} \oplus \cdots \\ & \vdots \\ & C_p \oplus \cdots \oplus C_p \oplus \cdots \end{aligned} \quad (3.3)$$

(infinitely many summands in each row) and with

$$\begin{aligned} N = & C_{p^s} \oplus \cdots \oplus C_{p^s} \oplus \\ & C_{p^{s-1}} \oplus \cdots \oplus C_{p^{s-1}} \oplus \cdots \\ & \vdots \\ & C_p \oplus \cdots \oplus C_p \oplus \cdots \end{aligned} \quad (3.4)$$

(finitely many summands in the first row and infinitely many summands in the rest of rows) will not generate the variety  $\mathfrak{A}_{p^k} \cdot \mathfrak{A}_{p^s}$ .

On the other hand, the wreath product

$$C_{p^k} \text{Wr} (C_{p^s} \oplus \cdots \oplus C_{p^s} \oplus \cdots) \quad (3.5)$$

generates the variety  $\mathfrak{A}_{p^k} \cdot \mathfrak{A}_{p^s}$ .

Continuing the examples, we see that if the (finite) exponents  $m$  and  $n$  of (finite or infinite) abelian groups  $N$  and  $H$ , respectively are coprime, then  $N \text{Wr} H$ , always generates the variety  $\mathfrak{A}_m \cdot \mathfrak{A}_n$ .

This means, in particular, that the criterion of Houghton is true not only for the case of finite cycles but also for arbitrary finite abelian groups. This fact, however, seems to be present in mathematical folklore.

**3.3. The general criterion for the case of wreath products of arbitrary sets of abelian groups.** The answer to [Question 2.2](#) for the case of abelian groups is given by [[12](#), Theorem 7.1]. Again, the mentioned theorem is based on special notation. We reformulate that result in a form depends only on traditional notions.

**THEOREM 3.2.** *For arbitrary sets  $\mathfrak{X}$  and  $\mathfrak{Y}$  of abelian groups,*

$$\text{var}(\mathfrak{X} \text{Wr} \mathfrak{Y}) = \text{var}(\mathfrak{X}) \cdot \text{var}(\mathfrak{Y}) \quad (3.6)$$

holds if and only if

- (1) at least one of the sets  $\mathfrak{X}$  and  $\mathfrak{Y}$  is of infinite exponent; or
- (2) if  $\text{exp} \mathfrak{X} = m$  and  $\text{exp} \mathfrak{Y} = n$  are finite, and for every prime number  $p$  dividing both  $m$  and  $n$ 
  - (a) either  $\mathfrak{Y}$  contains a group  $H$  such that the  $p$ -primary component  $H(p)$  of  $H$  is a direct sum  $H(p) = \sum_{i \in I} C_{p^{k_i}}$ , of cycles  $C_{p^{k_i}}$  with orders  $p^{k_i}$ ,  $i \in I$ , such that infinitely many of these summands are of order  $p^{k'}$ , where  $p^{k'}$  is the highest power of  $p$  dividing  $n$ ,
  - (b) or for arbitrary positive integer  $l$  the set  $\mathfrak{Y}$  contains a group  $H_l$  such that the  $p$ -primary component  $H_l(p)$  of  $H_l$  is a direct sum  $H_l(p) = \sum_{j \in J} C_{p^{k_j}}$ , of cycles  $C_{p^{k_j}}$  with orders  $p^{k_j}$ ,  $j \in J$ , such that at least  $l$  of these summands are of order  $p^{k'}$ .

We see that (3.6) can be falsified only if both  $\mathfrak{X}$  and  $\mathfrak{Y}$  are of finite exponents and if there is a prime divisor  $p$  of  $\text{exp} \mathfrak{X}$  and  $\text{exp} \mathfrak{Y}$ , and a positive integer  $h$  such that the  $p$ -primary component  $H(p)$  of every group  $H \in \mathfrak{Y}$  contains only finitely many direct summands  $C_{p^{k'}}$  (where, as above,  $p^{k'}$  is the highest power of  $p$  dividing  $n$ ), and the number of the mentioned summands  $C_{p^{k'}}$  is restricted by  $h$  for every group  $H \in \mathfrak{Y}$ . (Thus,  $h$  depends on  $p$ ,  $\mathfrak{X}$ , and  $\mathfrak{Y}$  but not on the group  $H$ .)

