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CONVERGENCE ANALYSIS OF A PERTURBED ITERATIVE SCHEME (PIS) FOR SOLUTION OF NONLINEAR SYSTEMS

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ABSTRACT. In the references [1, 2, 3] a perturbed iterative scheme (PIS) has been studied both theoretically and computationally to solve nonlinear equations. In this article a more general analysis of its convergence properties has been done.

KEY WORDS AND PHRASES. Perturbed Functional Iterations, Nonlinear Equations, Numerical Solution.

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1. INTRODUCTION.

A perturbed iterative scheme (PIS) has been developed in [1, 3] to solve nonlinear equations. It is a functional iterative scheme obtained by adding a predetermined perturbation parameter to nonlinear Gauss-Seidel iterations. PIS has a simple algorithm. Other similar iterations [6, 7, 8] had more restricted applications and complicated algorithms. Theorems on convergence properties of

PIS, derived in [1] for one nonlinear system, were modified in [3] for coupled nonlinear systems. In this article more generalized convergence properties of PIS have been analyzed by applying decaying matrices [5]. A practical demonstration of this concept is also given. Theorems on convergence of PIS derived in [1, 2, 3] may be interpreted as particular cases of those derived in this paper.

2. ALGORITHM OF PIS [1].

Let us consider a nonlinear system:

$$F_i(x_1, x_2, \ldots x_n) = 0, \; i = 1, 2, \ldots n \tag{2.1}$$

This equation may be expressed as $F(x) = 0$ where $x = (x_1\ x_2\ \ldots\ x_n)^T$, $F : D \subset R^n \to R^n$ (R^n is the real n-dimensional space). We assume that (2.1) admits a root $x = x^* \in D$. Our objective is to develop a perturbed iterative scheme (PIS) to solve (2.1) and compute x^*. If (2.1) is written as:

$$x_i = G_i(x_1, x_2, \ldots x_n), \; i = 1, 2, \ldots n \tag{2.2}$$

nonlinear Gauss-Seidel iteration at some kth step is:

$$x_i^k = G_i^k \tag{2.3}$$

where x_i^k = value of x_i at the kth (not exponent) iteration and

$$G_i^k = G_i(x_1^k, x_2^k \ldots x_{i-1}^k, x_i^{k-1} \ldots x_n^{k-1}). \tag{2.4}$$

Let $G : D \times D \subset R^n \times R^n \to D$. Then (2.3) may be expressed as

$$x^k = G(x^k, x^{k-1}) \tag{2.5}$$

and,

$$x^* = G(x^*, x^*) \tag{2.6}$$

DEFINITION 1: In such a case, x^* is called a <u>fixed image</u> of G on D x D. Now (2.3) may be perturbed as follows:

$$x_i^k = w_i^k + G_i^k \tag{2.7}$$

To compute the perturbation parameter w_i^k, we assume that $\forall$ i,k: (i) w_i^k's are small, such that terms of the order $(w_i^k)^2$ may be neglected, (ii) $(\partial G_i / \partial x_i)^k \neq 1$, and (iii) $(\partial^2 G_i / \partial x_i^2)^k$ is bounded.

If (2.7) converges after (k-1) iterations, $x_i^{k-1} = x_i^k = x_i^*$. Then, $w_i^k + G_i^k = G_i(x_1^k \ldots x_{i-1}^k, w_i^k + G_i^k, x_{i+1}^{k-1} \ldots x_n^{k-1})$. Expanding the right side by Taylor's series and using the above assumptions we get

$$w_i^k = (\overline{G}_i^k - G_i^k) / \{1 - \partial_i G_i^k\} \tag{2.8}$$

where $\overline{G}_i^k = G_i(x_1^k \ldots x_{i-1}^k, G_i^k, x_{i+1}^{k-1} \ldots x_n^{k-1})$ and

$$\partial_i G_i^k = (\partial G_i / \partial x_i)_{x_1^k \ldots x_{i-1}^k, G_i^k, x_{i+1}^{k-1} \ldots x_n^{k-1}}.$$

Thus in (2.7), w_i^k's are computed in terms of quantities known apriori. The convergence criterion is:

$$\max_i |w_i^k| < \varepsilon \tag{2.9}$$

where ε is positive and arbitrarily small.

3. ANALYSIS OF CONVERGENCE.

By convergence we mean $\lim_{k\to\infty} x^k = x^*$. Now (2.7) may be expressed as:

$$x^k = w^k + G(x^k, x^{k-1}) \tag{3.1}$$

where $w^k = (w_1^k \ldots w_n^k)^T \in R^n$. Let G be continuous on D x D.

