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ON SECTIONAL AND BISECTIONAL CURVATURE OF THE H-UMBILICAL SUBMANIFOLDS

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Let M be an H-umbilical submanifold of an almost Hermitian manifold $\tilde{M}$. Some relations expressing the difference of bisectional and of sectional curvatures of $\tilde{M}$ and of M are obtained. The geometric notion of related bases for a pair of oriented planes permits to write the second members in a completely geometrical form. When the planes are not orthogonal, more simple formulas are obtained. The paper ends with a remark, concerning the vector field JH, and some special cases.

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1. Introduction. In [2, 3] Chen introduces and studies the n-dimensional totally real H-umbilical submanifolds of the Kähler manifolds of real dimension $2n$. More generally, the present paper considers *the H-umbilical submanifolds of the almost Hermitian manifolds* (Section 4). Some remarks in Section 4 show that these submanifolds are very close to be weakly antiholomorphic submanifolds (see [4, Section 4]) and not far from being totally real submanifolds.

Let M be an H-umbilical submanifold of an almost Hermitian manifold $\tilde{M}$. The aim of this paper is to obtain *relations linking the bisectional (sectional) curvatures of M with the corresponding bisectional (sectional) curvature of $\tilde{M}$*.

Even if the relation (5.3) represents a first solution of our problem, the real difficulty was that of giving a clear geometrical meaning to its second member. The notion of *related bases* for a pair of planes (Section 3), introduced by Rizza in [8, Section 6], has been the main tool to overcome the above difficulty.

Finally, Propositions 6.1, 6.2, and 6.3 solve our problem in a complete and satisfactory way, since *in the formulas only geometrical elements occur*.

The relations of Propositions 6.1 and 6.2 may appear rather complicated, but when the two planes are not orthogonal more simple formulas can be proved (Propositions 6.4 and 6.5).

The paper ends with Section 8, containing some remarks about the vector field JH, that plays an essential role in the relations of Section 6. Other remarks, referring to some special cases, are also included in Section 8.

2. Geometric preliminaries. In this section as well as in the following one, we recall some geometric notions and fix some notations occurring in the sequel.

Let V be an m-dimensional real vector space ($m \geq 2$) and g an inner product on V. Let p, q be two oriented planes (2-dimensional subspaces) of V.

The planes p, q are said to be *orthogonal*, if there exists in p (in q) a line (1-dimensional subspace) orthogonal to q (to p). In particular, p, q are *strictly orthogonal*, if any line of p (of q) is orthogonal to q (to p).

Let A, Λ, p be a vector, a line, and an oriented plane of V, respectively. We denote by A_Λ and A_p the vectors obtained by *orthogonal projection* of the vector A on the line Λ and on the plane p. It is easy to check that, if L is a unit vector of Λ and X, Y is an orthonormal oriented basis of p, we have

$$A_\Lambda = g(A,L)L, \tag{2.1}$$

$$A_p = g(A,X)X + g(A,Y)Y. \tag{2.2}$$

It is worth remarking that A_Λ and A_p do not depend on the orientation of Λ and p.

3. Related bases. The main tool occurring in the proofs of this paper is the geometric notion of related bases for a pair p, q of oriented planes of V (see [8, Section 6]). Two oriented orthogonal bases X, Y and Z, W of p and q, respectively, are said to be *related bases*, if we have

$$g(X,W) = g(Y,Z) = 0. \tag{3.1}$$

The existence of related bases can be proved by an elementary calculation.

Starting from a pair of related bases of p, q and considering suitable rotations of $k(\pi/2)$ $(k = 0,1,2,3)$ of these bases in p and in q, we obtain 8 pairs of related bases for p, q, that will be regarded as *equivalent* in the sequel.

A simple investigation shows that when $|g(X,Z)| \neq |g(Y,W)|$ there exists, essentially, only one pair of related bases for the planes p, q. If we have $|g(X,Z)| = |g(Y,W)| \neq 0$, there exist ∞^1 pairs of related bases for p, q. Starting from one of these pairs, we can generate all the other ones by simultaneous rotations in p and in q. Finally, if we have $|g(X,Y)| = |g(Y,W)| = 0$, the planes p, q are strictly orthogonal. So there exist ∞^2 pairs of related bases for p, q. More explicitly, an oriented basis of p and an oriented basis of q are always related bases for p, q.

