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SUBSEMI-EULERIAN GRAPHS

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ABSTRACT. A graph is subeulerian if it is spanned by an eulerian supergraph. Boesch, Suffel and Tindell have characterized the class of subeulerian graphs and determined the minimum number of additional lines required to make a subeulerian graph eulerian.

In this paper, we consider the related notion of a subsemi-eulerian graph, i.e. a graph which is spanned by a supergraph having an open trail containing all of its lines. The subsemi-eulerian graphs are characterized and formulas for the minimum number of required additional lines are given. Interrelationships between the two problems are stressed as well.

KEY WORDS AND PHRASES. *Spanning walks, Eulerian graphs, Sub-Eulerian graphs, Semi-Eulerian graphs.*

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1. INTRODUCTION.

Although the study of eulerian graphs, initiated by Euler's solution of the Königsberg Bridge Problem in 1736, gave birth to graph theory, there remain interesting questions concerning related concepts.

The Chinese Postman Problem [1,2] is a celebrated example of such a question. This problem is concerned with the minimum number of repetitions of lines of a graph G which are required if one is to traverse the graph in such a way as to visit each line at least once. Boesch, Suffel and Tindell [3,4] considered the related question of when a non-eulerian graph can be made eulerian by the addition of lines. In this paper we consider the problem of when a nonsemi-eulerian graph can be made semi-eulerian by the addition of lines. We characterize "subsemi-eulerian" graphs, i.e., those nonsemi-eulerian graphs which are spanning subgraphs of semi-eulerian graphs. Generalizing the notion of a pairing of points of a graph first introduced by Goodman and Hedetniemi [5] in their study of the Chinese Postman Problem and then used by Boesch, Suffel and Tindell in their study of "subeulerian graphs", we specify the minimum number $s^+(G)$ of lines which must be added to a subsemi-eulerian graph to obtain a semi-eulerian spanning supergraph.

We will consider the following problems:

1) given a (possibly disconnected) multigraph, when is it possible to obtain a semi-eulerian spanning super multi-graph by the addition of lines and what is the minimum number of required lines?

2) given a (possibly disconnected) graph, when is it possible to obtain a semi-eulerian spanning supergraph by the addition of lines (necessarily from the complement of the graph) and what is the minimum number of required lines?

Although the multigraphs problem is rather easily answered, its solution provides valuable insight into the second problem. The second problem is treated both for connected and disconnected graphs.

An answer to the minimum number of additional lines question is given in both cases.

2. PRELIMINARIES.

In this paper, we make use of standard graph-theoretic notions, terminology, and notation set forth in the book by Harary [6]. For the sake of completeness, we repeat some of the basic ideas and reintroduce some pertinent notions and results from a previous paper [4].

A graph $G = (V,X)$ has a finite point set V and a set X whose elements, called lines, are two-point subsets of V. A multigraph is defined similarly except that more than one line is permitted between two points. A graph $H = (U,Y)$ is a subgraph of $G = (V,X)$ if $U \subset V$, $Y \subset X$ and the elements of Y join points of U. H spans G if $U = V$. The degree of a point is a multi-graph is the number of lines incident to it.

A walk W in a graph is an alternating sequence of points and lines $v_0, x_1, v_1, x_2, v_2, \ldots, v_{n-1}, x_n, v_n$ such that x_i has end points v_{i-1} and v_i for $i = 1, \ldots, n$. In a graph, the notation for a walk is redundant, so we shall shorten it to $v_o, v_1, \ldots, v_n$.

A graph is connected if, for each pair of points, there is a walk joining them. A maximally connected subgraph of a graph is a component of the graph. A walk W: $v_0, x_1, v_1, \ldots, v_{n-1}, x_n, v_n$ is closed if $v_0 = v_n$.

A trail is a walk for which $x_i \neq x_j$ whenever $i \neq j$, i.e., no line appears more than once.

An eulerian trail is a closed trail which contains all the lines of the graph and an eulerian graph is one which has an eulerian trail. In 1736, Euler proved the first theorem of graph theory: A multigraph is eulerian iff it is connected and each point has even degree. Of those multigraphs which are not eulerian, the ones which can be made eulerian by the addition of lines are called subeulerian. The minimum number of lines needed to produce a spanning eulerian supermultigraph of the multigraph M is called the eulerian completion number and is denoted by $e^+(M)$. A subeulerian graph G is a noneulerian graph which is contained in a spanning eulerian supergraph. The minimum number of lines which must be added to G is denoted by $e^+(G)$.