**3.4. Examples.** Actually, each one of the examples of [Section 3.2](#) can be considered as an illustration of [Theorem 3.2](#) for the case when the sets  $\mathfrak{X}$  and  $\mathfrak{Y}$  consist of one group each. It is not complicated to construct examples with sets of groups consisting of more than one group.

Continuing the arguments of the first example in [Section 3.2](#), we see that for an arbitrary set  $\mathfrak{X}$  generating the variety  $\mathfrak{A}_{p^k}$  and for the set  $\mathfrak{Y} = \{N_i \mid i \in I\}$ , where

$$\begin{aligned} N_i &= C_{p^s} \oplus \cdots \oplus C_{p^s} \oplus \\ &\quad C_{p^{s-1}} \oplus \cdots \oplus C_{p^{s-1}} \oplus \cdots \\ &\quad \vdots \\ &\quad C_p \oplus \cdots \oplus C_p \oplus \cdots \end{aligned} \tag{3.7}$$

( $h_i$  summands in the first row and infinitely many summands in the rest of rows,  $i \in I$ ), the wreath product  $\mathfrak{X} \text{Wr} \mathfrak{Y}$  will generate the variety  $\mathfrak{A}_{p^k} \cdot \mathfrak{A}_{p^s}$  if and only if the numbers  $h_i$ ,  $i \in I$ , are *not* restricted by a positive integer  $h$ .

It is easy to build other examples not consisting of  $p$ -groups only. If we add, say, the cycle  $C_{q^l}$  for arbitrary prime  $q \neq p$  to the set  $\mathfrak{Y}$ , then

$$\text{var}(\mathfrak{X} \text{Wr} \mathfrak{Y}) = \mathfrak{A}_{p^k} \cdot \mathfrak{A}_{p^s q^l} \tag{3.8}$$

holds if and only if, as above, the numbers  $h_i$ ,  $i \in I$ , are *not* restricted by a positive integer  $h$ .

**4. Techniques of the proofs.** Since our purpose is to pose problems and not to give proofs, we restrict ourselves by a brief outline of the argument used to deal with wreath products of abelian groups. It is important to note that the methods described below only are *some* of the many methods used in the theory of varieties of groups.

**4.1. The initial idea of Higman.** The elegance and beauty of Higman's initial idea described in his important paper [7] are already good reasons to start with its description.

As it is well known, every variety  $\mathfrak{V}$  can be generated by its finitely generated groups, for example, by its  $\mathfrak{V}$ -free groups of finite ranks:

$$F_r(\mathfrak{V}) = F_r / V(F_r), \quad r = 1, 2, \dots, \tag{4.1}$$

where  $F_r$  is the free group of rank  $r$ , and where  $V(F_r)$  is the verbal subgroup of  $F_r$  for the word set  $V$ . In particular, if the variety  $\mathfrak{V}$  is locally finite (i.e., if its groups are locally finite), then it can be generated by its finite groups. Moreover,  $\mathfrak{V}$  also can be generated by its finite *monolithic* groups, that is, by its finite groups which have one nontrivial normal subgroup only (the latter is called the *monolith* of  $G$ ). For, if the group  $G \in \mathfrak{V}$  has several minimal normal subgroups, say,  $M_1, \dots, M_t$ , then they intersect trivially and the group  $G$  is monomorphically embeddable into the direct product

$$G/M_1 \times \cdots \times G/M_t \tag{4.2}$$

by the rule  $g \mapsto (gM_1, \dots, gM_t)$ , for all  $g \in G$ . Assume now that  $G$  is a finite monolithic group in the variety  $\mathfrak{A}_p \cdot \mathfrak{A}_n$ , where  $p$  is a prime not dividing  $n$  (clearly  $\mathfrak{A}_p \cdot \mathfrak{A}_n$  is a locally finite variety) and show that  $G$  belongs to  $\text{var}(C_p \text{Wr} C_n)$ .