Assume first that the inequality

$$|g(X,Z)| > |g(Y,W)| \tag{3.2}$$

is satisfied. Consider a line in the plane p and let α $(0 \leq \alpha \leq \pi/2)$ be the angle that this line forms with the plane q. Denote by α_m, α_M the minimum and maximum value of α, as the line varies in p. Then, the values α_m, α_M are attained by sectioning p, q with the nonoriented planes t_m, t_M defined by X, Z and by Y, W, respectively (see [10, pages 69–74]). In other words, we have

$$\cos\alpha_m = |g(X,Z)|, \qquad \cos\alpha_M = |g(Y,W)|. \tag{3.3}$$

Since we have (cf. [7, (4), page 149])

$$\cos pq = g(X,Z)g(Y,W), \tag{3.4}$$

we can write $|\cos pq| = \cos\alpha_m \cos\alpha_M$.

We introduce also the symmetries $\sigma_p : p \to p$, $\sigma_q : q \to q$ defined by

$$\sigma_p X = Y, \qquad \sigma_p Y = X, \qquad \sigma_q Z = W, \qquad \sigma_q W = Z, \tag{3.5}$$

and useful in the sequel.

We assume now that

$$g(X,Z) = g(Y,W) \neq 0. \tag{3.6}$$

It is easy to show that any line of p (or q) forms the same angle α_* $(0 \leq \alpha_* \leq \pi/2)$ with the plane $q(p)$. So we say that p, q are *isoclinic planes*. Since we have $\alpha_m = \alpha_M = \alpha_*$, we denote by t_*, t^* the planes previously denoted by t_m, t_M, respectively. Under concord rotations of the bases in the oriented planes p, q the planes t_*, t^* generate two systems $\sum_*$, $\sum^*$ of ∞^1 planes and the correspondence $t_* \to t^*$ is one-to-one.

Some remarks are needed in the special case $|g(X,Z)| = 1$. When (3.2) is satisfied, the planes p, q have a line in common, $\alpha_m = 0$, t_m degenerates into the line $p \cap q$ and t_M is the normal plane. When (3.6) is satisfied, we have $q = p$ or $q = p'$, where p' denotes the same plane as p with opposite orientation. The planes t_*, t^* degenerate to lines and $\alpha_m = \alpha_M = \alpha_* = 0$.

Remark 3.1. If the inequality

$$|g(Y,W)| > |g(X,Z)| \tag{3.7}$$

replaces (3.2), then

$$\cos\alpha_m = |g(Y,W)|, \qquad \cos\alpha_M = |g(X,Z)|. \tag{3.8}$$

Consequently, the nonoriented planes t_m, t_M are defined by Y, W and by X, Z, respectively.

If the equation

$$g(X,Z) = -g(Y,W) \neq 0 \tag{3.9}$$

replaces (3.6), then p, q are again isoclinic planes. All goes as in the previous case, but the rotations of the bases in the oriented planes, leading to the systems $\sum_*$, $\sum^*$, are no more concord.

Finally, the remarks concerning the special case $|g(X,Z)| = 1$ hold true also in the special case $|g(Y,W)| = 1$.

Remark 3.2. Let p, q be isoclinic planes. Then, under the mentioned rotations of the bases in p and in q, the same condition, that is, (3.6) or (3.9), is satisfied.

Remark 3.3. Consider a pair of related bases of p, q varying in its equivalence class. If p, q are not isoclinic planes, then (3.2) holds in 4 cases and (3.7) in the other 4 ones. Correspondingly, the symmetries σ_p, σ_q do not change or change to $-\sigma_p$, $-\sigma_q$. If p, q are isoclinic planes, then the same relation, that is, (3.6) or (3.9), is satisfied. In both situations the nonoriented planes t_*, t^* do not change in 4 cases and interchange in the other 4 ones. Correspondingly, the symmetries σ_p, σ_q do not change or change to σ_q, σ_p.

We conclude the section, remarking that the notion of related bases is a *geometric notion (intrinsic notion)*. Consequently, the nonoriented planes t_m, t_M, t_*, t^*, as well as the isomorphisms σ_p, σ_q have a *geometrical meaning (intrinsic meaning)*.

