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OSCILLATION IN SECOND ORDER FUNCTIONAL EQUATIONS WITH DEVIATING ARGUMENTS

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ABSTRACT. For the pair of functional equations

(A) $$(r(t)y'(t))' + p(t)h(h(g(t))) = f(t)$$

and

(B) $$(r(t)y'(t))' - p(t)h(y(g(t))) = 0$$

sufficient conditions have been found to cause all solutions of equation (A) to be oscillatory. These conditions depend upon a positive solution of equation (B).

KEY WORDS AND PHRASES: Oscillatory, Nonoscillatory, Sublinear, Superlinear

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES:

1. INTRODUCTION.

Our main goal, in this work, is to seek the oscillatory behavior of the equation

$$(r(t)y'(t))' + p(t)h(y(g(t))) = f(t) \quad , \tag{1.1}$$

via the nonoscillation of the equation

$$(r(t)y'(t))' - p(t)h(y(g(t))) = 0. \tag{1.2}$$

Oscillation properties of equation (1.1) were studied by Kartsatos [3] and Kusano and Onose [4] by first "homogenizing" it and then using the techniques known for homogeneous equations. In fact a function $\lambda(t)$ was sought to satisfy

$$(r(t)(y(t) - \lambda(t))' = f(t). \tag{1.3}$$

A similar approach was later used by this author [9] in finding conditions for the oscillation of the equation

$$(r(t)y'(t))^{(n-1)} + (-1)^{n+1}p(t)y(g(t)) = f(t). \tag{1.4}$$

Recently Rankin [8] presented a new approach to study the oscillatory behavior of the ordinary differential equation

$$y''(t) + p(t)y(t) = f(t), \tag{1.5}$$

by using the transformation

$$y(t) = \phi(t)z(t) \quad , \tag{1.6}$$

where $\phi(t)$ is a positive solution of the equation

$$y''(t) + p(t)y(t) = 0. \tag{1.7}$$

Transformations usually do not carry over to functional equations (1.1) and (1.2). The failure in study of equation (1.1) leads us to this work in which we present a different approach to study the oscillation of equation (1.1) which may be sublinear, superlinear, retarded or advanced.

Since our results do not depend on the integral size of $p(t)$, they are different from those of Kartsatos [3], Kusano and Onose [4] and this author [9]. Our results are also different than those of Rankin [8]. In fact the following example shows that Rankin's results are not true for the pair of retarded equations

$$y''(t) + \frac{3}{16t^2(t-\pi)^{3/4}}\, y(t-\pi) = -100t \sin 10t + 20 \cos 10t + \frac{3}{8}\,\frac{(t-\pi)^{1/4}}{t^2} + \frac{3(t-\pi)^{1/4}}{16t^2} \sin 10t, \tag{1.8}$$

and

$$y''(t) + \frac{3}{16t^2(t-\pi)^{3/4}}\, y(t-\pi) = 0. \tag{1.9}$$

Equation (1.9) has the nonoscillatory solution $\phi(t) = t^{3/4}$ which satisfies the conclusion of Rankin's main theorem ([8, Theorem 2]) namely

$$\liminf_{t\to\infty} \int_T^t \frac{1}{\phi^2(x)} \int_T^x \phi(s) f(s)\, ds dx = -\infty, \tag{1.10}$$

$$\limsup_{t\to\infty} \int_T^t \frac{1}{\phi^2(x)} \int_T^x \phi(s) f(s)\, ds dx = \infty, \tag{1.11}$$

and $\displaystyle\int_T^\infty \frac{1}{\phi^2(x)}\, dx < \infty,$

for any large $T > 0$; where

$$f(t) = -100t \sin(10t) + 20 \cos(10t) + \frac{3(t-\pi)^{1/4}}{8t^2} + \frac{3(t-\pi)^{1/4}}{16t^2} \sin(10t).$$

But equation (1.8) has the nonoscillatory solution

$$y(t) = 2t + t \sin(10t).$$

2. DEFINITIONS AND ASSUMPTIONS

Throughout this study we assume the following:

(i) $g(t)$, $r(t)$, $p(t)$, $h(t)$ and $f(t)$ are $C[R,R]$ where R denotes the real line;

(ii) $r(t)>0$, $r'(t)\leq 0$ and $p(t)>0$ for $t>t_0>0$ where we shall assume t_0 to be fixed arbitrarily. t_0 will be referred to in this study without any further mention;

(iii) $g(t) \to \infty$ as $t \to \infty$;

(iv) sign $h(t)$ = sign t.