THEOREM 1: A necessary condition for convergence is that, for some norm

$$\lim_{k\to\infty} \| w^k \| = 0 \tag{3.2}$$

PROOF: Assuming that (2.7) converges to x^*, $\lim_{k\to\infty} x^k = x^*$ and $\lim_{k\to\infty} G(x^k, x^{k-1}) = G(x^*, x^*) = x^*$. Then, (3.1) gives $\lim_{k\to\infty} w^k = \emptyset$, which implies (3.2).

To prove that (3.2) may also be a sufficient condition for convergence, the following concept is used:

DEFINITION 2: A sequence of commuting square matrices $A_1, A_2, \ldots A_n$ is called a sequence of decaying matrices or simply D-matrices if

$$\lim_{k\to\infty} A_1 A_2 \ldots A_k = \emptyset \tag{3.3}$$

Each A_k is called a D-matrix. Obviously, the elements of A_k are variable and they change as k changes. Several properties of D-matrices are given in [5]. The one we need is:

LEMMA 1: A sufficient condition such that A_k is a D-matrix is that for some particular norm and $\forall\, k > K$,

$$\|A_k\| \leq \alpha < 1 \tag{3.4}$$

PROOF: For any norm, $\|A_1 A_2 \ldots A_k\| \leq \|A_1\|\,\|A_2\| \ldots \|A_k\|$. Also, $\|A_k\| = 0$ iff $A = \emptyset$. Proof is now rather trivial.

Let $H : D \times D \subset R^n \times R^n \to D$ $\forall$ (x^k, y^k), $(\alpha, \beta) \in D \times D$. Let us express

$$H(x^k, y^k) - H(\alpha, \beta) = A_k(x^k - \alpha) + B_k(y^k - \beta) \tag{3.5}$$

where A_k, B_k are square matrices $(n \times n)$ and, $a_{i,j}$ = an element of A_k = $a^k_{i,j}(x^k, y^k, \alpha, \beta)$; $b^k_{i,j}$ = an element of $B_k = b^k_{i,j}(x^k, y^k, \alpha, \beta)$. As k changes, x^k, y^k and hence $a^k_{i,j}$, $b^k_{i,j}$ change. Let $|x| = (|x_1|\ |x_2|\ \ldots\ |x_n|)^T$.

DEFINITION 2: If $\forall (x^k, y^k)$, $(\alpha, \beta) \in D \times D$, A_k, B_k are continuous,

$$C_k = |(I - A_k)^{-1}| \tag{3.6}$$

is bounded, and

$$E_k = |(I - A_k)^{-1} B_k| \tag{3.7}$$

is continuous and form a sequence of D-matrices, the mapping $H : D \times D \subset R^n \times R^n \to D$ is called a D-mapping on $D \times D$.

THEOREM 2: If in (3.1) G is a D-mapping and (3.2) is true, PIS (3.1) will converge to x^*, a fixed image of G on $D \times D$. Furthermore, if $\rho(E_k)$ (the spectral radius of E_k) < 1 $\forall k > K$, x^* is the unique fixed image of G on $D \times D$.

PROOF: Subtracting (2.6) from (3.1) and using (3.5) we get $x^k - x^* = w^k + A_k(x^k - x^*) + B_k(x^{k-1} - x^*)$.

Then,
$$|x^k - x^*| \leq C_k|w^k| + E_k|x^{k-1} - x^*| \tag{3.8}$$

where C_k and E_k are defined in (3.6) and (3.7) respectively. Now, using (3.8) recurrently,

$$|x^k - x^*| \le \sum_{j=1}^{k} E_k E_{k-1} \cdots E_{j+1} C_j |w^j| + E_k E_{k-1} \cdots E_1 |x^0 - x^1|$$

$$= (E_k E_{k-1} \cdots E_{k_0+1}) \sum_{j=1}^{k_0} (E_{k_0} E_{k_0-1} \cdots E_{j+1}) C_j |w^j|$$

$$+ \sum_{j=k_0+1}^{k} (E_k E_{k-1} \cdots E_{j+1}) C_j |w^j| + E_k E_{k-1} \cdots E_1 |x^0 - x^*|. \quad (3.9)$$

(3.2) implies that for some $k \ge k_0+1$, $|w^k| < \varepsilon$. Also, since E_k is a D-matrix, $\lim_{k\to\infty} E_1 E_2 \cdots E_k = \emptyset$ and $\lim_{k\to\infty} E_{k_0+1} E_{k_0+2} \cdots E_k = \emptyset$. Hence C_j being bounded $\forall j$, from (3.9) we get $\lim_{k\to\infty} |x^k - x^*| = \emptyset$ which establishes convergence.