The group  $G$  is an extension of a group  $A \in \mathfrak{A}_p$  by a group  $B \in \mathfrak{A}_n$ ;  $A$  is isomorphic with the additive group of a finite-dimensional space over the field  $\mathbb{F}_p$ :

$$A \cong \underbrace{C_p \oplus \cdots \oplus C_p}_l. \quad (4.3)$$

By Schur's theorem, we can think of the group  $B \cong G/A$  as of a subgroup of  $G$ . Thus, we can consider transformations of  $A$  by conjugations of elements of  $B$ . This defines a *linear representation* of the group  $B$  of degree  $l$  over  $\mathbb{F}_p$ . By Maschke's theorem, this representation is completely reducible and, since  $G$  is monolithical,  $A$  consists of only one  $B$ -irreducible direct summand which coincides with the monolith  $M$  of  $G$ . The representation defined is faithful, for, if a nontrivial element  $r$  of the abelian group  $B$  centralizes  $A = M$ , then it generates a cyclic subgroup  $\langle r \rangle$  normal in  $G$  (for, clearly,  $r$  also centralizes the abelian group  $B$ ). Then, clearly,  $\langle r \rangle$  contains a subgroup minimal and normal in  $G$  and different from  $M$ . Since the abelian group  $B$  has an irreducible and faithful representation, it must be a cycle and, thus, a subgroup of  $C_n$ . By Kaloujnine and Krassner's theorem,  $G$  is embeddable into the appropriate wreath product  $M \text{ wr } C_n$  which belongs to the variety  $C_p \text{ wr } C_n$ .

On the other hand, if  $n$  is divisible by  $p$  then  $\text{var}(C_p \text{ Wr } C_n)$  is not equal to  $\mathfrak{A}_p \cdot \mathfrak{A}_n$ , for, the latter contains the group

$$C_p \text{ Wr } (C_p \oplus \cdots \oplus C_p \oplus \cdots). \quad (4.4)$$

**4.2. The case of arbitrary finite abelian groups  $N$  and  $H$ .** The method of Higman cannot directly be applied for the case when  $N$  and  $H$  are arbitrary finite abelian groups. However, a combination of it with techniques connected with Hall  $\Pi$ -subgroups in groups  $G \in \mathfrak{A}_m \cdot \mathfrak{A}_n$  allows us to prove the above-mentioned criterion.

**THEOREM 4.1.** *For arbitrary finite abelian groups  $N$  and  $H$  of exponents  $m$  and  $n$ , respectively,*

$$\text{var}(N \text{ Wr } H) = \text{var}(N) \cdot \text{var}(H) = \mathfrak{A}_m \cdot \mathfrak{A}_n \quad (4.5)$$

*holds if and only if  $m$  and  $n$  are coprime.*

The proof of this theorem can be found in [13].

**4.3. The case of infinite abelian groups, a dichotomy.** The structure of infinite abelian groups of nonfinite exponent is much more complicated than that of finite abelian groups [5, 6], and, at first sight, their consideration may demand methods very different from that of varieties of groups. The following main dichotomy, however, enables us to reduce our consideration to direct sums of finite cycles by means of the important notion of *discriminating groups* (see the definition after [Theorem 4.2](#)), namely:

**THEOREM 4.2.** *Each abelian group*

- (1) *either is of finite exponent and, thus, is a direct sum of (possibly infinitely many) copies of some finitely many cycles of prime power orders, or*
- (2) *is a discriminating group.*

The group  $D$  is said to be discriminating, if for arbitrary finite-word set  $V$  with the property that, for each  $v \in V$  there exists a homomorphism  $\delta_v$  of a free group  $F_n$  into the group  $D$ , such that  $\delta_v(v) \neq 1$ , there exists a single homomorphism  $\delta$  of  $F_n$  into  $D$ , such that  $\delta(v) \neq 1$  for all  $v \in V$ . In other words,  $D$  is a discriminating group if every finite set of identities  $\{v \equiv 1 \mid v \in V\}$  that can be separately falsified in  $D$ , can also be simultaneously falsified in  $D$  for certain choice of values  $d_1, d_2, \dots, d_n \in D$ .