The term "solution" refers to nontrivial continuously extendable solutions of equations under consideration over the interval $[t_0,\infty)$. We call a function $Q(t) \in C[[t_0,\infty), R]$ oscillatory if $Q(t)$ has arbitrarily large zeros on $[t_0,\infty)$; otherwise $Q(t)$ is called nonoscillatory. Equations (1.1) and (1.2) are called sublinear or superlinear

$$0 < \frac{h(t)}{t^{\alpha}} \leq k$$

if $0 < \alpha \leq 1$ or $\alpha < 1$ respectively where k is constant and α is the ratio of odd integers.

3. MAIN RESULTS

THEOREM 1: In addition to (i)-(iv) suppose there exists a function $\phi(t)$ which is continuous for $t \geq t_0$ and satisfies $(r(t)\phi'(t))' \geq o$ ($\not\equiv 0$ in any interval),

$$\liminf_{t\to\infty} \int^{t} \frac{1}{\phi^2(s)} \int^{s} \phi(x)f(x)\,dx\,ds = -\infty, \tag{3.1}$$

$$\limsup_{t\to\infty} \int^{t} \frac{1}{\phi^2(s)} \int^{s} \phi(x)f(x)\,dx\,ds = \infty \tag{3.2}$$

and

$$\int^{\infty} \frac{1}{\phi^2(t)}\,dt < \infty . \tag{3.3}$$

Then all solutions of equation (1.1) are oscillatory.

PROOF: Suppose to the contrary that equation (1.1) has a nonoscillatory solution $y(t)$. Without any loss of generality suppose $T > t_0$ is large enough so that for $t > T$, $y(g(t)) > 0$ and $y(t) > 0$. Rewriting equation (1.1) after multiplication with $\phi(t)$ we have

$$(r(t)\phi(t)y'(t))' - (r(t)\phi'(t))y'(t) + p(t)\phi(t)h(y(g(t))) = \phi(t)f(t). \tag{3.4}$$

Integrating (3.4) for $t \geq T$ we have

$$r(t)\phi(t)y'(t) - r(T)\phi(T)y'(T) - r(t)\phi'(t)y(t)$$

$$+ r(T)\phi'(T)y(T) + \int_T^t (r(s)\phi'(s))'y(s)ds$$

$$+ \int_T^t p(s)\phi(s)h(y(g(s)))ds = \int_T^t \phi(s)f(s)ds. \tag{3.5}$$

Set

$$K = r(T)\phi'(T)y(T) - r(T)\phi(T)y'(T). \tag{3.6}$$

Dividing (3.5) by $\phi^2(t)$ and rearranging terms we have

$$\frac{r(t)y'(t)}{\phi(t)} + \frac{K}{\phi^2(t)} - \frac{r(t)\phi'(t)y(t)}{\phi^2(t)} + \frac{1}{\phi^2(t)}\int_T^t (r(s)\phi'(s))'yds$$

$$+ \frac{1}{\phi^2(t)}\int_T^t p(s)\phi(s)h(y(g(s)))ds = \frac{1}{\phi^2(t)}\int_T^t \phi(s)f(s)ds. \tag{3.7}$$

Integrating (3.7) between T and t we have

$$\frac{r(t)y(t)}{\phi(t)} - \frac{r(T)y(T)}{\phi(T)} + \int_T^t \frac{r(s)\phi'(s)y(s)}{\phi^2(s)}ds - \int_T^t \frac{r'(s)y(s)}{\phi(s)}ds$$

$$+ \int_T^t K/\phi^2(s)ds - \int_T^t \frac{r(s)\phi'(s)y(s)}{\phi^2(s)}ds$$

$$+ \int_T^t \frac{1}{\phi^2(x)}\int_T^x [(r(s)\phi'(s))'y(s) + p(s)\phi(s)h(y(g(s)))]dsdx$$

$$= \int_T^t \frac{1}{\phi^2(x)}\int_T^x \phi(s)f(s)dsdx \tag{3.8}$$

which leads to

$$\frac{r(t)y(t)}{\phi(t)} - \frac{r(T)y(T)}{\phi(T)} - \int_T^t \frac{r'(s)y(s)}{\phi(s)}ds$$

$$+ \quad K \int_T^t 1/\phi^2(s)\,ds$$

$$+ \quad \int_T^t \frac{1}{\phi^2(x)} \int_T^x [(r(s)\phi'(s))'y(s) + p(s)\phi(s)h(y(g(s)))]\,ds\,dx$$

$$= \quad \int_T^t \frac{1}{\phi^2(x)} \int_T^x \phi(s)f(s)\,ds\,dx. \tag{3.9}$$

Since third, fourth and fifth terms on the left hand side of (3.9) are either nonnegative or finite, we immediately reach a contradiction in view of (3.1) and (3.2). The proof is complete.

