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PARTIAL HENSELIZATIONS

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ABSTRACT. We define and note some properties of k H-pairs (k Henselian pairs), k N-pairs, and k N'-pairs. It is shown that the 2-Henselization and the 3-Henselization of a pair exist. Characterizations of quasi-local 2H-pairs are given, and an equivalence to the chain conjecture is proved.

KEY WORDS AND PHRASES. *k Henselian pair, k N-pair, k N'-pair, chain conjecture.*
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1. INTRODUCTION.

We define a pair (A,m) to be a k H-pair (a k Henselian pair) in case the ideal m is contained in the Jacobson radical of the commutative ring A and if for every monic polynomial f(X) of degree k in A[X] such that $\bar{f}(X) \in A/m\,[X]$ factors into $\bar{f}(X) = \bar{g}_o(X)\bar{h}_o(X)$ where $\bar{g}_o(X)$ and $\bar{h}_o(X)$ are monic and coprime, there exist monic polynomials g(x), $h(X) \in A[X]$ such that $f(X) = g(X)h(X)$, $\bar{g}(X) = \bar{g}_o(X)$, and $\bar{h}(X) = \bar{h}_o(X)$. It is shown that the 2-Henselization and the 3-Henselization of a pair (A,m) exist. Several properties of k H-pairs are noted. And an equivalence to the Chain Conjecture is also given.

2. k H-PAIRS, k N-PAIRS, AND k N'-PAIRS.

In this section we define and give some facts about k H-pairs, k N-pairs, and k N'-pairs. The main result, Theorem (2.10) states that (i) a k H-pair is a k N-pair, (ii) a k N-pair is a k N'-pair, and (iii) an k N'-pair is a j H-pair provided $k \geq \max \{C_{j,n} \mid n = 0,1,\dots,j\}$.

We begin be stating several definitions. In these definitions and throughout the paper a ring shall mean a commutative ring with an identity element, and J(A) denotes the Jacobson radical of the ring A.

DEFINITION 2.1. (A,m) is a <u>pair</u> in case A is a ring and m is an ideal in A.

DEFINITION 2.2. (A,m) is a <u>k H-pair</u> in case

(i) $m \subseteq J(A)$; and

(ii) for every monic polynomial f(X) of degree k in A[X] such that $f(X) \in A/m\,[X]$ factors into $\bar{f}(X) = \bar{g}_o(X)\,\bar{h}_o(X)$ where $\bar{g}_o(X)$ and $\bar{h}_o(X)$ are monic and coprime, there exist monic polynomials $g(X), h(X) \in A[X]$ such that $f(X) = g(X)h(X)$, $\bar{g}(X) = \bar{g}_o(X)$ and $\bar{h}(X) = \bar{h}_o(X)$.

DEFINITION 2.3. Let (A,m) be a pair. A monic polynomial $X^k + a_{k-1}X^{k-1} + \dots + a_1X + a_o$ of degree k is called a k N-polynomial over (A,m) in case $a_o \in m$ and a_1 is a unit mod m.

DEFINITION 2.4. (A,m) is a k N-pair in case

(i) $m \subseteq J(A)$; and

(ii) every k N-polynomial over (A,m) has a root in m.

The next results give some facts about k N-polynomials and k N-pairs.

LEMMA 2.5. Let f(X) be a k N-polynomial over the pair (A,m). If $m \subseteq J(A)$, then f(X) has at most one root in m.

PROOF. The proof follows from [5, Lemma 1.5], since a k N-polynomial is an N-polynomial.

REMARK. Every k N-polynomial over a k N-pair (A,m) has one and only one root in m.

PROPOSITION 2.6. If (A,m) is a k N-pair, then (A,m) is an j N-pair for 2 $2 \leq j \leq k$.

PROOF. Given a k N-pair (A,m), it suffices to show that (A,m) is a (k-1) N-pair. Let f(X) be a (k-1) N-polynomial over (A,m). Let u be a unit in A and

$g(X) = (X + u)f(X)$. Then $g(X)$ is a k N-polynomial and thus has a root r in m and $0 = g(r) = (r + u)f(r)$. Since $(r + u)$ is a unit, we have $f(r) = 0$. Therefore, (A,m) is a (k - 1) N-pair.

