Internat. J. Math. & Math. Sci.
Vol. 5 No. 1 (1982) 105-112

STABILITY IMPLICATIONS ON THE ASYMPTOTIC BEHAVIOR OF NONLINEAR SYSTEMS

KUO-LIANG CHIOU

Department of Mathematics
Wayne State University
Detroit, Michigan 48202

(Received July 20, 1978)

ABSTRACT. In this paper we generalize Bownds' Theorems (1) to the systems $\frac{dY(t)}{dt} = A(t)\,Y(t)$ and $\frac{dX(t)}{dt} = A(t)\,X(t) + F(t,X(t))$. Moreover, we also show that there always exists a solution X(t) of $\frac{dX}{dt} = A(t)X + B(t)$ for which $\limsup_{t \to \infty} ||X(t)|| > o\ (= \infty)$ if there exists a solution Y(t) for which $\limsup_{t \to \infty} ||Y(t)|| > o\ (= \infty)$.

KEY WORDS AND PHRASES. *stable, norm, linear systems, null solution, Schauder-Tychonoff Theorem, uniformly converges, equicontinuous.*

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. *Primary 34D05, 34D10, 34C10, Secondary 34A30, 34C15, 34D20.*

1. INTRODUCTION.

In this paper we shall study the stability behavior of the following systems

$$\frac{dY(t)}{dt} = A(t)Y(t)\ , \quad 0 \le t < \infty \tag{1.1}$$

and

$$\frac{dX(t)}{dt} = A(t)X(t) + F(t,X(t)), \quad 0 \le t < \infty \tag{1.2}$$

where A(t) is a continuous matrix on R^n for all $0 \le t < \infty$, F(t,X(t)) is a real valued continuous n-vector defined on $[0,\infty) \times R^n$ and X(t) and Y(t) are n-vectors.

Consider special equations of (1.1) and (1.2)

$$y'' + a(t)y = 0, \quad 0 \le t < \infty \tag{1.3}$$

and

$$x'' + a(t)x = g(t,x,x'), \quad 0 \le t < \infty \tag{1.4}$$

where $a(t) \in C[0,\infty)$ and $g(t,x,x')$ is continuous on $[0,\infty)\times R \times R$. From some theorems of stability theory, Bownds [1] showed that (1.3) has a solution $y(t)$ with property

$$\limsup_{t \to \infty} (|y(t)| + |y'(t)|) > 0 \qquad (1.5)$$

He also established that (1.4) has the property (1.5) provided that the zero solution of (1.3) is stable and there exists a function $\gamma(t) \in L[0,\infty)$ such that

$$|g(t,x,x')| \le \gamma(t) \quad (|x| + |x'|)$$

for $(t,x,x') \in [0,\infty) \times R \times R$.

Thus in the following section we shall extend the above results to systems (1.1) and (1.2). In section 3 we shall consider a nonhomogeneous system

$$\frac{dX(t)}{dt} = A(t)X(t) + B(t), \quad 0 \le t < \infty \qquad (1.6)$$

where $B(t)$ is a continuous vector for $0 \le t < \infty$. We shall prove that there always exists a solution $X(t)$ of (1.6) for which $\limsup_{t \to \infty} ||X(t)|| > 0 (= \infty)$, if there exists a solution $Y(t)$ of (1.1) for which $\limsup_{t \to \infty} ||Y(t)|| > 0 (= \infty)$. Here $|| \cdot ||$ is an appropriate vector (or matrix) norm.

2. ASYMPTOTIC BEHAVIOR FOR (1.1) AND (1.2).

Before stating main theorems, let us recall a theorem from Coppel [2, p. 60].

THEOREM 2.1. (Hartman [2, p. 60]). Suppose that, for every solution $Y(t)$ of (1.1), the limit

$$\lim_{t \to \infty} ||Y(t)|| \qquad (2.1)$$

exists and is finite. If there exists a nontrivial solution $Y(t)$ of (1.1) for which the limit (2.1) is zero, then

$$\int_{t_o}^{t} t_r A(s)ds \to -\infty \qquad \text{as } t \to \infty .$$

From the above theorem we will obtain the following corollary which is a generalization of Theorem 1 in [1].

