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RELATIVISTIC SIGNIFICANCE OF CURVATURE TENSORS

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ABSTRACT. In this paper new curvature tensors have been defined on the lines of Weyl's projective curvature tensor and it has been shown that the "distribution" (order in which the vectors in question are arranged before being acted upon by the tensor in question) of vector field over the metric potentials and matter tensors plays an important role in shaping the various physical and geometrical properties of a tensor viz the formulation of gravitational waves, reduction of electromagnetic field to a purely electric field, vanishing of the contracted tensor in an Einstein Space and the cyclic property.

KEY WORDS AND PHRASES. *Gravitational waves, Electromagnetic field, Einstein space, Cyclic property, Weyl's projective curvature tensor.*

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1. Introduction.

The relativistic significance of Weyl's projective curvature tensor has been explored by Singh, Radhakrishna and Sharan [1] and many other authors. In previous papers (Pokhariyal [4], Pokhariyal and Mishra [2,3], we have defined some curvature tensors and obtained their physical and geometrical properties. In this paper we have considered all other tensors that can be defined with the help of

Weyl's projective curvature tensor and shown how the distribution of vector fields over the metric potentials and matter tensors acts in shaping the properties of all such tensors. In section two, we define tensors W_5 and W_6. The tensor W_5 which is symmetric with the change of pair of the vector fields is broken into symmetric and skew-symmetric parts in two ways and various relationships are obtained. Further it is shown that the vanishing of the divergence of W_6 in an electromagnetic field implies a purely electric field. In the third part of this paper six more tensors are defined with the help of Weyl's projective curvature tensor. It is shown in the fourth part that distribution of the vector field X over all the metric potentials of a tensor is responsible for the vanishing of the gradient Θ_i of the Completion Θ defined by Misner and Wheeler [5]. In the fifth part the cyclic properties of the tensors are discussed while in the sixth part we discuss the vanishing of the contracted tensors in an Einstein space. The symmetric properties and formulation of the gravitational waves are explained with the help of these tensors in the seventh part. Lastly it is shown that Rainich conditions can be written in terms of the contracted parts of these tensors.

2. TWO TENSORS W_5 AND W_6 AND THEIR PROPERTIES.

Definition. (2.1): We define the tensors W_5 (X,Y,Z,T)

$$W_5(X,Y,Z,T) \underset{=}{\text{def}} R(X,Y,Z,T) + \frac{1}{n-1}\Big[g(X,Z)\ \mathrm{Ric}(Y,T) - g(Y,T)\ \mathrm{Ric}(X,Z)\Big] \tag{2.1}$$

and

$$W_6(X,Y,Z,T) \underset{=}{\text{def}} R(X,Y,Z,T) + \frac{1}{n-1}\Big[g(X,Y)\ \mathrm{Ric}(Z,T) - g(X,T)\ \mathrm{Ric}(Y,Z)\Big] \tag{2.2}$$

From (2.1), we notice that W_5 is symmetric with the change of pairs of the vector fields and does not satisfy the cyclic property.

We see from(2.2) that W_6 does not posses any symmetry but satisfies the cyclic property,

$$W_6(X,Y,Z,T) + W_6(X,Z,T,Y) + W_6(X,T,Y,Z) = 0 \tag{2.3}$$

We break W_5 into two parts

$$\mu(X,Y,Z,T) = \tfrac{1}{2}\Big[W_5(X,Y,Z,T) - W_5(Y,X,Z,T)\Big]$$

and

$$\nu(X,Y,Z,T) = \tfrac{1}{2}\left[W_5(X,Y,Z,T) + W_5(Y,X,Z,T)\right],$$

which are respectively skew-symmetric and symmetric in X,Y. From (2.1) it follows that

$$\mu(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{2(n-1)}\Big[g(X,Z)\ Ric(Y,T) - g(Y,T)\ Ric(X,Z) - g(Y,Z)\ Ric(X,T) + g(X,T)\ Ric(Y,Z)\Big] \tag{2.4}$$

and

$$\nu(X,Y,Z,T) = \frac{1}{2(n-1)}\Big[g(X,Z)\ Ric(Y,T) - g(Y,T)\ Ric(X,Z) + g(Y,Z)\ Ric(X,T) - g(X,T)\ Ric(Y,Z)\Big]. \tag{2.5}$$