More details about related bases can be found in [9].

4. H-umbilical submanifolds. Let $\tilde{M} = \tilde{M}(g, J)$ be an $\tilde{m}$-dimensional almost Hermitian manifold and M an m-dimensional submanifold of $\tilde{M}$ $(m \geq 4)$, with induced metric still denoted by g.

For the basic facts about the geometry of the submanifolds we refer to [1, Chapter 2], [6, Chapter 7], and [11, Chapter 2]. In the sequel, B denotes the *second fundamental form* and $H = 1/m$ trace B the *mean curvature vector field* of M.

Following Chen ([3, page 70] and [2, page 278]), we say that M is *H-umbilical* if there exists an open covering $\mathscr{C}$ on M such that in any open set U of the covering the second fundamental form B satisfies the condition

$$\begin{aligned} &B(e_1,e_1) = \lambda J e_1, \qquad B(e_2,e_2) = \cdots = B(e_m,e_m) = \mu J e_1, \\ &B(e_1,e_j) = \mu J e_j, \quad B(e_j,e_k) = 0 \quad j,k = 2,\dots,m \ j \neq k, \end{aligned} \tag{4.1}$$

for some suitable functions λ, μ and for some orthonormal system of fields $e_1, \dots, e_m$.

Remark 4.1. In [2, 3], $\tilde{M}$ is a Kähler manifold of real dimension $2n$ and M is an n-dimensional totally real H-umbilical submanifold of $\tilde{M}$.

From (4.1) it follows that

$$mH = (\lambda + (m-1)\mu) J e_1. \tag{4.2}$$

It is easy to prove that *if M is H-umbilical and totally umbilical, then M is totally geodesic, and vice versa.*

From $B(e_1,e_j) = \mu J e_j$ $(j = 2,\dots,m)$ it follows that $\mu = 0$. Hence we have $B(e_1,e_1) = \lambda J e_1 = mH$. On the other hand, we have $B(e_1,e_1) = H$ and since $m \geq 4$ we get $H = 0$ and $\lambda = 0$. The conclusion is now immediate. The converse is obvious.

From now on we assume that M is *H-umbilical and not minimal*. A first consequence is that *if we have $H \neq 0$ almost everywhere on M, then M is almost everywhere weakly antiholomorphic* (see [4, Section 4]).

The assumption on H implies that for any U of the open covering we have $\lambda \neq 0$ or $\mu \neq 0$ almost everywhere in U. So, from the first row of (4.1) we derive that almost everywhere on U there exists a non-null tangent vector e_1, such that Je_1 results to be orthogonal to M. This leads to the conclusion.

The submanifold M is called $\mathscr{C}$-regular if for any U of $\mathscr{C}$ we have $\mu \neq 0$ almost everywhere in U. It is now worth remarking that *if M is $\mathscr{C}$-regular, then M is almost everywhere antiholomorphic (almost everywhere totally real)*.

Let x be a point of U where $\mu \neq 0$, then by virtue of (4.1) the fields $e_1, \dots, e_m$ at x are such that the fields $Je_1, \dots, Je_m$ at x belong to $T_x(M)^\perp$. Consequently, we have $JT_x(M) \subset T_x(M)^\perp$ and this proves the statement.

We complete the section remarking that at any point x of U where $H \neq 0$ and for any pair X, Y of vectors of $T_x(M)$ we have

$$\begin{aligned} B(X,Y) = \alpha g(JX,H)g(JY,H)H + \beta g(H,H)g(X,Y)H \\ + \beta g(H,H)[g(JX,H)JY + g(JY,H)JX], \end{aligned} \tag{4.3}$$

where

$$\alpha = \frac{\lambda - 3\mu}{\gamma^3}, \qquad \beta = \frac{\mu}{\gamma^3}, \qquad \gamma = \frac{\lambda + (m-1)\mu}{m}. \tag{4.4}$$

Of course in (4.3) and (4.4), the second fundamental form B, the mean curvature field H, the Riemannian structure g, and the almost complex structure J are considered at the point x and the functions λ, μ are evaluated at x.