COROLLARY 2.1. Suppose that

$$\int_{t_o}^{\infty} t_r A(s)ds < \infty .$$

Then there exists a nontrivial solution $Y(t)$ of (1.1) for which

$$\limsup_{t \to \infty} ||Y(t)|| > 0.$$

PROOF. Suppose, to the contrary, that all solutions Y(t) of (1.1) satisfy

$$\limsup_{t \to \infty} ||Y(t)|| = 0.$$

From Theorem 2.1 we obtain $\int_{t_0}^{t} t_r A(s)ds \to -\infty$ as $t \to \infty$. This leads to a contradiction. The corollary then follows.

Throughout this paper we shall denote $\Phi(t)$, the fundamental matrix of (1.1) with initial condition $\Phi(0) = I$ (identity matrix).

Now we shall prove the following theorem via the Schauder-Tychonoff Theorem [2, p. 9].

THEOREM 2.2. Suppose that the null solution of (1.1) is stable and that there exists a solution Y(t) of (1.1) for which

$$\limsup_{t \to \infty} ||Y(t)|| > 0. \tag{2.2}$$

Suppose also that there exists $\gamma(t) \in L_1[t_o,\infty)$ such that for some positive constant ℓ

$$||F(t,x)|| \le \gamma(t)||x||^{\ell}\ . \tag{2.3}$$

Then there exists a nontrivial solution X(t) of (1.2) for which

$$\limsup_{t \to \infty} ||X(t)|| > 0.$$

PROOF. Since the null solution of (1.1) is stable, there exists a positive constant k such that

$$||\Phi(t)\ \Phi^{-1}(s)|| \le k \tag{2.4}$$

for all $0 \le t \le s$ and there exists a nontrivial solution Y(t) of (1.1) for which (2.2) holds and

$$||Y(t)|| \le 1 - \varepsilon \tag{2.5}$$

for $t \ge t_o$ and for given small positive constant ε (<1).

Since $\gamma(t) \in L_1[t_o,\infty)$, there exists $T_o (> t_o)$ such that

$$k \int_{t}^{\infty} \gamma(s)ds < \varepsilon \qquad \text{for all } t \ge T_o\ . \tag{2.6}$$

Via the Schauder-Tychonoff Theorem we shall establish the existance of a solution of the integral equation

$$X(t) = Y(t) - \Phi(t) \int_t^\infty \Phi^{-1}(s)\ f(s,X(s))ds, \quad t \geq T_o . \tag{2.7}$$

Consider the set

$F = \{U;\ U(t) = X(t)$ is continuous on $J_o = [T_o,\infty)$ and

$||U(t)|| \leq 1$ for $t \geq T_o\}$

and define the operator T by

$$TU(t) = Y(t) - \int_t^\infty \Phi(t)\ \Phi^{-1}(s)\ f(s,U(s))ds. \tag{2.8}$$

First, we shall show that $TF \subset F$. Taking the norm to both sides of (2.8) and using (2.3), (2.4), (2.5), and (2.6), we obtain for $U \in F$

$$\begin{aligned} ||TU(t)|| &\leq ||Y(t)|| + \int_t^\infty ||\Phi(t)\ \Phi^{-1}(s)\ f(s,U(s))||ds \\ &\leq 1 - \varepsilon + k \int_t^\infty ||f(s,U(s))||ds \\ &\leq 1 - \varepsilon + k \int_t^\infty \gamma(s)\ ||U(s)||^{\ell} ds \\ &\leq 1 - \varepsilon + k \int_t^\infty \gamma(s)ds \\ &\leq 1 - \varepsilon + \varepsilon = 1. \end{aligned}$$

It is clear that $TU(t)$ is continuous on J_o. This proves $TF \subset F$.

