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GRADED RADICAL W TYPE LIE ALGEBRAS I

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We get a new $\mathbb{Z}$-graded Witt type simple Lie algebra using a generalized polynomial ring which is the radical extension of the polynomial ring $\mathbf{F}[x]$ with the exponential function e^x.

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1. Introduction. Let $\mathbf{F}$ be a field of characteristic zero (not necessarily algebraically closed). Throughout this paper, $\mathbb{Z}_+$ and $\mathbb{Z}$ denote the nonnegative integers and the integers, respectively. Let $\mathbf{F}[x]$ be the polynomial ring in indeterminate x. Let $\mathbf{F}(x) = \{f(x)/g(x) \mid f(x), g(x) \in F[x],\ g(x) \neq 0\}$ be the field of rational functions in one variable. We define the $\mathbf{F}$-algebra $V_{\sqrt{m},e}$ spanned by

$$\Big\{e^{dx} f_1^{a_1/b_1} \cdots f_m^{a_m/b_m} x^t \mid d, a_1, \ldots, a_m, t \in \mathbb{Z},\ f_i \neq x,\\ (a_1,b_1) = 1, \ldots, (a_m,b_m) = 1,\ 1 \leq i \leq m\Big\}, \tag{1.1}$$

where $b_1, \ldots, b_m$ are fixed nonnegative integers, and $(a_i, b_i) = 1$, $1 \leq i \leq m$, means that a_i and b_i are relatively primes, and $f_1, \ldots, f_n$ are the fixed relatively prime polynomials in $\mathbf{F}[x]$. The $\mathbf{F}$-subalgebra $V^+_{\sqrt{m},e}$ of $V_{\sqrt{m},e}$ is spanned by

$$\Big\{e^{dx} f_1^{a_1/b_1} \cdots f_m^{a_m/b_m} x^t \mid d, a_1, \ldots, a_m \in \mathbb{Z},\ t \in \mathbb{Z}_+,\ f_i \neq x,\\ (a_1,b_1) = 1, \ldots, (a_m,b_m) = 1,\ 1 \leq i \leq m\Big\}. \tag{1.2}$$

Let $W_{\sqrt{m},e}(\partial)$ be the vector space over $\mathbf{F}$ with elements $\{f\partial \mid f \in V_{\sqrt{m},e}\}$ and the standard basis $\{e^{dx} f_1^{a_1/b_1} \cdots f_m^{a_m/b_m} x^t \partial \mid e^{dx} f_1^{a_1/b_1} \cdots f_m^{a_m/b_m} x^t \partial \in W_{\sqrt{m},e}\}$. Define a Lie bracket on $W_{\sqrt{m},e}(\partial)$ as follows:

$$[f\partial, g\partial] = f(\partial(g))\partial - g(\partial(f))\partial, \quad f, g \in V_{\sqrt{m},e}. \tag{1.3}$$

It is easy to check that (1.3) defines a Lie algebra $W_{\sqrt{m},e}(\partial)$ with the underlying vector space $W_{\sqrt{m},e}(\partial)$ (see also [1, 3, 5]). Similarly, we define the Lie subalgebra $W^+_{\sqrt{m},e}(\partial)$ of $W_{\sqrt{m},e}(\partial)$ using the $\mathbf{F}$-algebra $V^+_{\sqrt{m},e}$ instead of $V_{\sqrt{m},e}$.

The Lie algebra $W_{\sqrt{m},e}(\partial)$ has a natural $\mathbb{Z}$-gradation as follows:

$$W_{\sqrt{m},e}(\partial) = \bigoplus_{d \in \mathbb{Z}} W^d_{\sqrt{m},e}, \tag{1.4}$$

where $W^d_{\sqrt{m},e}$ is the subspace of the Lie algebra $W_{\sqrt{m},e}(\partial)$ generated by elements of the form $\{e^{dx} f_1^{a_1/b_1} \cdots f_m^{a_m/b_m} x^t \partial \mid f_1, \ldots, f_n \in \mathbf{F}[x],\ a_1, \ldots, a_m, t \in \mathbb{Z},\ m \in \mathbb{Z}_+\}$. We call the subspace $W^d_{\sqrt{m},e}$ the d-homogeneous component of $W_{\sqrt{m},e}(\partial)$.