THEOREM 1. (SUBEULERIAN MULTIGRAPHS) Every noneulerian multigraph is subeulerian. If p_0 denotes the number of points of the noneulerian multigraph M having odd degree and k_e the number of components of M containing only points of even degree, then

$$e^+(M) = \frac{1}{2} p_0 + k_e.$$

Now a path in a multigraph is a walk P: $v_0, x_1, v_1, \ldots, v_{n-1}, x_n, v_n$ with property that $v_i \neq v_j$ whenever $i \neq j$; i.e., no repeated points. The points v_0 and v_n are called the end points of P and we write $\partial p = \{v_0, v_n\}$. A collection $\mathcal{P}$ of paths from the graph L with $S = \bigcup_{P \in \mathcal{P}} \partial P$ and $\partial P \cap \partial P' = \emptyset$ for all pairs of distinct paths $P, P' \in \mathcal{P}$ is referred to as a pairing of S on L. If G = (V,X) is a graph, then $\overline{G} = (V, \overline{X})$ denotes the complement of G, i.e., $x \in X$ iff $x \notin \overline{X}$. We will denote by $\Theta(G)$ the set of all odd degree points of G. A pairing $\mathcal{P}$ of $\Theta(G)$ on $\overline{G}$ will henceforth be referred to simply as a pairing. Pairings are needed to "complete" connected subeulerian graphs to eulerian graphs.

A minimum pairing of $\Theta(G)$ on $\overline{G}$ is a pairing $\mathcal{P}$ with the property that $\sum_{P \in \mathcal{P}} |X(P)|$ is minimum over all pairings of $\Theta(G)$ on $\overline{G}$ ($|X(P)|$ denotes the number of lines in P). The number of lines in such a pairing is denoted by $m(\Theta(G), \overline{G})$.

The complete bipartite graph $K_{m,n}$ consists of a point set V partitioned into two parts U and W with $|U| = m$, $|V| = n$ and all possible lines joining a point of U to a point of W but no connections internal to U or W.

Complete multipartite graphs are defined similarly; the only difference being that the point set V is partitioned into more than two disjoint parts.

THEOREM 2. (SUBEULERIAN CONNECTED GRAPHS) The following are equivalent for a connected graph G:

1) G is not subeulerian.
2) G is spanned by $K_{2m+1,2n+1}$ for some value of m.
3) G has evenly many points and $\overline{G}$ has at least two components with an odd number of points.

If G is subeulerian, then $e^+(G) = m(\Theta(G), \overline{G})$.. If a graph is disconnected, then each pair of its points may be joined in the complement by a path of length no greater than two. Thus, pairings of odd degree points of a disconnected graph may always employ paths of length no more than two.

The complete graph K_p consists of a point set with p points and all possible lines between distinct pairs of points.

The union of two graphs $H = (V_H, X_H)$ and $L = (V_L, X_L)$ is the graph $H \cup L = (V_H \cup V_L,\ X_H \cup X_L)$.

THEOREM 3. (SUBEULERIAN DISCONNECTED GRAPHS) With the sole exception of $K_1 \cup K_{2n+1}$, all disconnected graphs are subeulerian. If r denotes the number of paths of length two required in a minimum pairing of the odd degree points of the disconnected graph G, which is not the union of two eulerian components, then

$$e^+(G) = \begin{cases} m(\Theta(G), \overline{G}) & \text{if } r > k_e \\ m(\Theta(G), \overline{G}) = \frac{1}{2} p_0 + k_e & \text{if } r = k_e \\ \frac{1}{2} p_0 + k_e & \text{if } r < k_e \end{cases}$$

If $G = \neq K_1 \cup K_{2n+1}$ and $G = E_1 \cup E_2$ where E_1 and E_2 are eulerian, then $e^+(G) = 4$ if E_1 and E_2 are complete; 3 otherwise.