As in the case considered by Chen in [2, 3], relation (4.3) is an easy consequence of condition (4.1).

5. A first curvature relation. Our aim now is to obtain some information about the curvature of the H-umbilical submanifold M. The basic facts about sectional and bisectional curvatures are recalled in [7].

Consider first the classical Gauss formula

$$\tilde{R}(X,Y,Z,W) - R(X,Y,Z,W) = g(B(X,W),B(Y,Z)) - g(B(X,Z),B(Y,W)), \tag{5.1}$$

where R, $\tilde{R}$ are the curvature tensor fields of M, $\tilde{M}$ respectively and X, Y, Z, W are vector fields of M.

Let U be an open set of the covering $\mathscr{C}$ of M and x a point of $U \subset M \subset \tilde{M}$, where H does not vanish. Since we know that the second fundamental form at the point x is given by (4.3), we can evaluate the second member of (5.1) at X. A long but elementary calculation leads to the relation

$$\begin{aligned} & g(B(X,W),B(Y,Z)) - g(B(X,Z),B(Y,W)) \\ &\quad = \delta(g(H,H))^2[g(X,W)g(JY,H)g(JZ,H) + g(Y,Z)g(JX,H)g(JW,H)] \\ &\qquad - \delta(g(H,H))^2[g(X,Z)g(JY,H)g(JW,H) + g(Y,W)g(JX,H)g(JZ,H)] \\ &\qquad - \beta^2(g(H,H))^3[g(X,Z)g(Y,W) - g(X,W)g(Y,Z)], \end{aligned} \tag{5.2}$$

where $\delta = \beta(\alpha+\beta)$.

Now, let p, q be two oriented planes of $T_x(M) \subset T_x(\tilde{M})$. Denote by χ_{pq} and $\tilde{\chi}_{pq}$ the *bisectional curvatures* of M and of $\tilde{M}$ with respect to p, q. If X, Y and Z, W are oriented orthonormal bases of p and of q respectively, then from (5.1), (5.2) we derive

$$\begin{aligned} \tilde{\chi}_{pq} - \chi_{pq} = \beta^2(g(H,H))^3[g(X,W)g(Y,Z) - g(X,Z)g(Y,W)] \\ + \delta(g(H,H))^2[g(X,W)g(JY,H)g(JZ,H) + g(Y,Z)g(JX,H)g(JW,H)] \\ - \delta(g(H,H))^2[g(X,Z)g(JY,H)g(JW,H) + g(Y,W)g(JX,H)g(JZ,H)], \end{aligned} \tag{5.3}$$

where $\delta = \beta(\alpha+\beta)$.

Relation (5.3) is a first expression for the difference of the bisectional curvatures of $\tilde{M}$ and of M. The tensor fields H, g, J occurring at second member of (5.3) must be considered at the point x; similarly the functions δ, α, β must be evaluated at x.

6. Main results. The notations introduced in Sections 2 and 3 now permit to state some results.

Proposition 6.1. *If the planes p, q have no lines in common, then*

$$\tilde{\chi}_{pq} - \chi_{pq} = -\beta^2 \big(g(H,H)\big)^3 \cos pq - \delta\big(g(H,H)\big)^2 \mathscr{E}, \tag{6.1}$$

where

$$\mathscr{E} = g\big(\sigma_p (JH)_{p\cap t_m}, \sigma_q (JH)_{q\cap t_m}\big) + g\big(\sigma_p (JH)_{p\cap t_M}, \sigma_q (JH)_{q\cap t_M}\big) \tag{6.2}$$

and $\delta = \beta(\alpha+\beta)$. When p, q are isoclinic planes, then t_*, t^* replace t_m, t_M.