And in analogy with this, the group set  $\mathfrak{D}$  is said to be discriminating, if for arbitrary finite word set  $V$  with the property that, for each  $v \in V$  there exists a homomorphism  $\delta_v$  of a free group  $F_n$  into a group  $D_v \in \mathfrak{D}$ , such that  $\delta_v(v) \neq 1$ , there exists a single homomorphism  $\delta$  of  $F_n$  into a single group  $D_V \in \mathfrak{D}$ , such that  $\delta(v) \neq 1$  for all  $v \in V$ . In other words,  $\mathfrak{D}$  is a discriminating group if every finite set of identities  $\{v \equiv 1 \mid v \in V\}$  that can be separately falsified in some groups  $D_v \in \mathfrak{D}$  can also be simultaneously falsified in a single group  $D_V \in \mathfrak{D}$  for certain choice of values  $d_1, d_2, \dots, d_n \in D_V$ .

If the abelian group  $H$  is a discriminating group, then the wreath product of  $N$  and  $H$  generates  $\text{var}(N) \cdot \text{var}(H)$ . The same is true for sets  $\mathfrak{X}$  and  $\mathfrak{Y}$ , where  $\mathfrak{Y}$  is a discriminating set of groups [12]. This allows us only to consider the abelian groups which, according to our dichotomy, are direct sums of the above-mentioned type.

**4.4. The functions  $\lambda(N, H, t)$ .** To deal with the case mentioned above we define special functions:

$$\lambda(N, H, t). \quad (4.6)$$

For given abelian  $p$ -groups  $N$  and  $H$  of finite exponents and for given positive integer  $t$  the value  $\lambda(N, H, t)$  is defined to be the *maximum of the nilpotency classes of the  $t$ -generated groups* in the variety  $\text{var}(N\text{Wr}H)$ .

Note the fact that these functions seem to be very handy tools to detect the cases when  $\text{var}(N\text{Wr}H)$  is different from  $\text{var}(N) \cdot \text{var}(H)$ . Namely, for the given  $N$  and  $H$  we calculate the upper bound of  $\lambda(N, H, t)$  and we find in  $\text{var}(N) \cdot \text{var}(H)$  a nilpotent group, the last of which is greater than the bound we calculated.

For example, let  $N$  be an abelian group of exponent  $p^u$  (i.e.,  $N$  generates the variety  $\mathfrak{A}_{p^u}$ ) and let the abelian group  $H$  of exponent  $p^k$  have the direct decomposition

$$H = C_{k_1} \oplus C_{k_2} \oplus \dots \oplus C_{k_t} \quad (k = k_1 \geq k_2 \geq \dots \geq k_t). \quad (4.7)$$

Then,

$$\lambda(N, H, t) \leq \sum_{i=1}^t (p^{k_i} - 1) + (u - 1)(p - 1)p^{k-1} + 1. \quad (4.8)$$

For further details refer to [12, 14].

**5. The case of non-abelian groups.** Problems 1.1 and 1.2 offered for non-abelian groups, as we see, naturally follow from Theorems 3.1 and 3.2.

Consider the examples which illustrate our problems and show that the criteria of Theorems 3.1 and 3.2 no longer work on non-abelian groups. Moreover, there are no direct analogs of our criteria even for the case of “small” finite nilpotent or metabelian groups, that is, for classes of groups “nearest” to the abelian groups.



**EXAMPLE 5.1.** Let  $N = F_2(\mathfrak{N}_2 \cap \mathfrak{B}_3)$  be the free group of rank 2 in the variety of all nilpotent groups of class at most 2 and exponent dividing 3 and let  $H = C_2$ . Then the following equation holds:

$$\mathfrak{V} \cdot \mathfrak{A}_2 \neq \text{var}(N \text{ wr } H) (= \text{var}(N \text{ Wr } H)). \quad (5.1)$$