In order to prove uniqueness, we assume that $x = y^*$ is a second root. Then, $x^* - y^* = G(x^*, x^*) - G(y^*, y^*) = A_*(x^* - y^*) + B_*(x^* - y^*)$. Thus, $|x^* - y^*| = |(I - A_*)^{-1} B_*|\ |x^* - y^*| = E_*\ |x^* - y^*|$. Since $\rho(E_k) < 1$, $\forall k > K$ and $E_k \to E_*$ as $k \to \infty$, $\rho(E_*) < 1$. This gives $(I - E_*)^{-1} = \sum_{j=0}^{\infty} E_*^j > \emptyset$. Hence $(I - E_*)|x^* - y^*| = \emptyset$, implies $x^* = y^*$.

It may be observed that A_k and B_k are respectively lower and upper triangular matrices with variable elements. If $A_k + B_k$ is a tridiagonal matrix it is rather easy to check when G is a D-mapping [9]. However, in general $(I - A_k)^{-1}$ is not quite simple to find. In such cases, theorem 2 cannot be used and the following approach may be taken.

THEOREM 3: Let $\forall k$,

$$G(x^k, x^{k-1}) - G(x^*, x^*) = A_k(x^k - x^*) + B_k(x^{k-1} - x^*) \quad (3.10)$$

where $(I - A_k)^{-1}$ is bounded. Let for some particular norm and $\forall k > K$, $\|A_k\| + \|B_k\| \le \alpha < 1$. Furthermore if (3.2) is true, PIS given by (3.1) will converge to x^* and x^* is the unique fixed image of G on D x D.

PROOF: Let $M_k = (I - A_k)^{-1} B_k$. Since $\|A_k\| + \|B_k\| \le \alpha < 1$, $\|A_k\| \le \alpha < 1$ and $(\alpha - \|A_k\|)(1 - \|A_k\|)^{-1} < \alpha$. Also,

$\| M_k \| = \| (I - A_k)^{-1} B_k \| \leq \| B_k \| / (1 - \| A_k \|) \leq (\alpha - \| A_k \|) / (1 - \| A_k \|)$ $< \alpha < 1$. Hence M_k is a D-matrix. Subtracting (2.6) from (3.1) and using (3.10) we get: $x^k - x^* = w^k + A_k(x^k - x^*) + B_k(x^{k-1} - x^*)$. Thus, $\| x^k - x^* \|$ $\leq \| (I - A_k)^{-1} \| \; \| w^k \| + \| M_k \| \; \| x^{k-1} - x^* \|$. If $\theta = \max_k \| (I - A_k)^{-1} \|$,

$$\| x^k - x^* \| \leq \theta \| w^k \| + \alpha \| x^{k-1} - x^* \| . \tag{3.11}$$

(3.2) suggests that for some $k \geq k_0+1$, $\| w^k \| < \varepsilon$. Hence applying (3.11) recurrently we get:

$$\| x^k - x^* \| \leq \theta \cdot \alpha^{k-k_0} \sum_{j=1}^{k_0} \alpha^{k_0-j} \| w^j \| + \varepsilon \cdot \theta \cdot (1 - \alpha)^{-1} + \alpha^k \| x^0 - x^* \| .$$

Since $0 < \alpha < 1$, as $k \to \infty$, $\| x^k - x^* \| \to 0$.

To prove uniqueness, let $x = y^*$ be the other root. Then, $x^* - y^*$ $= G(x^*, x^*) - G(y^*, y^*) = A_*(x^* - y^*) + B_*(x^* - y^*)$ or $(1 - \| M_* \|) \| x^* - y^* \| \leq 0$. Since $\| M_k \| < \alpha < 1 \; \forall k > K$ and as $k \to \infty$, $M_k \to M_*$, we have $\| x^* - y^* \| \leq 0$ giving $x^* = y^*$.

It may be easy to prove:

THEOREM 4: A sufficient condition such that nonlinear Gauss-Seidel iterations $x^k = G(x^k, x^{k-1})$ will converge to $x = x^*$ is that $E_k = |(I - A_k)^{-1} B_k|$ is a D-matrix where A_k and B_k are defined in (3.5).

PROOF: In (3.9) we set $w^k = 0$ and the proof follows.

4. AN APPLICATION.

Solve: $0.25x_1^2 - x_1 + 0.75x_2 = 0$, $0.5x_1x_2 - x_2 + 0.005 = 0$. In $[-0.5, 0.5] \times [-0.5, 0.5]$, this system has a root given by $x_1 = 3.7606 \times 10^{-3}$ and $x_2 = 5.00942 \times 10^{-3}$. We express the system as $x_1 = G_1(x_1, x_2)$, $x_2 = G_2(x_1, x_2)$, where $G_1(x_1, x_2) = 0.25x_1^2 + 0.75x_2$; $G_2(x_1, x_2) = 0.5x_1x_2 + 0.005$. Denoting the roots by x_1^*, x_2^*: $G_1(x_1^{k-1}, x_2^{k-1}) - G_1(x_1^*, x_2^*) = 0.25(x_1^{k-1} + x_1^*)(x_1^{k-1} - x_1^*)$ $+ 0.75(x_2^{k-1} - x_2^*)$, and $G_2(x_1^k, x_2^{k-1}) - G_2(x_1^*, x_2^*) = 0.5x_2^{k-1}(x_1^k - x_1^*)$ $+ 0.5x_1^*(x_2^{k-1} - x_2^*)$. Thus, $G(x^k, x^{k-1}) - G(x^*, x^*) = A_k(x^k - x^*) + B_k(x^{k-1} - x^*)$