We decompose the d-homogeneous component $W^d_{\sqrt{m},e}$ as follows:

$$W^d_{\sqrt{m},e} = \bigoplus_{s_1,\dots,s_m\in\mathbb{Z}} W_{(d,s_1,\dots,s_m)}, \tag{1.5}$$

where $W_{(d,s_1,\dots,s_m)}$ is the subspace of $W^d_{\sqrt{m},e}$ spanned by

$$\{e^{dx} f_1^{s_1/b_1}\cdots f_m^{s_m/b_m} x^q \partial \mid q\in\mathbb{Z}\}. \tag{1.6}$$

Note that $W_{(0,0,\dots,0)}$ is the Witt algebra $W(1)$ as defined in [3].

The two radical-homogeneous components $W_{(d,a_1,\dots,a_m)}$ and $W_{(d,r_1,\dots,r_m)}$ are equivalent if $a_1-r_1,\dots,a_m-r_m\in\mathbb{Z}$. This defines an equivalence relation on $W^d_{\sqrt{m},e}$. Thus we note that the equivalent class of $W_{(d,a_1,\dots,a_m)}$ depends only on $a_1,\dots,a_m$. From now on $W_{(d,a_1,\dots,a_m)}$ will represent the radical homogeneous equivalent class of $W_{(d,a_1,\dots,a_m)}$ without ambiguity. It is possible to choose the minimal positive integers $a_1,\dots,a_m$ for the radical homogeneous equivalent component $W_{(d,a_1,\dots,a_m)}$.

We give the lexicographic order on all the radical homogeneous equivalent components $W_{(d,a_1,\dots,a_m)}$ using $\mathbb{Z}\times\mathbb{Z}_+^m$.

The radical equivalent homogeneous component $W^d_{\sqrt{m},e}$ can be written as follows:

$$W^d_{\sqrt{m},e} = \sum_{(a_1,\dots,a_m)\in\mathbb{Z}_+^m} W_{(d,a_1,\dots,a_m)}. \tag{1.7}$$

Thus for any element $l\in W_{\sqrt{m},e}(\partial)$, l can be written uniquely as follows:

$$l = \sum_{(d,a_1,\dots,a_m)\in\mathbb{Z}\times\mathbb{Z}_+^m} l_{(d,a_1,\dots,a_m)}. \tag{1.8}$$

For any such element $l\in W_{\sqrt{m},e}(\partial)$, $H(l)$ is defined as the number of different homogeneous components of l as in (1.4), and $L_d(l)$ as the number of nonequivalent radical d-homogeneous components of l in (1.8). For each basis element $e^{dx}f_1^{a_1/b_1}\cdots f_m^{a_m/b_m}x^t\partial$ of $W_{\sqrt{m},e}(\partial)$ (or $W^+_{\sqrt{m},e}(\partial)$), define $\deg_{\text{Lie}}(e^{dx}f_1^{a_1/b_1}\cdots f_m^{a_m/b_m}x^t\partial)=t$. Since every element l of $W_{\sqrt{m},e}(\partial)$ is the sum of the standard basis element, we may define $\deg_{\text{Lie}}(l)$ as the highest power of each basis element of l. Note that the Lie algebra $W_{\sqrt{m},e}(\partial)$ is self-centralized, that is, the centralizer $C_l(W_{\sqrt{m},e}(\partial))$ of every element l in $W_{\sqrt{m},e}(\partial)$ is one dimensional [1]. We find the solution of

$$1^{1/3} = y \tag{1.9}$$

in $\mathbb{Z}_7$. Equation (1.9) implies that

$$1 \equiv y^3 \bmod 7. \tag{1.10}$$

The solutions of (1.10) are 1, 2, or 4. Thus $1^{1/3}=1$, 2, or $4 \bmod 7$. Thus the radical number in $\mathbb{Z}_p$ is not uniquely determined generally. So we may not consider the Lie algebras in this paper over a field of characteristic p differently from the Lie algebras in [2, 3, 4]. It is easy to prove that the Lie algebra $W_{(0,\dots,0)}$ is simple [3].

2. Main results. We need several lemmas for Theorem 2.5.

Lemma 2.1. *For any element l in the $(d,a_1,\dots,a_m)$-radical-homogeneous component of $W_{\sqrt{m}}(\partial)$, and for any element $l_1 \in W_{(0,0,\dots,0)}$, $[l,l_1]$ is an element in the $(d,a_1,\dots,a_m)$-radical homogeneous equivalent component.*

The proof of Lemma 2.1 is straightforward.