DEFINITION 2.7. Let (A,m) be a pair. A monic polynomial $X^k + d_1 X^{k-1} + d_2 X^{k-2} + \dots + d_k$ of degree k is called a k N'-polynomial over (A,m) in case d_1 is a unit mod m and $d_2, \dots, d_k$ belong to m.

DEFINITION 2.8. (A,m) is a k N'-pair in case

(i) $m \subseteq J(A)$; and

(ii) every k N'-polynomial over (A,m) has a root in A, which is a unit.

We note that if (A,m) is a k N'-pair, $f(X) = X^k + d_1X^{k-1} + \dots + d_k$ is a k N'-polynomial over (A,m) and $r \in A$ is a root of f(X) given by the definition of a k N'-pair, then $\bar{r} = -\bar{d}_1$, and $f'(r)$ is a unit.

PROPOSITION 2.9. Let (A,m) be a k N'-pair, then (A,m) is an j N'-pair for $2 \leq j \leq k$.

PROOF. Given a k N'-pair (A,m), it suffices to show that (A,m) is a (k-1) N'-pair. Let f(X) be a (k-1)N'-polynomial over (A,m). Then Xf(X) is a k N'-polynomial and has a root u, which is a unit. and $uf(u) = 0$ implies that $f(u) = 0$, therefore (A,m) is a (k-1)N'-pair.

THEOREM 2.10. (i) A kH-pair is a kN-pair

(ii) A kN-pair is a kN'-pair

(iii) A kN'-pair is a jH-pair, provided

$$k \geq \max \{C_{j,n} \mid n = 0, 1, \dots, j\}$$

PROOF. Part (i) follows from the definitions.

The proof of (ii) follows from the proof of [10, Lemma 7]

The proof of (iii) follows from Crépeaux's proof of [3, Prop. 1]

3. k N-CLOSURE.

In this section we construct the k N-closure for a given pair (A,m). That is, we find the "smallest" k N-pair which "contains" (A,m). The development of this section parallels Greco's development in [5].

In order to construct the k N-closure we need the following definitions.

DEFINITION 3.1. A morphism (of pairs) $\emptyset:(A,m) \to (B,n)$ is a ring homomorphism $\emptyset:A \to B$, such that $\emptyset^{-1}(n) = m$.

DEFINITION 3.2. A morphism (of pairs) $\emptyset:(A,m) \to (B,n)$ is strict in case $n = \emptyset(m)B$ and $\emptyset$ induces an isomorphism $A/m \to B/n$.

DEFINITION 3.3. Let (A,m) be a pair. A k N-pair (B,n) together with a morphism $\emptyset:(A,m) \to (B,n)$ is a k N-closure of (A,m) if for any k N-pair (B',n') and any morphism $\Psi:(A,m) \to (B',n')$ there exists a unique morphism $\Psi':(B,n) \to (B',n')$ such that $\Psi' \circ \emptyset = \Psi$.

DEFINITION 3.4. Let (A,m) be a pair and f(X) a k N-polynomial over (A,m). Let $A[x] = A[X]/(f(X))$, $S = 1 + (m,x)A[x]$ and $B = S^{-1}A[x]$. Then (B,mB) is called a simple k N-extension of (A,m).

DEFINITION 3.5. A k N-extension of (A,m) is a pair obtained from (A,m) by a finite number of simple k N-extensions.

The next two results give some useful properties of simple k N-extensions and k N-extensions.

LEMMA 3.6. Let (B,n) be a simple k N-extension of (A,m). Let $\emptyset:A \to B$ be the canonical morphism. Then:

(i) $x \in n$.

(ii) $\emptyset^{-1}(n) = m$ and $\emptyset:(A,m) \to (B,n)$ is a morphism of pairs.

(iii) $\emptyset:(A,m) \to (B,n)$ is strict.