COROLLARY 1. Suppose (i)-(iv) hold. Further suppose that equation a positive solution $\phi(t)$ satisfying (3.1), (3.2) and (3.3). Then all solutions of equation (1.1) are oscillatory.

PROOF. Since $(r(t)\phi'(t))' \geq 0$, conclusion follows from Theorem 1.

EXAMPLE 1. Consider the equations

$$y''(t) + e^{\pi}y(t-\pi) = 4e^{2t}\cos t + 3e^{2t}\sin t - e^{2t-\pi}\sin t\ , \tag{3.10}$$

and

$$y''(t) - e^{\pi}y(t-\pi) = 0\ , \tag{3.11}$$

for $t > \pi$. Equation (3.11) has $y(t) = e^t$ as a solution which satisfies (3.1), (3.2) and (3.3). Thus all solutions of equation (3.10) are oscillatory. In fact $y(t) = e^{2t}\sin t$ is one such solution.

REMARK. In Rankin's work $\phi''(t) < 0$ where as here $\phi''(t) > 0$ when $r(t) \equiv 1$.

THEOREM 2. Suppose $r(t) \equiv 1$ and (i)-(iv) hold. Further suppose that equation (1.2) has a positive solution $\phi(t)$ such that $\phi'(t) \geq 0$ ($\not\equiv 0$ in any subinterval) for $t > t_0$. Let (3.1) and (3.2) of Theorem (1) hold. Then all solutions of equation (1.1) are oscillatory.

PROOF: Since

$$\phi''(t) \geq 0, \quad \phi'(t) \geq 0 \quad \text{and} \quad \phi(t) \geq 0 , \tag{3.12}$$

for $t > t_0$, there exist positive numbers c_1 and c_2 such that $\phi(t) \geq c_1 t + c_2$, and consequently $\phi(t)$ satisfies (3.3). The proof is complete. We now have the following corollary.

COROLLARY 2: Suppose equation (1.2) has a positive nonoscillatory solution $z(t)$ such that $z'(t) > 0$. Further suppose that equation (1.1) has a nonoscillatory solution. Then either

$$\liminf_{t\to\infty} \int^t \frac{1}{z^2(s)} \int^s z(x) f(x)\,dxds > -\infty \tag{3.13}$$

or

$$\limsup_{t\to\infty} \int^\infty \frac{1}{z^2(s)} \int^s z(x) f(x)\,dxds < \infty . \tag{3.14}$$

EXAMPLE 2. The equation

$$y''(t) + \frac{2}{t^2} y(t) = -\sin t + \frac{4}{t^2} + \frac{2\sin t.}{t^2} \tag{3.15}$$

has the nonoscillatory solution $y(t) = 2 + \sin t$. Now consider

$$y''(t) - \frac{2}{t^2} y(t) = 0 , \tag{3.16}$$

which has $z(t) = t^2$ as a nonoscillatory solutions satisfying the conditions and conclusion of Corollary 2.

4. ASYMPTOTIC NONOSCILLATION

Example 2 shows that when (3.1) and (3.2) are relaxed then equation (1.1) may have nonoscillatory solutions. In this section we give conditions when nonoscillatory solutions of (1.1) approach limits.

THEOREM 3: Suppose (i)-(iv) hold. Let $\phi(t)$ be a positive solution of equation (1.2) such that $\phi'(t) \geq 0$ ($\not\equiv 0$ in any subinterval of t for $t > t_0$),

$$\liminf_{t\to\infty} \int^{t} \frac{1}{\phi^2(x)} \int^{x} \phi(s)f(s)\,ds\,dx < 0 , \tag{4.1}$$

and

$$\limsup_{t\to\infty} \int^{t} \frac{1}{\phi^2(x)} \int^{x} \phi(s)f(s)\,ds\,dx > 0. \tag{4.2}$$

Let $y(t)$ be a bounded solution of equation (1.1). If $y(t)$ is nonoscillatory then $y(t)$ tends to a finite limit.