In particular, if p, q are orthogonal, but not strictly orthogonal, we have $\cos pq = 0$ *and*

$$\mathscr{E} = g\big(\sigma_p (JH)_{p\cap t_M}, \sigma_q (JH)_{q\cap t_M}\big). \tag{6.3}$$

If p, q are strictly orthogonal, then

$$\tilde{\chi}_{pq} = \chi_{pq}. \tag{6.4}$$

Proposition 6.2. *If the planes p, q have one and only one line in common, then (6.1) holds true and*

$$\mathscr{E} = g\big(\sigma_p (JH)_{p\cap q}, \sigma_q (JH)_{p\cap q}\big) + g\big(\sigma_p (JH)_{p\cap \nu}, \sigma_q (JH)_{q\cap \nu}\big), \tag{6.5}$$

where ν is the normal plane. In particular, if p, q are orthogonal, then $\cos pq = 0$ *and*

$$\mathscr{E} = g\big(\sigma_p (JH)_{p\cap \nu}, \sigma_q (JH)_{q\cap \nu}\big). \tag{6.6}$$

Proposition 6.3. *Let $\tilde{K}_p$, K_p be the sectional curvatures of $\tilde{M}$, M with respect to the plane p. Then*

$$\tilde{K}_p - K_p = -\beta^2 \big(g(H,H)\big)^3 - \delta\big(g(H,H)\big)^2 g\big((JH)_p, (JH)_p\big). \tag{6.7}$$

Moreover, if p' denotes the same plane as p with opposite orientation, then

$$\tilde{\chi}_{pp'} - \chi_{pp'} = \beta^2 \big(g(H,H)\big)^3 + \delta\big(g(H,H)\big)^2 g\big((JH)_p, (JH)_p\big). \tag{6.8}$$

It is worth remarking that Propositions 6.1, 6.2, and 6.3 exhaust all possible cases. More expressive formulas can be obtained when the planes are not orthogonal.

Proposition 6.4. *If the planes p, q are not orthogonal and have no lines in common, then relation (6.1) holds true where*

$$\mathscr{E} = \cos pq\left[\frac{g((JH)_{p\cap t_m},(JH)_{q\cap t_m})}{\cos^2\alpha_m}+\frac{g((JH)_{p\cap t_M},(JH)_{q\cap t_M})}{\cos^2\alpha_M}\right], \tag{6.9}$$

and $\delta = \beta(\alpha+\beta)$.

In particular, if p, q are isoclinic planes, then

$$\mathscr{E} = \pm\left[g((JH)_{p\cap t_*},(JH)_{q\cap t_*})+g((JH)_{p\cap t^*},(JH)_{q\cap t^*})\right] \tag{6.10}$$

according to $\cos pq > 0$, *or* $\cos pq < 0$.

Proposition 6.5. *If the planes p, q are not orthogonal and have one and only one line in common, then relation (6.1) holds true where*

$$\mathscr{E} = \frac{1}{\cos pq}\left[g((JH)_p,(JH)_q)-g((JH)_{p\cap q},(JH)_{p\cap q})\sin^2 pq\right]. \tag{6.11}$$

We conclude the section by remarking that all the formulas occurring in Propositions 6.1, 6.2, 6.3, 6.4, and 6.5 have a clear geometrical meaning. Moreover, taking into account of Remarks 3.1, 3.2, and 3.3, we can assure that the above results do not depend on the choice of the pair of related bases in p, q.

7. Proofs. In order to evidence the geometrical meaning of relation (5.3) and to prove the propositions of Section 6, we assume that X, Y and Z, W are *related bases* of p, q (Section 3). Then, using [7, (4)], we can write relation (5.3) in the form (6.1) where

$$\mathscr{E} = g(X,Z)g(JH,Y)g(JH,W)+g(Y,W)g(JH,X)g(JH,Z) \tag{7.1}$$

and $\delta = \beta(\alpha+\beta)$.

To prove Proposition 6.1, assume first that (3.2) is satisfied. So, using the notations of Section 2, we have

$$\begin{aligned} (JH)_{p\cap t_m} &= g(JH,X)X, \qquad (JH)_{p\cap t_M} = g(JH,Y)Y,\\ (JH)_{q\cap t_m} &= g(JH,Z)Z, \qquad (JH)_{q\cap t_M} = g(JH,W)W. \end{aligned} \tag{7.2}$$

Now, recalling definition (3.5) of the symmetries σ_p, σ_q, we immediately realize that (7.1) can be written in the form (6.2).

When (3.7) replaces (3.2), we are led to the same conclusion.

We can use the previous proceeding also in the case when p, q are isoclinic planes (Section 3). We have only to change the notations, that is, to replace t_m, t_M with t_*, t^*.