Lemma 2.2. *A Lie ideal I of $W_{\sqrt{m},e}(\partial)$ which contains ∂ is $W_{\sqrt{m},e}(\partial)$.*

Proof. Let I be the ideal in the lemma. The Lie subalgebra which has the standard basis $\{x^i\partial \mid i \in \mathbb{Z}_+\}$ is simple. Let I be any ideal of $W_{\sqrt{m},e}(\partial)$ which contains ∂. Then for any $f\partial \in W_{\sqrt{m},e}(\partial)$,

$$[x\partial, f\partial] = x\partial(f)\partial - f\partial \in I. \tag{2.1}$$

On the other hand,

$$[\partial, xf\partial] = f\partial + x\partial(f)\partial \in I. \tag{2.2}$$

Thus by subtracting (2.2) from (2.1) we get $2f\partial \in I$. Therefore, we have proven the lemma, since $I \cap W_{(0,0,\dots,0)}$ contains nonzero elements and so $I \supset W_{(0,0,\dots,0)}$. □

Lemma 2.3. *A Lie ideal I of $W_{\sqrt{m},e}(\partial)$ which contains a nonzero element in $W_{(d,a_1,\dots,a_m)}$ is $W_{\sqrt{m},e}(\partial)$, for a fixed $(d,a_1,\dots,a_m) \in \mathbb{Z}\times\mathbb{Z}_+$.*

Proof. Let I be a Lie ideal of $W_{\sqrt{m},e}(\partial)$ and l a nonzero element in the ideal I. Then we take an element $l_1 = e^{-dx} f_1^{-a_1/b_1}\cdots f_m^{-a_m/b_m} x^p\partial$ with p a sufficiently large positive integer such that $[l,l_1] \neq 0$. Then $[f\partial,[l,l_1]]$ is a nonzero element in $W_{(0,0,\dots,0)}$ by taking an element $f_1^{t_1}\cdots f_m^{t_m} \in \mathbf{F}[x]$, where $t_1,\dots,t_m$ are sufficiently large integers. Thus $I \cap W_{(0,0,\dots,0)}$ contains nonzero elements, and hence, $\partial \in I \cap W_{(0,0,\dots,0)}$ by simplicity of $W_{(0,0,\dots,0)}$. Then the lemma follows from Lemma 2.2. □

Throughout this paper, $a \gg b$ means that a is a number sufficiently larger than b.

Lemma 2.4. *Let I be any nonzero Lie ideal of $W_{\sqrt{m},e}(\partial)$. For any nonzero element $l \in I$, there is an element $x^s\partial$, $s \gg 0$, such that $[x^s\partial, l]$ is the sum of elements in $W_{\sqrt{m},e}(\partial)$ with $\deg_{\text{Lie}}([x^s\partial,l]) > 0$.*

Proof. It is straightforward by choosing a sufficiently large positive integer s. □

Theorem 2.5. *The Lie algebra $W_{\sqrt{m},e}(\partial)$ is simple.*

Proof. Let I be a nonzero Lie ideal of $W_{\sqrt{m},e}(\partial)$. Let l be a nonzero element of I. By Lemma 2.4, we may assume that l has polynomial terms with positive powers for each basis element of l. We prove this theorem in several steps.

Step 1. If l is in the 0-homogeneous component, then the theorem holds. We prove this step, by induction on the number $L_0(l)$ of nonequivalent radical-homogeneous components of the element l of I. If $L_0(l)$ is 1 and $l \in W_{(0,0,\dots,0)}$, then the theorem holds by Lemmas 2.2, 2.3, and the fact that $W_{(0,0,\dots,0)}$ is simple.

Assume that $l \in W_{(0,0,\dots,0,a_r,\dots,a_m)}$ with $a_r \neq 0$. If we take an element $f_1^{h_r/k_r} \cdots f_n^{h_m/k_m} x^{h_{m+1}}\partial$ such that $h_r \gg k_r, \dots, h_n \gg k_r$ and $(h_r + k_r)/k_r \in \mathbb{Z}_+, \dots, (h_m + k_m)/k_m \in \mathbb{Z}_+$, then we have $l_1 = [f_1^{h_r/k_r} \cdots f_m^{h_m/k_m} x^{h_{m+1}}\partial, l] \neq 0$. This implies that l_1 is in $W(0,0,\dots,0)$. Thus we have proven the theorem by Lemma 2.2.