A multigraph may fail to be eulerian and yet may have a trail which includes all the lines of the multigraph. This is the case when the trail is not closed. Such a graph is referred to as semi-eulerian and it is known that a multigraph is semi-eulerian iff it is connected and has exactly two points of odd degree. Of those nonsemi-eulerian multigraphs, the ones which are contained in some spanning semi-eulerian super-multigraph are called subsemi-eulerian. The minimum number of additional lines required by a subsemi-eulerian multigraph M to produce a semi-eulerian super-multigraph is called the semi-eulerian completion number and is denoted by $s^+(M)$. The notion of subsemi-eulerian graph and the symbol $s^+(G)$ are defined similarly.

3. SUBSEMI-EULERIAN MULTIGRAPHS.

As was stated in Theorem 1, every non-eulerian multigraph M is sub-eulerian and $e^+(M) = p_0/2 + k_e$. Thus, by the addition of all but one of the lines required to make M eulerian, i.e., with $p_0/2 + k_e - 1$ lines, we may create a semi-eulerian spanning super-multigraph of M. Furthermore, if M is a semi-eulerian spanning super-multigraph of M, only one line need be added to M to obtain a spanning eulerian super-multigraph of M. Thus, it follows that

$$s^+(M) \geq e^+(M) - 1 = \frac{1}{2} p_0 + k_e - 1$$

and we may state

THEOREM 4. (SUBSEMI-EULERIAN MULTIGRAPHS) Every noneulerian multigraph M, not already semi-eulerian, is subsemi-eulerian and $s^+(M) = \frac{1}{2} p_0 + k_e - 1$. Any eulerian multigraph M is also subsemi-eulerian with $s^+(M) = 1$.

4. CONNECTED SUBSEMI-EULERIAN GRAPHS.

First, we observe that if G is complete then it is certainly impossible to add lines to G from $\overline{G}$ to make G semi-eulerian. On the other hand, it is clear that any subeulerian graph, connected or not, which is not already semi-eulerian, can be made semi-eulerian by the addition of all but one of the lines of a set which would render the graph eulerian. What if the connected graph G is neither subeulerian not semi-eulerian? Then, by Theorem 2, G is spanned by some $K_{2m+1,2n+1}$, or , equivalently, G has an even number of points and at least two of the components of $\overline{G}$ have an odd number of points. Suppose that $\overline{G}$ has exactly two components with an odd number of points. Then, denoting the components of $\overline{G}$ with an odd number of points by G_1 and G_2, those with an even number of points by $G_3,\ldots,G_n$, and the odd degree points of G lying in G_i by Θ_i for $i = 1,\ldots,n$, we note that Θ_1 and Θ_2 have odd cardinality while $\Theta_3,\ldots,\Theta_n$ each have an even number of points. Thus, if P_1 is a pairing of all but one point of Θ_1 on G_1, P_2 is a pairing of all but one point of Θ_2 on G_2 and P_i is a pairing of the set Θ_i on K for $i = 3,\ldots,n$, it follows that $G \cup \bigcup_{P\in P_i} P$ is semi-eulerian. It is natural to conjective that, in this case,

$$s^+(G) = \min_{u\in\Theta_1} m\,(\Theta_1 - \{u\}, G_1) + \min_{u\in\Theta_2} m(\Theta_2 = \{u\}, G_2) + \sum_{i=3}^{n} m\,(\Theta_i, G_i), \quad (4.1)$$

This is indeed the case but, before we can be sure of this, we must first show that the additional lines needed to make a connected subsemi-eulerian graph a semi-eulerian graph must include a pairing of all but two of the odd degree points of G on $\overline{G}$. The reason for concern in this regard is that it is possible for a spanning semi-eulerian supergraph of a graph G to have two odd degree points which were originally even in G.

DEFINITION 1. (SEMI-PAIRINGS) If $P = \{P_1,\ldots,P_n\}$ is a collection of paths with lines from $\overline{G}$, with $\bigcup_{P\in P} P$ consisting of all but two points of $\Theta(G)$, and $\partial P_i \cap \partial P_j = \emptyset$ for $i \neq j$, then <u>P is a semi-pairing of $\Theta(G)$ on $\overline{G}$</u>.

Now, with the aid of this formal notion, we establish

LEMMA 5a. (SEMI-EULERIAN SUPERGRAPHS CONTAIN SEMI-PAIRINGS) If $\hat{G}$ is a spanning semi-eulerian supergraph of the subsemi-eulerian (possibly disconnected) non-eulerian graph G, then $\hat{G} - G$ contains a semi-pairing of $\Theta(G)$ on $\overline{G}$.