PROOF. The proof follows from [5, Lemmas 2.3,2.4, and 2.5] since a simple k N-extension is a simple N-extension.

COROLLARY 3.7. If (B,n) is a k N-extension of (A,m), then the canonical morphism $\emptyset:(A,m) \to (B,n)$ is strict.

We note that a k N-extension of a quasi-local ring (A,m) is a quasi-local ring.

The following lemma is used to show that the partial order defined in Definition (3.9) is well defined.

LEMMA 3.8. Let (A',m') be a k N-extension of (A,m) and let (B,n) be a pair with $n \subseteq J(B)$. Let $\emptyset:(A,m) \to (A',m')$ be the canonical morphism. Then for any morphism $\Psi:(A,m) \to (B,n)$ there is at most one morphism $\Psi':(A',m') \to (B,n)$ such that $\Psi' \circ \emptyset = \Psi$.

PROOF. The proof follows from [5, Lemma 3.1] since a k N-extension is an N-extension.

In particular, the above lemma holds when (B,n) is a k N-extension of (A,m).

DEFINITION 3.9. Define a partial order on the set of k N-extensions of (A,m) as follows: If (A',m') and (A",m") are two k N-extensions of (A,m), then $(A',m') \leq (A'',m'')$ if and only if there is a morphism $\Psi:(A',m') \to (A'',m'')$ such that $\Psi \circ \emptyset = \emptyset''$, where $\emptyset:(A,m) \to (A',m')$ and $\emptyset'':(A,m) \to (A'',m'')$ are the canonical morphisms.

PROPOSITION 3.10. Let (A,m) be a pair. Then the k N-extensions of (A,m) form a directed set with the order relation and the morphisms defined above.

PROOF. The proof is analogous to [5, Prop. 3.3].

LEMMA 3.11 Let (A',m') be a k N-extension of (A,m) and let $\emptyset:(A,m) \to (A',m')$ be the canonical morphism. Let (B,n) be a k N-pair and let $\Psi:(A,m)\to(B,n)$ be a morphism. Then there is a unique morphism $\Psi':(A',m')\to(B,n)$ such that $\Psi = \Psi' \circ \emptyset$.

PROOF. The proof is analogous to [5, Prop. 3.4].

THEOREM 3.12. Let (A,m) be a pair and let (A^{kN},m^{kN}) be the direct limit of the set of all k N-extensions. Then (A^{kN},m^{kN}) with the canonical morphism $(A,m) \to (A^{kN},m^{kN})$ is a k N-closure of (A,m).

PROOF. The proof is analogous to [5, Thm. 3.5].

We note that if (A,m) is a quasi-local ring; then a k N-closure (A^{kN},m^{kN}) of (A,m) is quasi-local, since the direct limit of quasi-local rings is quasi-local.

4. k H-CLOSURES AND AN EQUIVALENCE TO THE CHAIN CONJECTURE.

In this section, we note the existence of a 2H-closure and of a 3H-closure, we give some characterization of a quasi-local 2H-pair, and we observe that the H-closure (or Henselization) of a pair (A,m) can be written as the direct limit or union of k H-pairs, k = 2,3,4,... . We also give an equivalence to the Chain Conjecture.

DEFINITION 4.1. Let (A,m) be a pair. A k H-pair (B,n), together with a morphism $\emptyset:(A,m)\to(B,n)$ is a <u>k H-closure</u> of (A,m) if for any k H-pair (B',n') and any morphism $\Psi:(A,m)\to(B'n')$, there exists a unique morphism $\Psi':(B,n)\to(B',n')$ such that $\Psi' \circ \emptyset = \Psi$.

THEOREM 4.2. Let (A,m) be a pair. Then:

(i) a 2 H-closure of (A,m) is (A^{2N}, m^{2N}).

(ii) a 3 H-closure of (A,m) is (A^{3N}, m^{3N}).

PROOF. It suffices to show that a k N-closure (k = 2,3) is a k H-pair. And by Theorem 2.10, we have that a 2N-pair is a 2H-pair, and that a 3N-pair is a 3H-pair.