Further the cyclic property,

$$\nu(X,Y,Z,T) + \nu(X,Z,T,Y) + \nu(X,T,Y,Z) = 0 \tag{2.6}$$

is satisfied. We now break W_5 into two other parts

$$\gamma(X,Y,Z,T) = \tfrac{1}{2}\left[W_5(X,Y,Z,T) - W_5(X,Y,T,Z)\right]$$

and

$$\delta(X,Y,Z,T) = \tfrac{1}{2}\left[W_5(X,Y,Z,T) + W_5(X,Y,T,Z)\right]$$

which are respectively skew-symmetric and symmetric in Z,T. From(2.1), we get

$$\gamma(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{2(n-1)}\Big[g(X,Z)\ Ric(Y,T) - g(Y,T)\ Ric(X,Z) - g(X,T)\ Ric(Y,Z) + g(Y,Z)\ Ric(X,T)\Big] \tag{2.7}$$

and

$$\delta(X,Y,Z,T) = \frac{1}{2(n-1)}\Big[Ric(Y,T)\ g(X,Z) - g(Y,T)\ Ric(X,Z) + {} + g(X,T)\ Ric(Y,Z) - g(Y,Z)\ Ric(X,T)\Big] \tag{2.8}$$

from (2.7), we notice that the cyclic property,

$$\gamma(X,Y,Z,T) + \gamma(X,Z,T,Y) + \gamma(X,T,Y,Z) = 0 \tag{2.9}$$

is satisfied. From (2.4) and (2.8), we get

$$\mu(X,Y,Z,T) = R(X,Y,Z,T) + \delta(X,Y,Z,T) \tag{2.10}$$

Similarly from (2.5) and (2.7), we have

$$\gamma(X,Y,Z,T) = R(X,Y,Z,T) + \nu(X,Y,Z,T) \tag{2.11}$$

Equations (2.10) and (2.11) reduce to the following equation

$$R(X,Y,T,Z) + R(Y,X,Z,T) = W_5(X,Y,T,Z) + W_5(Y,X,Z,T) \tag{2.12}$$

The vector

$$\Theta_i = \frac{g_{ij}\ \varepsilon^{jklm}\ R^P_k\ R_{pl;m}}{\sqrt{-g}\ R_{ab}R^{ab}} \tag{2.13}$$

is called the gradient of the completion Θ, of a non-null electromagnetic field with no matter by Misner and Wheeler and its vanishing implies that field is purely electrical. A semi-colon stands for covariant differentiation.

Interchanging the dummy indices l,m (2.13) can be written as

$$= \frac{g_{ij}\ \varepsilon^{jkml}\ R^P_k\ R_{pm;l}}{\sqrt{-g}\ R_{ab}\ R^{ab}} \tag{2.14}$$

$$= -\frac{g_{ij}\ \varepsilon^{jklm}\ R^P_k\ R_{pm;l}}{\sqrt{-g}\ R_{ab}\ R^{ab}}$$

By setting $W^h_{6\ pml;m} = 0$; we get

$$R_{pm;l} = R_{pl;m} \tag{2.15}$$

which on substitution in (2.14) implies that $\Theta_i = 0$.

Thus the vanishing to the divergence of W_6 in an electromagnetic field implies a purely electric field.

OTHER TENSORS

In this section we shall define all other tensors that can be defined with the help of the Wely's projective tensor mention other tensors defined in earlier papers.

Definiti · (3.1). We define the tensors

$$W_7(X,Y,Z,T) \ \underline{\underline{def}}\ R(X,Y,Z,T) + \frac{1}{n-1}\left[g(Y,Z)\ Ric(X,T) - g(X,T)\ Ric(Y,Z)\right] \tag{3.1}$$

$$W_8(X,Y,Z,T) \ \underline{\underline{def}}\ R(X,Y,Z,T) + \frac{1}{n-1}\left[g(Z,T)\ Ric(X,Y) - g(X,T)\ Ric(Y,Z)\right] \tag{3.2}$$

$$W_9(X,Y,Z,T) \ \underline{\underline{def}}\ R(X,Y,Z,T) + \frac{1}{n-1}\left[g(Z,T)\ Ric(X,Y) - g(Y,Z)\ Ric(X,T)\right] \tag{3.3}$$

Three more tensors W_{10},W_{11} and W_{12} can be defined in a similar manner to complete the set of such tensors.