Second, we shall show that t is continuous. Suppose that the sequence $\{U_n\}$ in F converges uniformly to U in F on every compact subinterval of J_o. We claim that TU_n converges uniformly to TU on every compact subinterval of J_o. Let ϵ_1 be a small positive number satisfying $\epsilon_1 < 1$. Since $\gamma(t)$ $L_1[t_o,\infty)$, there exists $T_1 > T_o$ so that for $t \geq T_1$

$$k \int_t^\infty \gamma(s)ds < \frac{\epsilon_1}{4} . \tag{2.9}$$

By the uniform convergence, there is an $N = N(\epsilon_1, T_1)$ such that if $n \geq N$, then

$$||f(s,U_n(s)) - f(s,U(s))|| < \frac{\epsilon_1}{2kT_1} , \quad T_o \leq s \leq T_1 . \tag{2.10}$$

Then using (2.8), (2.9), (2.10), (2.3), (2.4), and the fact that $||U_n(t)|| \leq 1$ and $||U(t)|| \leq 1$ for $T_o \leq t < \infty$, we obtain the following inequalities

$$||TU_n(t) - TU(t)|| = || \int_t^\infty \Phi(t)\Phi^{-1}(s)f(s,U_n(s))ds - \int \Phi(t)\Phi^{-1}(s)f(s,U(s)ds||$$

$$\leq \int_t^{T_1} ||\Phi(t)\Phi^{-1}(s)||\ ||f(s,U_n(s)) - f(s,U(s))||ds$$

$$+ \int_{T_1}^\infty ||\Phi(t)\Phi^{-1}(s)||\ ||f(s,U_n(s))||ds + \int_{T_1}^\infty ||\Phi(t)\Phi^{-1}(s)||\ ||f(s,U(s))||ds$$

$$\leq k \int_t^{T_1} ||f(s,U_n(s)) - f(s,U(s))||ds + 2k \int_{T_1}^\infty \gamma(s)ds$$

$$< \frac{\epsilon_1}{2} + \frac{\epsilon_1}{2} = \epsilon_1 \qquad \text{for } n \geq N.$$

This shows that TU_n converges uniformly to TU on every compact subinterval of J_o. Hence T is continuous.

Third, we claim that the functions in the image set TF are equicontinuous and bounded at every point of J_o. Since $TF \subset F$, it is clear that the functions in TF are uniformly bonded. Now we show that they are equicontinuous at eact point of J_o. For each $U \in F$, the function $z(t) = TU(t)$ is a solution of the linear system

$$\frac{dV}{dt} = A(t)V + f(t,U(t))$$

Since $||z(t)|| = ||TU(t)|| \leq 1$ and $||f(t,U(t))||$ is uniformly bounded for $U \in F$ on any finite t interval, we see that $\frac{dz}{dt}$ is uniformly bounded on any finite interval. Therefore, the set of all such z is equicontinuous at each point of J_o (see [2, p.6]).

All of the hypotheses of the Schauder-Tychonoff Theorem are satisfied. Thus there exists a $U \in F$ such that $U(t) = TU(t)$; that is, there exists a solution X(t) of

$$X(t) = Y(t) - \Phi(t) \int_t^\infty \Phi^{-1}(s)\ f(s,x(s))ds$$

Thus, from the hypotheses and the above equality, we obtain .

$$\limsup_{t \to \infty} ||X(t) - Y(t)|| = 0 \tag{2.11}$$

Since $\limsup_{t \to \infty} ||Y(t)|| > 0$, (2.11) implies that $\limsup_{t \to \infty} ||X(t)|| > 0$. This proves the theorem.

It is clear that (1.4) can be written as the form (1.2) with

$$A(t) = \begin{pmatrix} 0 & 1 \\ -a(t) & 0 \end{pmatrix} \quad \text{and} \quad F(t,X) = \begin{pmatrix} 0 \\ g(t,x,x') \end{pmatrix}$$

where X = colum(x,x'). Thus we can apply Theorem 2.2 to (1.4) to obtain the following corollary which is a generalization of Theorem 2 in [1].

COROLLARY 2.2. Suppose that the null solution of (1.3) is stable and that there exists $\gamma(t) \in L_1[t_o,\infty)$ such that for some positive constant ℓ

$$||g(t,x,x')|| \le \gamma(t)\ (|x| + |x'|)^{\ell} .$$

Then there exists a nontrivial solution x(t) of (1.4) for which

$$\limsup_{t \to \infty} (|x| + |x'|) > 0 .$$

PROOF. Since $t_r A(t) = 0$ for Corollary 2.1, we know that there exists a solution Y(t) of (1.1) for which

$$\limsup_{t \to \infty} ||Y(t)|| > 0 .$$

If we take $||X|| = |x| + |x'|$, then the corollary follows from Theorem 2.2.