DEFINITION 4.3. If $\emptyset:A\to B$ is a ring homomorphism, then B is said to be <u>k-integral</u> over A in case each $b \in B$ satisfies a monic polynomial of degree k over $\emptyset(A)$.

REMARK. If B is k-integral over A, then B is also j-integral over A for all $j \geq k$.

In the next three items we give examples of rings and elements which are k-integral over a given ring A.

LEMMA 4.4. If A is an integrally closed domain and $f(X) \in A[X]$ is a monic polynomial of degree k, then $A[X]/(f(X))$ is k-integral over A.

PROOF. Let $A[x] = A[X]/(f(X))$ and let L be the quotient field of A. Then $[L(x):L] \leq k$ and thus each $\alpha \in A[x]$ satisfies a monic polynomial $g(X) \in L[X]$ of degree $\leq k$. Since α is integral over A and A is integrally closed, it follows that $g(X) \in A[X]$. Therefore A[x] is k-integral over A.

LEMMA 4.5. Let A be a ring and let $f(X) = X^2 + \alpha X + \beta \in A[X]$. Then $A[X]/(f(X))$ is 2-integral over A.

PROOF. Let $A[x] = A[X]/(f(X))$ and then all of the elements of A[X] are of the form $ax + b$ where $a,b \in A$. To show that A[x] is 2-integral over A, we need to find $F, G \in A$ such that

$$(ax + b)^2 + F(ax + b) + G = 0.$$

By expanding the left side, we see that $F = a\alpha - 2b$ and $G = a^2\beta - b^2 - Fb = a^2\beta + b^2 - ab\alpha$ are the needed values. Therefore A[X] is 2-integral over A.

EXAMPLE 4.6. Each element of $\mathrm{End}_A(A^k)$ is k-integral over A by [1, Proposition 2.4].

In fact, if M is any A-module generated by k elements, each element of $End_A(M)$ is k-integral over A.

DEFINITION 4.7. (A,m) is a <u>(≤k)H-pair</u> in case (A,m) is a j H-pair for $2 \le j \le k$.

It follows by Theorem 2.10 that if (A,m) is a j N-pair (or j H-pair), then (A,m) is a (≤k)H-pair provided $j \ge \max \{C_{k,n} \mid n = 0, 1, \ldots k\}$. In particular we have that for k = 2,3, or 4, a k H-pair is also a (≤k)H-pair.

LEMMA 4.8. Let (A,m) be a quasi-local domain which is a (≤k)H-pair. Then every k-integral extension domain of A is quasi-local.

PROOF. The proof is analogus to [6, (30.5)]

DEFINITION 4.9. A ring A is <u>decomposed</u> if A is the product of finitely many quasi local rings.

THEOREM 4.10. Let (A,m) be a quasi local ring. Then the following statements are equivalent.

(i) Every finite 2-integral A-algebra B is decomposed.

(ii) Every finite free 2-integral A-algebra B is decomposed.

(iii) Every A-algebra of the form A[X]/(f(X)), where $f(X) \in A[X]$ is monic and of degree 2, is decomposed.

(iv) (A,m) is a 2 H-pair.

PROOF. (i)⇒(ii) is clear. (ii)⇒(iii) is clear by (4.5). The proofs that (iii)⇒(i) and that (iii)⇔(iv) follow classical lines; for example, see [9, Prop. 5, p.2].

THEOREM 4.11. A quasi local domain (A,m) is a 2H-pair if and only if every 2-integral extension domain A' of A is quasi-local.

PROOF. (⇒) is true by (4.8).