The other tensors defined in previous papers (Pokhariyal and Mishra [2-3] are given by

$$W_1(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}\left[g(X,T)\ Ric(Y,Z) - g(Y,T)\ Ric(X,Z)\right] \quad (3.4)$$

$$W_2(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}\left[g(X,Z)\ Ric(Y,T) - g(Y,Z)\ Ric(X,T)\right] \quad (3.5)$$

$$W_3(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}\left[g(Y,Z)\ Ric(X,T) - g(Y,T)\ Ric(X,Z)\right] \quad (3.6)$$

and

$$W_4(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}\left[g(X,Z)\ Ric(Y,T) - g(X,Y)\ Ric(Z,T)\right]. \quad (3.7)$$

The Weyl's projective curvature tensor is given by (Eisenhart [6])

$$W(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}\left[g(X,Z)\ Ric(Y,T) - g(X,T)\ Ric(Y,Z)\right]. \quad (3.8)$$

We now look into the distribution of the vector fields in all these tensors and show that it shapes these tensors to yield different physical and geometrical properties.

4. Reduction of Electromagnetic field to a purely electric field

The vanishing of the gradient θ_i of the completion θ of a non-null electromagnetic field with no matter, defined by equation (2.13), implies that the field is purely electrical. From the set of these tensors, we see that the vanishing of the divergence, of either of the tensors W, W_4 and W_6 implies $\theta_i = 0$. Looking into the structure of these tensors from (3.8), (3.7) and (2.2), we notice that the vector field X is distributed in such a fashion that it is always present in the metric potentials. Thus we have a theorem.

Theorem (4.1): The vanishing of the divergence of all those tensors in which the vector field X is present in all the metric potentials, reduce the electromagnetic field to a purely electric field.

5. Cyclic Property

We study the cyclic properties of the tensors by fixing the vector field present in the first place. If we look into the structure of the tensors W, W_3,W_4, W_6 and W_9, we find that the vector field X is distributed in such a way that it is present either in both the metric potential terms or in both the matter tensor terms. Thus these tensors satisfy the cyclic properties with fixed X. Hence we have the following theorem:

Theorem (5.1): All the tensors, in which the vector field X is distributed in such a fashion that it is present either in both the metric potentials or in both the matter tensors alone,satisfy the cyclic properties with fixed X.

6. Vanishing in Einstein Space

The space for which

$$R(X,Y) = \frac{R}{n} g(X,Y) \tag{6.1}$$

holds is called an Einstein space (Eisenhart [6]). If the contracted tensor after subsituting (6.1) vanishes , we say that it vanishes in an Einstein space.

Looking into the tensors W, W_2, W_6, W_7, and W_8 we find that the combination of the vector fields X and T or the combination of vector fields Y and Z is present in the metric potential terms with a negative sign. Thus the contracted part of these tensors, with vector fields X and T or Y and Z respectively, vanish in the Einstein space.

7. Symmetry and Formulation of Gravitational Waves

Inspecting the symmetric properties of the tensors we notice that W and W_3 are skew-symmetric in Z,T while W_1 and W_2 are skew-symmetric in X,Y, and W_5 is symmetric with change pairs of vector fields.

We break these skew-symmetric tensors into symmetric and skew-symmetric parts and notice that the skew-symmetric parts of those tensors, in which the combinations of the vector fields X,T and Y,Z are distributed in a term among the metric potentials and matter tensors in such a way that a negative sign is always present with this term, on contraction vanish identically in an Einstein space. All such tensors enable us to extend the Pirani formulation of the gravitational waves to the Einstein space with the help of the defined skew-symmetric parts.

It is seen that the skew-symmetic parts defined with the help of other symmetric or skew-symmetric tensors, do not vanish identically in an Einstein space. Hence the Pirani formulation of gravitational waves cannot be extended to the Einstein space with the help of these tensors. Thus we have the following theorem:

Theorem (7.1): The skew-symmetric parts defined by those skew-symmetric tensors, in which the combinations of vector fields X, T and Y,Z, are distributed in a term among the metric potentials and matter tensors with a negative sign, enable to extend the Pirani formulation of gravitational waves to the Einstein space. The Rainich conditions [7] for the existence of the non-null electrovariance, can be obtained by the defined tensors, if we replace the matter tensor by tensors obtained after contraction of the defined tensors in an electromagnetic field.

Discussion: We conclude from the above results that the physical and geometrical properties of a tensor are mainly determined by the distribution of vector fields over the metric potentials and the matter tensors. Thus the various properties of Weyl's projective curvature tensor are due to a particular type of distribution of vector fields contained in it and not due to its invariance in two spaces V_n and $\bar{V}_n$. In view of the above results the properties of the tensors studied by Singh, Radhakrishna and Sharan [1], Pokhariyal and Mishra [2, 3] and by other authors can be obtained by looking into the structure of these tensors.

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