However, the exponents  $\exp N$  and  $\exp H$  are coprime. To prove this we represent  $N$  as

$$N = F_2(\mathfrak{V}) = \langle x_1, x_2 \mid [x_1, x_2, x_1] = [x_1, x_2, x_2] = x_1^3 = x_2^3 = 1 \rangle \quad (5.2)$$

and define the group  $R$  to be the extension of  $N$  by means of the group of operators generated by automorphisms  $v_1, v_2 \in \text{Aut}(N)$  defined as follows:

$$\begin{aligned} v_1 : x_1 &\mapsto x_1^{-1}, & v_1 : x_2 &\mapsto x_2; \\ v_2 : x_1 &\mapsto x_1, & v_2 : x_2 &\mapsto x_2^{-1}. \end{aligned} \quad (5.3)$$

Clearly  $\langle v_1, v_2 \rangle \cong C_2 \oplus C_2 \in \mathfrak{A}_2$ . As it is shown in [4],  $R$  is a critical group, that is, a finite group that does not belong to the variety generated by its proper factors. Every one of its proper factors, but not  $R$  itself, satisfies the identity

$$[[x_1, x_2], [x_3, x_4], x_5] \equiv 1. \quad (5.4)$$

On the other hand, the wreath product  $N \text{ wr } H$  satisfies this identity because its second commutator subgroup lies in the center.

**EXAMPLE 5.2.** Let  $N$  and  $H$  be arbitrary finite groups generating varieties  $\mathfrak{U} = \mathfrak{A}_p \cdot \mathfrak{A}_q$  and  $\mathfrak{A}_r$  respectively, where  $p, q, r$  are arbitrary pairwise different primes. Then the following equation holds:

$$\mathfrak{U} \cdot \mathfrak{A}_r \neq \text{var}(N \text{ wr } H) (= \text{var}(N \text{ Wr } H)). \quad (5.5)$$

Again,  $\exp A$  and  $\exp B$  are coprime.

We simply have to recall that the product of three nontrivial varieties  $\mathfrak{U} \cdot \mathfrak{A}_r = \mathfrak{A}_p \cdot \mathfrak{A}_q \cdot \mathfrak{A}_r$  cannot, by the theorem of Šmelkin on product varieties generated by finite groups [17], be generated by only one finite group  $N \text{ wr } H$ .

**6. An application: subvarieties generated by wreath products in the subgroup lattice of a product variety.** We conclude this paper by an example showing the role of the subvarieties generated by wreath products of abelian groups in the subgroup lattice of the variety  $\mathfrak{A}_p^2 = \mathfrak{A}_p \cdot \mathfrak{A}_p$  ( $p > 2$  is a prime). Wreath products  $C_p \text{ Wr } \sum_{i=1}^s C_p$  ( $s = 1, 2, \dots$ ) generate infinitely many subvarieties of  $\mathfrak{A}_p^2$ . The subgroup lattice of this variety is described by Kovács and Newman in [9]. Figure 6.1 illustrates this subgroup lattice where the subvarieties which can be generated by wreath products of abelian groups are circled.

In Figure 6.1,  $\mathfrak{A}_{p^*}$  is the subvariety defined in the variety  $\mathfrak{A}_p$  by the additional identity:

$$\prod_{s=2}^p [x_s, x_1, \dots, x_{s-1}, x_{s+1}, \dots, x_p] \equiv 1. \quad (6.1)$$

