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A NOTE ON LOGHARMONIC MAPPINGS

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We consider the problem of minimizing the moments of order p for a subclass of logharmonic mappings.

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1. Introduction. Let $H(U)$ be the linear space of all analytic functions defined on the unit disc $U = \{z = x+iy : |z| < 1\}$. A logharmonic mapping is a solution of the nonlinear elliptic partial differential equation

$$\overline{f_{\bar{z}}} = (a\overline{f}/f)f_z, \tag{1.1}$$

where the second delatation function a is in $H(U)$ and $|a(z)| < 1$ for all $z \in U$. If f does not vanish on U, then f is of the form

$$f = H \cdot \overline{G}, \tag{1.2}$$

where H and G are in $H(U)$. On the other hand, if f vanishes at 0 but has no other zeros in U, then f admits the representation

$$f(z) = z^m |z|^{2\beta m} h(z)\overline{g(z)}, \tag{1.3}$$

where

(a) m is nonnegative integer,

(b) $\beta = \overline{a(0)}(1+a(0))/(1-|a(0)|^2)$ and therefore, $\Re\beta > -1/2$,

(c) h and g are analytic in U, $g(0) = 1$, and $h(0) \neq 0$.

Univalent logharmonic mappings on the unit disc have been studied extensively. For details see [1, 2, 3, 4, 5, 6, 7, 8]. Suppose that f is a univalent logharmonic mapping defined on the unit disc U. Then, if $f(0) = 0$, the function $F(\zeta) = \log(f(e^{\zeta}))$ is univalent and harmonic on the half plane $\{\zeta : \operatorname{Re}\zeta < 0\}$. For more details on univalent harmonic mappings defined in the unit disc U, see [9, 10, 11, 12].

In this note, we consider the problem of minimizing the moments of order p over a subclass of the class logharmonic mappings defined over the unit disc U. It is interesting to note that the extremal functions are univalent starlike logharmonic mappings.

2. Moments of order p

Theorem 2.1. *Let $f = zh(z)\overline{g(z)}$ be logharmonic mapping defined on the unit disc U such that $h(0) = g(0) = 1$. Let $M_p(r,f)$ denote the moment of order p, $p \geq 0$. Then,*

$$M_p(r,f) \geq 2\pi\left(\frac{r^{p+2}}{p+2} - \frac{r^{p+4}}{p+4}\right). \tag{2.1}$$

Equality holds if and only if

$$f_1(z) = z\frac{(1+((p+2)/(p+4))\overline{z})}{(1+((p+2)/(p+4))z)} \tag{2.2}$$

or one of its rotations $\overline{\eta} f_1(\eta z)$.

Remark 2.2. If $p = 0$ in Theorem 2.1, then we have the problem of minimizing the area. Moreover, if $p = 2$, then we obtain the minimum of the moment of inertia.

Proof. Let $f = zh(z)\overline{g(z)}$ be logharmonic mapping defined on the unit disc U. Then, f satisfies (1.1) for some $a \in H(U)$ such that $|a(z)| < 1$ and $a(0) = 0$. Hence, using Schwarz's lemma, we have

$$\begin{aligned} M_p(r,f) &\geq \int_0^r \int_0^{2\pi} |f|^p \left(|f_z|^2 - |f_{\overline{z}}|^2\right)\rho\, d\theta\, d\rho \\ &= \int_0^r \int_0^{2\pi} |f|^p |f_z|^2 (1-|a|^2)\rho\, d\theta\, d\rho \\ &\geq \int_0^r \rho(1-\rho^2)\int_0^{2\pi} |f|^p |f_z|^2 d\theta\, d\rho. \end{aligned} \tag{2.3}$$

Writing $(h\cdot g)^{p/2}\cdot(zh)'\cdot g = 1+\sum_{k=1}^{\infty} c_k z^k$, we have

$$\int_0^{2\pi} |f|^p |f_z|^2 d\theta = 2\pi\rho^p\left(1+\sum_{k=1}^{\infty}|c_k|^2\rho^{2k}\right) \tag{2.4}$$

and therefore,

$$M_p(r,f) \geq 2\pi\int_0^{2\pi}\rho^p(1-\rho^2)d\rho = 2\pi\left(\frac{r^{p+2}}{p+2}-\frac{r^{p+4}}{p+4}\right). \tag{2.5}$$

Equality holds if and only if

$$(h)^{p/2}\cdot(g)^{(p+2)/2} \equiv 1 \tag{2.6}$$

and $a(z) = \eta z$, $|\eta| = 1$. This implies that

$$(h)^{(p+2)/2}\cdot(g)^{p/2} g' = \eta \tag{2.7}$$

and then,

$$z\cdot\frac{\partial(h\cdot g)^{(p+2)/2}}{\partial z} = \frac{(p+2)(1-(h\cdot g)^{(p+2)/2}+\eta z)}{2}. \tag{2.8}$$

The solution of the differential equation

$$z\cdot u(z)' + \frac{(p+2)\cdot u(z)}{2} = \frac{(p+2)(1+\eta z)}{2};\quad u(0) = 1 \tag{2.9}$$

is $u(z) = (h(z)g(z))^{(p+2)/2} = 1 + ((p+2)/(p+4))\eta z$. Together with (2.6), we get

$$\frac{g(z)'}{g(z)} = \frac{\eta}{(1 + ((p+2)/(p+4))\eta z)} \tag{2.10}$$

and therefore,

$$\begin{aligned} g(z) &= \left(1 + \frac{p+2}{p+4}\eta z\right)^{(p+4)/(p+2)}, \\ zh(z) &= \frac{z}{(1 + ((p+2)/(p+4))\eta z)}, \end{aligned} \tag{2.11}$$

which leads to the solution $\overline{\eta} f_1(\eta z)$. Since

$$\phi(z) = \frac{zh(z)}{g(z)} = \frac{z}{(1 + ((p+2)/(p+4))\eta z)^{(2p+6)/(p+2)}} \tag{2.12}$$

is a starlike univalent analytic, it follows from [4, Theorem 2.1] that f_1 is a starlike univalent logharmonic mapping. □

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