where $A_k = \begin{bmatrix} 0 & 0 \\ 0.5x_2^{k-1} & 0 \end{bmatrix}$, $B_k = \begin{bmatrix} 0.25(x_1^{k-1} + x_1^*) & 0.75 \\ 0 & 0.5x_1^* \end{bmatrix}$

Then, $E_k = |(I - A_k)^{-1}B_k| = \begin{bmatrix} 0.25|x_1^{k-1} + x_1^*| & 0.75 \\ 0.125|x_2^{k-1}(x_1^{k-1} + x_1^*)| & |0.375x_2^{k-1} + 0.5x_1^*| \end{bmatrix}$

Obviously, if $\forall k$ $(x_1^k, x_2^k) \in D \times D$, $D = [-0.5, 0.5]$, $\|E_k\|_\infty \leq \alpha < 1$ (since $x_1^* \approx 0$). Thus E_k is continuous and a D-matrix. Also, since $C_k = |(I - A_k)^{-1}|$ is bounded, G is a D-mapping. Hence, iterations will converge iff (3.2) is true. The algorithm, now, requires the following to be computed sequentially at any kth iteration: (i) $G_1^k = 0.25(x_1^{k-1})^2 + 0.75x_2^{k-1}$; (ii) $[\partial G_1 / \partial x_1]_{G_1^k, x_2^{k-1}} = 0.5G_1^k$; (iii) $\overline{G}_1^k = 0.25(G_1^k)^2 + 0.75x_2^{k-1}$; (iv) $w_1^k = (\overline{G}_1^k - G_1^k) / \{1 - [\partial G_1 / \partial x_1]_{G_1^k, x_2^{k-1}}\}$; (v) $x_1^k = w_1^k + G_1^k$; (vi) $G_2^k = 0.5x_1^k x_2^{k-1} + 0.005$; (vii) $[\partial G_2 / \partial x_2]_{x_1^k, G_2^k} = 0.5x_1^k$; (viii) $\overline{G}_2^k = 0.5x_1^k G_2^k + 0.005$; (ix) $w_2^k = (\overline{G}_2^k - G_2^k) / \{1 - [\partial G_2 / \partial x_2]_{x_1^k, G_2^k}\}$; (x) $x_2^k = w_2^k + G_2^k$. Convergence follows iff $\max_{i=1,2} |w_i^k| < \varepsilon$. If $\varepsilon = 10^{-5}$ and $(x_1^0, x_2^0) \in D \times D$, PIS converges in 5 iterations. This simple example explains the convergence principle as developed in theorem 2. It shows that in order to prove that E_k is a D-matrix, some _prior_ knowledge of the roots is required.

5. DISCUSSIONS.

PIS has some limitations [1]. In order to prove that G is a D-mapping some apriori knowledge of x^* is required (as is evident in section 4). PIS is not quite effective to solve equations with multiple roots. For example the system: $x = \sin(x)\cos(y) + \sin(z)$, $y = \sin(y)\cos(z) + \sin(x)$. $z = \sin(z)\cos(x) + \sin(y)$ has three roots: (0, 0, 0), (1.249, 1.249, 1.249) and (-1.249, -1.249, -1.249). With (x^0, y^0, z^0) = (1, 1, 1), (9999, 9999, 9999), (0.1, -0.1, 0.005), (-9.0, -9.0, 0.00003), (76, 900, 8000), (-3, -3, 3) it converged to the second root, with

(x^0, y^0, z^0) = (5555, 3333, 1111), (56, 92, -9), (-1, -1, 13), (49, 78, 100) it converged to the third root but the first root was not found. The reason for this is still under investigation. If, in some cases, w_i^k's are identically equal to zero, PIS becomes less effective [1]. PIS may be interpreted as a combination of Gauss-Seidel and Lieberstein's methods [1]. It was compared with other similar methods in [1, 3]. Although Newton's method has a quadratic rate of convergence it requires in general initial estimates to be close to the root and evaluations of Jacobians at each iteration. PIS requires none. Thus for large systems whereas Newton's method is not practical to use, PIS can be easily used [1, 3]. In [1] it has been proved that PIS has a quadratic rate of convergence. Also, since PIS-algorithm is relatively simple, computer programming is rather easy. Since no matrices are stored, requirement for computer memory storage is also small.

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