PROOF. Remove all cycles of $\hat{G} - G$ from $\hat{G}$ and denote the resulting graph by H. Now H is a spanning (possibly disconnected) supergraph of G which has precisely two odd degree points. Furthermore, H-G is a forest so there are line disjoint paths $P_1, \ldots, P_n$ with $\partial P_i \cap \partial P_j = \emptyset$ for each pair of distinct indices i and j and $H-G = \bigcup_{i=1}^{n} P_i$. To see that this is the case, choose P_1 to be a longest path in H-G, P_2 to be a longest path in $H-(G \cup P_1)$, and so on until all lines of H-G have been accounted for. Now, if $\partial P_i = \{u_i, v_i\}$ and each of the points u_i and v_i have odd degree in G, then $P_1, \ldots, P_n$ is a semi-pairing of $\Theta(G)$ on $\overline{G}$. Suppose there exists an index i such that u_i and v_i have even degree in G. Then $P = \{P_1, \ldots, P_{i-1}, P_{i+1}, \ldots P_n\}$ is a pairing of $\Theta(G)$ on $\overline{G}$ for all the points distinct from u_i and v_i must be even in H. Of course, removal of any path from P yields a semi-pairing of $\Theta(G)$ on $\overline{G}$. Finally, if there are indices i and j such that u_i and u_j have even degree in G and v_i and v_j have odd degree in G, then $P' = \{P_k | k \neq i,j\}$ is a semi-pairing of $\Theta(G)$ on $\overline{G}$.

Now, in returning to the original problem; we see that, if $\overline{G}$ has exactly two components with an odd number of points, a semi-pairing of $\Theta(G)$ on $\overline{G}$ must leave exactly one point in each of the odd components with odd degree. Furthermore, it is also clear that, if the graph $\overline{G}$ on an even number of points has four or more components with an odd number of points, no semi-pairing exists. Finally, if G is subeulerian and noneulerian, then a semi-pairing would have to leave two odd degree points in a single component of $\overline{G}$. Thus, in this case,

$$s^{+}(G) = \min_{1 \le i \le n} \min_{u,v \in \Theta_1} \left(m(\Theta_i - \{u,v\}, G_i) + \sum_{j \neq i} m(\Theta j, G_j)\right) \qquad (4.2)$$

(where Θ_i = the collection of odd degree points of G within the component G_i of $\overline{G}$). Summarizing the foregoing discussion, we have

THEOREM 5. (SUBSEMI-EULERIAN CONNECTED GRAPHS) The following are equivalent for a connected graph G:

(1) G is not subsemi-eulerian.

(2) G is complete or G is spanned by a complete multipartite graph having four odd parts.

(3) G is complete or G has an even number of points and $\overline{G}$ has four or more components with an odd number of points.

If G is subsemi-eulerian, then $s^{+}(G) = 1$ if G is eulerian or is given by (2) or (1), depending on whether G is subeulerian or not.

Before proceeding to the disconnected case, let us consider the natural question of whether, for a subeulerian graph, a minimum semi-pairing may be extracted from a minimum pairing by removing a longest path from the minimum pairing? Doubt is cast on the validity of such a conjecture when the possibility of completing a subeulerian graph to a semieulerian nonsubeulerian graph is granted. We illustrate this situation in our first example.

EXAMPLE 1. A CONNECTED SUBEULERIAN GRAPH WITH A SEMIEULERIAN SPANNING SUPERGRAPH WHICH IS NOT SUBEULERIAN.

Consider the connected graph G with complement $\overline{G}$ shown in the accompanying figure.