(⇐). We will show that (A,m) is a 2H-pair by showing that every finite free 2-integral A-algebra is decomposed. Let B be a finite free 2-integral A-algebra. Since B is decomposed if and only if B/nil rad B is decomposed, we may assume that B is reduced. Since B is flat over A, regular elements of A are also regular in B. Thus the minimal primes of B contract to {0} in A. Let $\{P_i\}_{i\in I}$ be the minimal primes of B. Then for each $i\in I$, B/P_i is a 2-integral extension domain of A and is quasi local by the hypothesis. Thus each minimal prime P_i is contained in a unique maximal ideal. By [2, Proposition 3, p. 329], the set of minimal primes of B is finite. Let $I_j = \cap_{P_i \subseteq M_j} P_i$ where M_j, j=1,..., n, are the maximal ideals of B. Then the I_j are coprime, and $\cap_{j=1}^{n} I_j = 0$ since B is reduced. So by the Chinese Remainder Theorem $B \cong \pi_{j=1}^{n} B/I_j$ and each B/I_j is quasi local. Thus B is decomposed and therefore (A,m) is a 2H-pair.

COROLLARY 4.12. Let (A,m) be a quasi local domain which is 2H-pair. Let A' be an integral extension domain of A. If $b \in A'$ is 2-integral over A, then $b \in J(A')$ or b is a unit.

PROOF. A[b] is a 2-integral extension domain of A and is thus quasi local. The result follows since all the maximal ideals of A' contract to the unique maximal ideal of A[b].

We will now show that the N-closure of a pair (A,m) is the direct limit of the k N-closures of (A,m). It will follow from this result that the H-closure of (A,m) can be written as the direct limit of k H-pairs.

DEFINITION 4.13. Let (A,m) be a pair. Then (A,m) is an N-pair (respectively, a H-pair) in case (A,m) is a k N-pair (respectively, a k H-pair) for k = 2,3,... .

DEFINITION 4.14. Let (A,m) be a pair. An N-pair (respectively, an H-pair) (B,n), together with a morphism $\emptyset:(A,m)\to(B,n)$ is an N-closure (respectively, an H-closure) of (A,m) if for any N-pair (respectively, any H-pair) (B',n'), and any morphism Ψ: $(A,m)\to(B',n')$, there exists a unique morphism Ψ': $(B,n)\to(B',n')$ such that $\Psi' \circ \emptyset = \Psi$.

THEOREM 4.15. Let (A,m) be a pair. Then the H-closure of (A,m) is isomorphic to the N-closure.

PROOF. See [5, Lemma 1.4 and Theorem 5.10].

PROPOSITION 4.16. Let (A^N,m^N) be an N-closure of (A,m). Then $(A^N,m^N) \cong \text{dir lim } (A^{kN},m^{kN})$, where the directed system $\{(A^{kN},m^{kN}),\mu_{kj}\}$ of k N-closures of (A,m), k=2,3,4,..., is ordered by $(A^{kN},m^{kN}) \leq (A^{jN},m^{jN})$ iff $k \leq j$ and if $k \leq j$, then μ_{kj}: $(A^{kN},m^{kN})\to(A^{jN},m^{jN})$ is the unique morphism which makes the following diagram commute:

$$\begin{array}{ccc} (A,m) & \xrightarrow{\ \emptyset_k\ } & (A^{kN},m^{kN}) \\ & \underset{\emptyset_j}{\searrow} & \downarrow \mu_{kj} \\ & & (A^{jN},m^{jN}) \end{array}$$

where $\emptyset_j$ and $\emptyset_k$ are the canonical morphisms.

PROOF. The proof follows immediately from Definitions (3.3) and (4.14) and the definition of a direct limit.

COROLLARY 4.17. Let (A^H,m^H) be the H-closure of (A,m). Then $(A^H,m^H) \cong$ dir lim (A_i,M_i) where (A_i,m_i) is an i <u>H-pair</u> for i = 2,3,... .

PROOF. For a given i, let $(A_i,m_i) = (A^{kN},m^{kN})$ where $k = \max' \{C_{j,n} | n=0,1,\ldots,j\}$. Then the corollary follows by results (2.10), (4.15) and (4.16).

We now give an equivalence to the Chain Conjecture. The terminology used is the same as in [8] or [10].

THEOREM 4.18. The following statements are equivalent:

(i) The Chain Conjecture holds.

(ii) Every 2 Henselian local domain A, such that the integral closure of A is quasi-local, is catenary.

PROOF. (i) $\Rightarrow$ (ii). This follows by [8, Thm. 2.4].