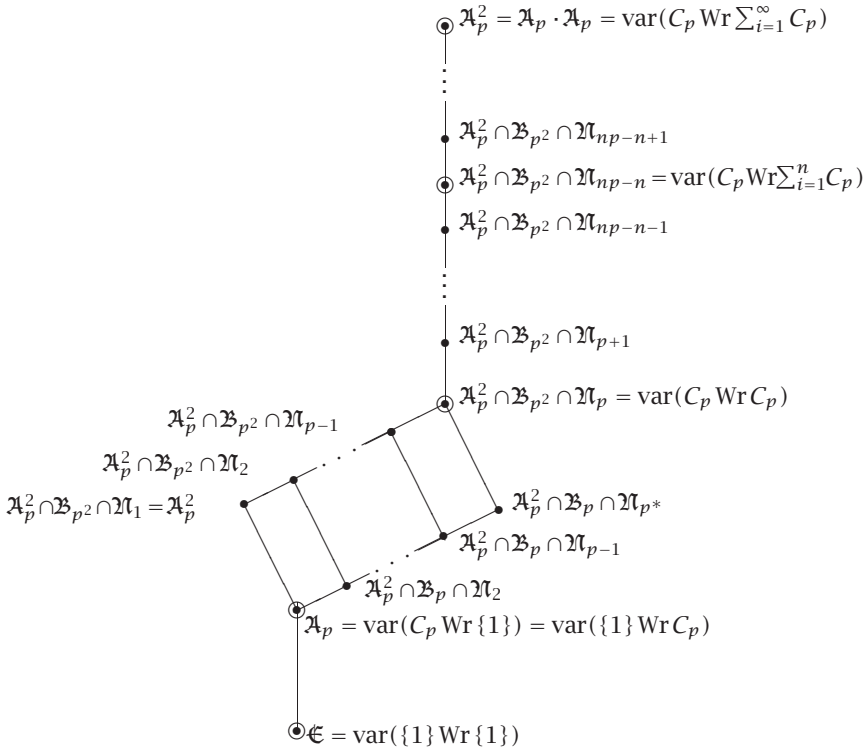


FIGURE 6.1

Clearly  $\mathfrak{A}_p^2 = \text{var}(C_p \text{Wr} \sum_{i=1}^{\infty} C_p)$  holds. Further, as it is easy to see from Figure 6.1, all the proper subvarieties  $\mathfrak{V}$  of  $\mathfrak{A}_p^2$  containing  $\mathfrak{A}_p^2 \cap \mathfrak{B}_{p^2} \cap \mathfrak{N}_p$  form a chain. These subvarieties  $\mathfrak{V}$  can be characterized by their nilpotency class:

$$\mathfrak{V} = \mathfrak{A}_p^2 \cap \mathfrak{B}_{p^2} \cap \mathfrak{N}_c, \tag{6.2}$$

where  $c = p, p + 1, \dots$ . When  $c$  can be presented as

$$c = np - n = n(p - 1) \tag{6.3}$$

for some positive integer  $n$ , then  $\mathfrak{V}$  can be presented as

$$\mathfrak{V} = \text{var} \left( C_p \text{Wr} \sum_{i=1}^n C_p \right). \tag{6.4}$$

This means that for every subvariety  $\mathfrak{V}$  contained in  $\mathfrak{A}_p^2$  and containing  $\mathfrak{A}_p^2 \cap \mathfrak{B}_{p^2} \cap \mathfrak{N}_p$  there is an integer  $s \geq 1$  such that

$$\text{var} \left( C_p \text{Wr} \sum_{i=1}^s C_p \right) \subseteq \mathfrak{V} \subseteq \text{var} \left( C_p \text{Wr} \sum_{i=1}^{s+1} C_p \right). \tag{6.5}$$

Moreover, for any  $s \geq 1$  there are the following  $p - 2$  subvarieties of  $\mathfrak{A}_p^2$  “between”  $\text{var}(C_p \text{Wr} \sum_{i=1}^s C_p)$  and  $\text{var}(C_p \text{Wr} \sum_{i=1}^{s+1} C_p)$ .

In addition, there are  $2p - 3$  subvarieties of  $\mathfrak{A}_p^2$  “between” the varieties  $\mathfrak{A}_p^2 \cap \mathfrak{B}_{p^2} \cap \mathfrak{A}_p$  and  $\mathfrak{A}_p$ . None of them can be generated by wreath product of abelian groups.

Notice that the two final subvarieties, that is, the subvariety  $\mathfrak{A}_p$  and the trivial subvariety  $\mathfrak{E}$  can be generated by wreath products of abelian groups.

The condition, that  $\mathfrak{V}$  is generated by a wreath product of *abelian* groups, clearly, cannot be replaced by the fact that  $\mathfrak{V}$  is generated by a wreath product (of any groups), for, each variety  $\mathfrak{V}$  can be generated by a wreath product, namely, by  $T \text{Wr} \{1\} = \{1\} \text{Wr} T$ , where  $T$  is an arbitrary group generating  $\mathfrak{V}$ .

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