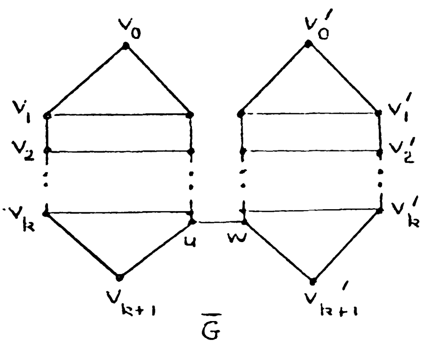


Figure 1

G has $2k + 6$ points with $k \geq 3$. The points v_0, v_0', v_{k+1}, and v_{k+1}' have odd degree in G and it is easy to see that the paths $P_1 = v_0, v_1, \ldots, v_{k+1}$ and $P_1' = v_0', v_1', \ldots, v_{k+1}'$ constitute a minimum pairing of the odd degree points of G on $\overline{G}$. Thus $m(\Theta(G), \overline{G}) = 2(k+1)$ and it readily follows that $m(\Theta(G), \overline{G})$ less the length of a longest path in the minimum pairing $P = \{P_1, P_1'\}$ is $k + 1$. On the other hand, the path $P = v_{k+1}, u, w, v_{k+1}'$ serves as a minimum semi-pairing of $\Theta(G)$ on $\overline{G}$ which when added to G yields a non-subeulerian graph. Thus $s^{+}(G) = 3 < k + 1$. Furthermore, the difference between the numbers $k + 1$ and 3 attains all positive integral values as k ranges over the integers ≥ 3.

Next suppose $\mathcal{P}$ is a minimum semi-pairing of $\Theta(G)$ on $\overline{G}$ such that $G \cup \bigcup_{P \in \mathcal{P}} P$ is subeulerian. Can $\mathcal{P}$ be enlarged to a pairing of $\Theta(G)$ on $\overline{G}$ which contains a path of length long enough to make the size of the pairing larger than $m(\Theta(G), \overline{G})$?

EXAMPLE 2. MINIMUM SEMI-PAIRING WHICH CANNOT BE EXTENDED TO MINIMUM PAIRINGS.

A minor modification of the graph given in Example 1 yields a graph G on $2k + 6$ points ($k \geq 3$) with complement as shown in Figure 2. All the conclusions reached in Example 1, save for one, are valid here.

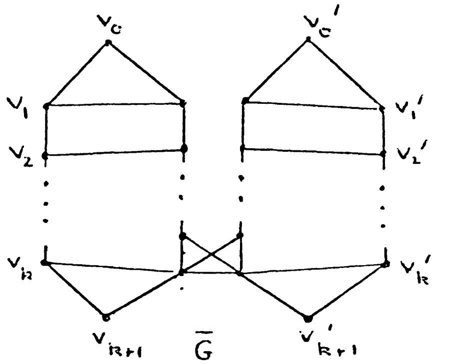


Figure 2.

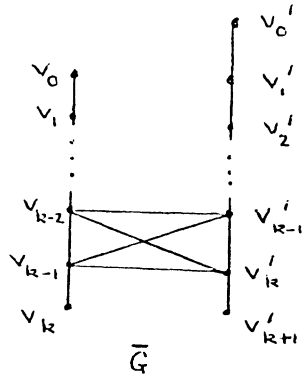


Figure 3.

In this case, the addition of $P = v_{k+1}, u, w, v_{k+1}'$ to G yields a semi-eulerian graph which is subeulerian. The graph G with complement shown in Figure 3 is an example of subsemi-eulerian graph on an odd number of points which contains a minimum semi-pairing that cannot be extended to minimum pairing. In this case, $m(\Theta(G),\overline{G}) = 2k+1$ ($k \geq 4$) so that the value of $m(\Theta(G),\overline{G})$ less the length of a longest path in a minimum pairing is k. Of course, $s^+(G) = 3$ so that $k > s^+(G)$ whenever $k \geq 4$.

5. DISCONNECTED SUBSEMI-EULERIAN GRAPHS.

We recall that every disconnected graph with the sole exception of $K_1 \cup K_{2n+1}$ ($n \geq 0$) is subeulerian (Theorem 3). Because the addition of any line from K_1 to a point of K_{2n+1} yields a semi-eulerian graph, all disconnected graphs are subsemi-eulerian. Furthermore, since graphs are also multigraphs, it follows by Theorem 1 that $s^+(G) \geq \frac{1}{2}p_0 + k_e - 1$. In fact, in some instances, $s^+(G) = \frac{1}{2} p_0 + k_e - 1$.

In order to completely determine $s^+(G)$, we first verify the following facts: if G is disconnected, then

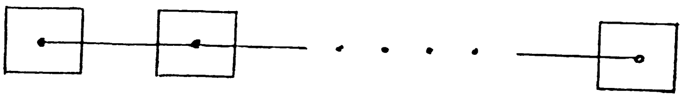
(1) every minimum pairing or semi-pairing of (G) on $\overline{G}$ must use paths of length at most two, and

(2) the size of a minimum semi-pairing of $\Theta(G)$ on $\overline{G}$ is $m(\Theta(G),\overline{G}) - 2$ or $m(\Theta(G),\overline{G}) - 1$, the latter being the case only when $\overline{G}$ contains a complete matching of $\Theta(G)$.