(ii) $\Rightarrow$ (i). By [8, Thm. 2.4] it suffices to show that every Henselian local domain is catenary. Let A be a Henselian local domain. Then A is also 2 Henselian and the integral closure of A is quasi-local by [6, (43.12)]. Thus by the hypothesis A is catenary.

5. <u>EXAMPLES</u>.

In this section we show that there exist k N-pairs which are not N-pairs and there exist k H-pairs which are not H-pairs. More precisely, for each prime number p we give an example of a pair which is not a p N-pair but is a k N-pair for $2 \le k < p$. This example also shows that for any integer $k \ge 2$, there exists a k H-pair which is not a p H-pair for some sufficiently large prime number p.

Let $p > 2$ be a prime number. Let (R,q) be a normal quasi-local domain such that there exists an $f(X) = X^p + \ldots + a_1X + a_o \in R[X]$, where $a_1 \notin q$, $a_o \in q$ and f(X) is irreducible over R[X].

In particular, let $R = Z_{(2)}$ and let $f(X) = X^p + 3X + 6$. Then by Eisenstein's Criterior, f(X) is irreducible in Q[X], and thus irreducible in $Z_{(2)}[X]$ since f(X) has content 1.

Let K be the quotient field of R and let $\overline{K}$ be an algebraic closure of K. Let R' be the integral closure of R in $\overline{K}$ and P' any maximal ideal in R'. Now f(X) as an element of R'[X] factors completely, and since $P' \cap R = q$, f(X) has a unique root $\alpha \in P'$. Let L be the least normal extension of K containing α. Then $p \mid [L:K]$ and by [7, Thm. 6] there is a maximal field M without α of exponent p with $K \subseteq M \subset \overline{K}$. Let $A = R' \cap M$ and let $m = P' \cap A$.

Now (A,m) is not a p N-pair since f(X) is a p N-polynomial over (A,m) which does not have a root in m. But (A,m) is a k N-pair for $2 \leq k < p$. For, let g(X) be a (p - 1)N-polynomial over (A,m). Then g(X) as an element of R'[X] has a unique root $\beta \in P'$. Now $[M(\beta):M] \leq p - 1$, but by [7, Thm. 2], $[M(\beta):M] = p^i$ for some $i \geq 0$. So $[M(\beta):M] = 1$ and $\beta \in M$. Thus $\beta \in m = P' \cap A$ and (A,m) is a (p - 1)N-pair. It follows by (2.6) that (A,m) is a k N-pair for $2 \leq k < p$.

REMARK. If j and the prime number p are closen such that $p > \max \{C_{j,n} \mid n=0,1,\ldots,j\}$, then by Theorem 2.10, the above example is an example of a pair (A,m) such that (A,m) is not a p H-pair, but (A,m) is a k H-pair for $2 \leq k \leq j$.

Let the notation be as in the above example. Then (A_m, mA_m) is as an example of a normal quasi-local domain which is not a p N-pair, but is a k N-pair for $2 \leq k < p$.

6. PROPERTIES OF k N-PAIRS.

We conclude this paper by noting that many of the properties of the Hensilization or N-closure of a pair which S. Greco proved in [5] also hold for a k N-closure and thus also for a 2 H-closure and a 3 H-closure. Some of these results are: direct limits commute with k N-closures, cf. [5, Cor. 3.6]; a k N-closure of (A,m) is flat over A and is faithfully flat over A iff $m \subseteq J(A)$, cf. [5, Thm. 6.5]; a k N-closure of a noetherian ring is noetherian, and if a k N-closure of (A,m) is Noetherian and $m \subseteq J(A)$, then A is Noetherian, cf. [5, Cor. 6.9]; if A is Noetherian and A has one of the properties R_k, S_k, regular, or Cohen-Macaulay, then a k N-closure of (A,m) also has that property, and the converse is also true provided $m \subseteq J(A)$, cf. [5, Cor. 7.7]; a k N-closure preserves locally normal, cf. [5, Thm. 9.7]; and a k N-closure of a reduced ring is reduced, cf. [5, Thm. 8.7].

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