PROOF. Without any loss of generality, let $T \geq t_0$ be large enough so that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq T$. Suppose to the contrary that

$$\liminf_{t\to\infty} y(t) < \limsup_{t\to\infty} y(t). \tag{4.3}$$

Then there exists a sequence $\{T_n\}_{n=1}^{\infty}$ such that $T_n \to \infty$ as $n \to \infty$ and $y'(T_n) = o$. Let k be a large positive integer such that

$$\frac{y(T_k)r(T_k)}{\phi(T_k)} < \text{Min}\left[-\liminf_{t\to\infty} \int_{T_k}^{t} \frac{1}{\phi^2(x)} \int_{T_k}^{x} \phi(s)f(s)\,ds\,dx ,\right.$$

$$\left. \limsup_{t\to\infty} \int_{T_k}^{t} \frac{1}{\phi^2(x)} \int_{T_k}^{x} \phi(s)f(s)\,ds\,dx \right]. \tag{4.4}$$

Following the proof of Theorem 1, we obtain from (3.9)

$$\frac{r(t)y(t)}{\phi(t)} - \int_{T_k}^{t} \frac{r'(s)y(s)\,ds}{\phi(s)} + r(T_k)\phi'(T_k)y(T_k) \int_{T_k}^{t} \frac{1}{\phi^2(x)}\,dx$$

$$+ \int_{T_k}^{t} \frac{1}{\phi^2(x)} \int_{T_k}^{x} p(s)\,(y(s)h(\phi(g(s))) + \phi(s)h(y(g(s))))\,ds\,dx$$

$$= \int_{T_k}^{t} \frac{1}{\phi^2(x)} \int_{T_k}^{x} \phi(s)f(s)\,ds\,dx + \frac{y(T_k)r(T_k)}{\phi(T_k)} . \tag{4.5}$$

In view of (4.1), (4.2) and (4.4), we reach a contradiction in (4.5). The proof is complete.

REMARK. Example 2 shows that conditions (4.1) and (4.2) cannot be weakened.

COROLLARY 3. Suppose conditions of Theorem 3 hold. Let $y(t)$ be any solution of equation (1.1) such that $\frac{y(t)}{\phi(t)} \to 0$ as $t \to \infty$. If $y(t)$ is nonoscillatory then $y(t)$ tends to a finite or infinite limit as $t \to \infty$.

REMARK. Recently Graef and Spikes [1], Hammett [2], Kusano and Onose [5,6], Philos and Starkos [7], this author [10,11] have studied asymptotic nonoscillation with regard to equation (1.1). However all these results make use of an integral condition on $p(t)$. Theorem 3 and Corollary 3 present a different approach.

REFERENCES

1. J. Graef and P. Spikes, Asymptotic behavior of solutions of a second order nonlinear differential equation, J. Differential Equations, 17 (1975), 461-476.

2. M. Hammett, Nonoscillation properties of a nonlinear differential equation, Proc. Amer. Math. Soc., 30 (1971), 92-96.

3. A.G. Kartsatos, On the maintenance of oscillations of nth order equations under the effect of a small forcing term, J. Differential Equations, 10 (1971), 355-563.

4. T. Kusano and H. Onose, Oscillations of functional differential equations with retarded arguments, 15 (1974), 269-277.

5. T. Kusano and H. Onose, Asymptotic behavior of nonoscillatory solutions of functional differential equations of arbitrary order, J. London Math. Soc., 14 (1976), 106-112.

6. T. Kusano and H. Onose, Nonoscillation theorems for differential equations with deviating argument, Pacific J. Math., 63 (1976), 185-192.

7. Ch. G. Philos and V.A. Staikos, Asymptotic properties of nonoscillatory solutions of differential equations with deviating argument, Pacific J. Math., 70 (1977), 221-242.

8. S. Rankin, Oscillation results for a nonhomogeneous equations, Pacific J. Math., 80 (1979), 237-244.

9. B. Singh, Impact of delays on oscillation in general functional equations, Hiroshima Math. J., 5 (1975), 351-361.

10. B. Singh, Forced nonoscillations in fourth order functional equations, Funk. Ekva., 19 (1976), 227-237.

11. B. Singh, Nonoscillations of forced fourth order retarded equations, SIAM J. Appl. Math., 28 (1975), 265-269.

12. B. Singh and T. Kusano, On asymptotic limits of nonoscillations in functional equations with retarded arguments, Hiroshima Math. J., (to appear).