In particular, if p, q are orthogonal but not strictly orthogonal (Section 2), there exists a unit vector Y of p (W of q), that results to be orthogonal to any vector of q (of p). Choose a unit vector X of p (Z of q) such that X, Y (Z, W) be an oriented orthonormal basis of p (of q). It is easy now to realize that X, Y and Z, W are related bases of p, q (Section 3).

Referring to these bases, we have $g(Y,W) = 0$. Since p, q are not strictly orthogonal, we have $g(X,Z) \neq 0$. So inequality (3.2) is satisfied. It is now immediate to check that the first addend of (6.2) vanishes and that $\cos pq = 0$ by (3.4). Therefore (6.3) is proved.

Finally, let p, q be strictly orthogonal (Section 2).Then any orthonormal basis X, Y of p and any orthonormal basis Z, W of q form a pair of related bases of p, q. Since we have $g(X,Z) = g(Y,W) = 0$, the second member of (7.1) vanishes as well as $\cos pq$ and (6.1) reduces to (6.4).

To prove Proposition 6.2, we consider a unit vector $X = Z$ on the line $p \cap q$ and choose Y and W in such a way that X, Y and Z, W are oriented orthonormal bases of p and of q, respectively. These bases are related bases and the plane defined by Y, W is the normal plane ν. The proceeding used to prove (6.2) now leads from (7.1) to (6.5).

Equality (6.5) can be also regarded as a limiting case of (6.2), the plane t_m degenerating into the line $p \cap q$ and t_M tending to the normal plane ν.

In particular, if p, q are orthogonal, then $g(Y,W) = 0$. Consequently, since we have $\sigma_p(JH)_{p\cap q} \in p \cap \nu$, $\sigma_q(JH)_{p\cap q} \in q \cap \nu$, we get $\cos pq = 0$ and the first addend of (6.5) vanishes.

In order to prove Proposition 6.3, we consider the special case $q = p$, recalling that $\chi_{pp} = K_p$ and $\tilde{\chi}_{pp} = \tilde{K}_p$ (see [7, page 149]). Let X, Y and Z, W where $Z = X$ and $W = Y$, be orthonormal oriented bases of p and of $q = p$, respectively. Since these bases are related bases of p, q (Section 3), we have relation (6.1) where $\mathscr{E}$ is given by (7.1). Remarking that in the present case (7.1) reduces to $\mathscr{E} = (g(JH,X))^2 + (g(JH,Y))^2$, by virtue of (2.2) we have $\mathscr{E} = g((JH)_p,(JH)_p)$. It is now immediate to check that (6.1) reduces to (6.7). Finally, the relation (6.8) is a direct consequence of (6.7) and of [7, (3)].

We prove now Proposition 6.4. As we have seen at the beginning of the section, if X, Y and Z, W are related bases of p, q then we have (6.1) where $\mathscr{E}$ is given by (7.1). We assume first that (3.2) is satisfied. So we can use (7.2). Since p, q are not orthogonal, taking account of (3.1) we find $g(X,Z) \neq 0$ and $g(Y,W) \neq 0$. Then, recalling (3.3) and (3.4), we have

$$\frac{\cos pq}{\cos^2 \alpha_m} = \frac{g(Y,W)}{g(X,Z)}, \qquad \frac{\cos pq}{\cos^2 \alpha_M} = \frac{g(X,Z)}{g(Y,W)}. \tag{7.3}$$

It is now easy to check that (7.1) can be written in the form (6.9). When (3.7) replaces (3.2) we arrive to the same conclusion.

If p, q are isoclinic planes, that is, $\alpha_m = \alpha_M = \alpha_*$ (Section 3), then we have either (3.6) or (3.9). Relation (7.3) shows that the two cases occur when $\cos pq > 0$, or $\cos pq < 0$, respectively. Using (3.6), (3.9), and (7.2) where t_*, t^* replace t_m, t_M (Section 3), we see immediately that (7.1) reduces to (6.10).