3. ASYMPTOTIC BEHAVIOR FOR (1.6).

In this section we shall show that if there exists a solution Y(t) of (1.1) for which $\limsup_{t \to \infty} ||Y(t)|| > 0$ (= ∞), then there exists a solution X(t) of (1.6) for which $\limsup_{t \to \infty} ||X(t)|| > 0$ (= ∞).

THEOREM 3.1. Suppose that there exists a solution Y(t) of (1.1) for which

$$\limsup_{t \to \infty} ||Y(t)|| > 0. \tag{3.1}$$

Then there exists a solution X(t) of (1.6) for which

$$\limsup_{t \to \infty} ||X(t)|| > 0.$$

PROOF. From the variation of constants formula we know that any solution X(t) of (1.6) can be written as the form below

$$X(t) = \Phi(t)c + \Phi(t) \int_0^t \Phi^{-1}(s)\ B(s)ds \tag{3.3}$$

Hence we shall choose c so that $Y(t) = \Phi(t)c$ satisfies (3.1).

First, let us suppose

$$\limsup_{t \to \infty} ||\ \Phi(t) \int_0^t \Phi^{-1}(s)\ B(s)ds|| > 0. \tag{3.4}$$

Let $X_1(t) = X(t) - Y(t)$. It is clear that $X_1(t)$ is a solution of (1.6). Thus from (3.3) and (3.4) we obtain

$$\limsup_{t \to \infty} ||X_1(t)|| = \limsup_{t \to \infty} ||X(t) - Y(t)||$$

$$= \limsup_{t \to \infty} ||\Phi(t) \int_0^t \Phi^{-1}(s)\ B(s)ds|| > 0\ .$$

Thus there exists a solution $X_1(t)$ of (1.6) for which (3.2) holds.

Second, suppose that

$$\lim_{t \to \infty} ||\ \Phi(t) \int_0^t \Phi^{-1}(s)\ B(s)ds|| = 0\ . \qquad (3.5)$$

Taking the norm to both sides of (3.3) and using (3.1) and (3.5) we obtain

$$\limsup_{t \to \infty} ||X(t)|| \geq \limsup_{t \to \infty} (||Y(t)|| - ||\Phi(t) \int_0^t \Phi^{-1}(s)\ B(s)ds||)$$

$$\geq \limsup_{t \to \infty} ||Y(t)|| - \limsup_{t \to \infty} ||\Phi(t) \int_0^t \Phi^{-1}(s)\ B(s)ds||$$

$$\geq \limsup_{t \to \infty} ||Y(t)|| > 0\ .$$

This shows that $X(t)$ satisfies (3.2). The theorem then follows.

Using the same argument as Theorem 3.1 we also can obtain the following theorem.

THEOREM 3.2. Suppose that there exists a solution $Y(t)$ of (1.1) for which

$$\limsup_{t \to \infty} ||Y(t)|| = \infty\ .$$

Then there exists a solution $X(t)$ cf (1.6) for which

$$\limsup_{t \to \infty} ||X(t)|| = \infty\ .$$

PROOF. Since the proof is almost the same as Theorem 3.1, we shall omit the detail.

REMARKS. It is interesting to note that Hatvani and Pintêr [3] have studied this type of problem for equation (1.4).

ACKNOWLEDGEMENTS. This work was supported by the U.S. Army Research Office Grant DAAG29-78-G-0042 at Research Triangle Park, N.C.

REFERENCES

1. Bownds, J. M. Stability Implications on the Asymptotic Behavior of Second Order Differential Equations, Proc. Amer. Math. Soc. 39 (1973), 169-172, MR 47 #2150.

2. Coppel, W. A. Stability and Asymptotic Behavior of Differential Equations, Heath, Boston, 1965. MR 32 #7875.

3. Hatvani, L. and L. Pintér. On Pertubation of Unstable Second Order Linear Differential Equations, Proc. Amer. Math. Soc. 61 (1976), 36-38.