By induction, we may assume that the theorem holds for $l \in I$ such that $L_0(l) = k$, for some fixed nonnegative integer $k > 1$. Assume that $L_0(l) = k+1$. If l has a $W_{(0,0,\dots,0)}$ radical-homogeneous equivalent component, we take $l_2 \in W_{(0,0,\dots,0)}$ such that $[l, l_2]$ can be written as follows: $[l, l_2] = l_3 + l_4$ where l_3 is a sum of nonzero radical-homogeneous components, and $l_4 = f\partial$ with $f \in \mathbf{F}[x]$. Thus we have the nonzero element

$$\partial, [\cdots, [\partial, l] \cdots] = l_2 \in I \tag{2.3}$$

which has no terms in the homogeneous equivalent component $W_{(0,0,\dots,0)}$, where we applied Lie brackets until l_2 has no terms in the radical homogeneous equivalent component $W_{(0,0,\dots,0)}$. Then $l_2 \in I$ such that $H(l_2) \leq k$. Therefore, we have proven the theorem by Lemmas 2.2, 2.3, and induction. If l has no terms in the radical homogeneous equivalent component $(0,0,\dots,0)$, then l has a term in the radical homogeneous equivalent component $W_{(0,a_1,\dots,a_n)}$. Take an element $l_3 = f_1^{c_1/p_1} \cdots f_m^{c_m/p_m} x^{c_{m+1}}\partial$ such that $c_1, \dots, c_{m+1}$ are sufficiently large positive integers such that $c_1 + a_1 \in \mathbb{Z} \cdots c_m + a_m \in \mathbb{Z}$, and which is in a radical homogeneous equivalent component $W_{(0,a_1,\dots,a_m)}$. Then $[l_3, l]$ is nonzero and which has a term in the radical homogeneous equivalent component $W_{(0,0,\dots,0)}$. So in this case we have proven the theorem by induction.

Step 2. Assume that l is in the d-homogeneous component such that $0 \neq d$ and $L_0(l) = 1$, then the theorem holds. By taking $e^{-dx}x^t\partial$, we have $0 \neq [e^{-dx}x^t\partial, l] \in W_{(0,0,\dots,0)}$ by taking a sufficiently large positive integer t. Thus we have proven the theorem by Step 1.

Step 3. If l is the sum of $(k-1)$ nonzero homogeneous components and 0-homogeneous component, then the theorem holds. We prove the theorem by induction on the number of distinct homogeneous components by Steps 1 and 2. Assume that we have proven the theorem when l has $(k-1)$ radical-homogeneous components. Assume that l has terms in $W_{(0,0,\dots,0)}$. By Step 1, we have an element $l_1 \in I$, such that $l_1 = l_2 + f\partial$, where l_2 has $(k-1)$ homogeneous components and $f \in \mathbf{F}[x]$. Then $0 \neq \partial, [\cdots, [\partial, l_1] \cdots] \in I$ has $(k-1)$ homogeneous components, where we applied the Lie bracket until it has no terms in $W_{(0,0,\dots,0)}$. Therefore, we have proven the theorem by induction.

Assume that l has a (k) homogeneous equivalent components. We may assume l has the terms which is in $0 \neq d$-homogeneous component. By taking a sufficiently large positive integer r, we have $[e^{-dx}x^r\partial, l] \neq 0$ and it has (k) homogeneous components with a term in the radical-homogeneous component $W_{(0,0,\dots,0)}$. Therefore, we have proven the theorem by Step 3. □

Corollary 2.6. *The Lie algebra $W^+_{\sqrt{m},e}(\partial)$ is simple.*

Proof. It is straightforward from Theorem 2.5 without using Lemma 2.4. □

Corollary 2.7. *The Lie subalgebra $W^0_{\sqrt{m},e}$ of $W_{\sqrt{m},e}(\partial)$ is simple.*

Proof. It is straightforward from Step 1 of Theorem 2.5. □

Proposition 2.8. *For any nonzero Lie automorphism θ of $W^+_{\sqrt{m},e}(\partial)$, $\theta(\partial) = \partial$ holds.*

Proof. It is straightforward from the relation $\theta([\partial, x\partial]) = \theta(\partial)$ and the fact that $W^+_{\sqrt{m},e}(\partial)$ is self-centralized and $\mathbb{Z}$-graded. □

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