To prove Proposition 6.5, we consider related bases X, Y and Z, W of p, q such that $X = Z$ is a unit vector of $p \cap q$ (see proof of Proposition 6.2). Taking into account (2.2) and (3.1), respectively, we have

$$g\big((JH)_p,(JH)_q\big) = \big(g(JH,X)\big)^2 + g(Y,W)g(JH,Y)g(JH,W). \tag{7.4}$$

On the other hand, from (2.1) it follows

$$g\big((JH)_{p\cap q},(JH)_{p\cap q}\big) = \big(g(JH,X)\big)^2. \tag{7.5}$$

Since (3.4) reduces to $g(Y,W) = \cos pq \neq 0$, it is now easy to check that (7.1) can be written in the form (6.11).

8. Remarks and special cases. Let U be an open set of the covering $\mathscr{C}$ (Section 4) and x a point of U, where H does not vanish.

The first result of the section is an inequality concerning the sectional curvatures.

Remark 8.1. If at the point x we have $\delta \geq 0$ or $\delta \leq 0$, then for any plane of $T(M) \subset T(\tilde{M})$ we have

$$K_p \leq \tilde{K}_p + (\beta^2 + \delta)\big(g(H,H)\big)^3, \tag{8.1}$$

or

$$K_p \geq \tilde{K}_p + (\beta^2 + \delta)\big(g(H,H)\big)^3, \tag{8.2}$$

respectively.

The results of Section 6 underline the essential role played by the vector field JH in the present research. We add now the following remarks.

Remark 8.2. The vector field JH is tangent to M.

Remark 8.3. Assume first $q \neq p$ and $q \neq p'$. Then, if at the point x the plane p (the plane q) is orthogonal to JH, we have

$$\tilde{\chi}_{pq} - \chi_{pq} = -\beta^2\big(g(H,H)\big)^3 \cos pq, \tag{8.3}$$

and consequently

$$\tilde{\chi}_{pq} - \beta^2\big(g(H,H)\big)^3 \leq \chi_{pq} \leq \tilde{\chi}_{pq} + \beta^2\big(g(H,H)\big)^3. \tag{8.4}$$

We assume now $q = p$ or $q = p'$. If at x the plane p is orthogonal to JH, we have

$$K_p = \tilde{K}_p + \beta^2\big(g(H,H)\big)^3, \qquad \chi_{pp'} = \tilde{\chi}_{pp'} - \beta^2\big(g(H,H)\big)^3, \tag{8.5}$$

respectively. Then we have

$$K_p \geq \tilde{K}_p, \qquad \chi_{pp'} \leq \tilde{\chi}_{pp'}. \tag{8.6}$$

To prove Remark 8.1, just note that

$$g((JH)_p,(JH)_p) \le g(JH,JH) = g(H,H). \tag{8.7}$$

Then, use (6.7).

At the point x of U, relation (4.2) implies

$$JH = -\frac{\lambda+(m-1)\mu}{m} e_1 \in T_x(M). \tag{8.8}$$

So Remark 8.2 is proved.

Finally, Remark 8.3 is an easy consequence of Propositions 6.1, 6.2, and 6.3.

The next remark refers to some special cases for the functions λ and μ, considered by Chen [2, page 282].

Remark 8.4. If at the point x we have $\lambda = 3\mu$ ($\lambda = 0$), then the left (right) inequality of Remark 8.1 holds with $\delta = \beta^2$ ($\delta = -2\beta^2$).

If at x we have $\lambda = 2\mu$, then $\delta = 0$. Consequently, we have relation $K_p = \tilde{K}_p + \beta^2(g(H,H))^3$ and $K_p \ge \tilde{K}_p$. Finally, if at x we have $\lambda = \mu$, since $\delta = -\beta^2$, the right inequality of Remark 8.1 reduces to $K_p \ge \tilde{K}_p$.

We end the paper with another special case.

Remark 8.5. We consider the H-umbilical submanifolds, such that for any U of $\mathscr{C}$ we have $\mu = |H|$. Then the relations giving the differences $\tilde{\chi}_{pq} - \chi_{pq}$ of the bisectional curvatures for the above mentioned submanifolds and for the totally umbilical submanifolds differ only for an additional term. In particular, the expressions of $\tilde{\chi}_{pq} - \chi_{pq}$ are the same in both cases, when in U we have $\lambda = 2\mu = 2|H|$.

To prove Remark 8.5, just note that $\beta^2(g(H,H))^3 = \mu^2$ and compare (6.1) with [5, (8), page 74].

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