To begin with, we shall show that semi-pairings and pairings may be found using paths of length at most two. Let M denote a maximum matching of $\Theta(G)$ in $\overline{G}$: if $M \neq \emptyset$, we write $M = \{e_i\}_{i=1}^{m}$. Denote the endpoints of e_i by u_i and v_i for each $i = 1,\ldots,m$. Of the remaining vertices in $\Theta(G)$, of which there are $2n = |\Theta(G)|-2m$ ($m = 0$ if $M = \emptyset$), no two are adjacent. If $\{u_{m+1},v_{m+1}\}$, $\{u_{m+2},v_{m+2}\},\ldots,\{u_{m+n},v_{m+n}\}$ denotes an arbitrary pairing of the vertices of $\Theta(G) - \bigcup_{i=1}^{m} \{u_i,v_i\}$, then there are paths $P_{m+1},\ldots,P_{m+n}$, of length two, which join the respective pairs. Of course, the set of internal points of $P_{n+1},\ldots,P_{n+m}$ is disjoint from $\bigcup_{i=n+1}^{n+m} \{u_i,v_i\}$ so that these paths are mutually edge-disjoint. Furthermore, the paths of $P = M \cup \{P_i\}_{i=m+1}^{m+n}$ are edge-disjoint. Hence P is a pairing of $\Theta(G)$ on $\overline{G}$ while, P' - obtained from P by deleting P_{m+n} if P contains paths of length two or e_m if $P = M$ - is a semi-pairing of $\Theta(G)$ on $\overline{G}$.

In due course, we shall see that P and P' are minimum. Suppose that $S = \{S_1,\ldots,S_{|\Theta(G)|/2}\}$ is a minimum pairing of $\Theta(G)$ on $\overline{G}$. If each path of S has length at least two, then it readily follows that P is a minimum pairing and each path of S has length exactly two. On the other hand, if $S_1,\ldots,S_k$ denote the paths of S with length one, then they constitute a matching of $\Theta(G)$ in $\overline{G}$ and, therefore, $k \leq m$. Again it readily follows that P is a minimum pairing. In fact, $k < m$ is impossible so that every minimum pairing must include a maximum matching of $\Theta(G)$. Furthermore, it is clear that each path $S_{m+1},\ldots,S_{m+n}$ must have length two. The same argument applies for semi-pairings. Indeed, every minimum semi-pairing must use paths of length at most two, P'-described above - is a minimum semi-pairing, and every minimum semi-pairing includes a maximum matching of $\Theta(G)$ in the event that P has at least one path of length two. Moreover, the size of a minimum semi-pairing is $m(\Theta(G),\overline{G})-2$ if P contains a path of length two and $m(\Theta(G),\overline{G})-1$ otherwise.

As the final major step in the development, we show that one of the two values $\frac{1}{2}p_0 + k_e - 1$ and $m(\Theta(G),\overline{G}) - 2$ is always attained. Indeed if $p_0 = 0$, so that G is the union of two or more eulerian components, G may be made semi-eulerian by the addition of a path of length $k_e - 1 = \frac{1}{2}p_0 + k_e - 1$, (See Figure (4)).



k_e eulerian components

Figure 4

Now, it was stated in the section of preliminaries that, whenever the number of paths of length two of a minimum pairing of G is less than or equal to k_e, $e^+(G) = \frac{1}{2}p_0 + k_e$. Hence, in this case, $s^+(G) = p_{0/2} + k_e - 1$. On the other hand, if the number of paths of lengths in a minimum pairing exceeds k_e, then $e^+(G) = m(\Theta(G),\overline{G})$. Furthermore, if $\mathcal{P}$ is a minimum pairing of $\Theta(G)$ on $\overline{G}$ such that $H = G \cup_{P \in \mathcal{P}} P$ is eulerian, then there must be a path P of length two in P with all three points of P in the same component of G. Thus removal of P from H yields a semi-eulerian graph and $s^+(G) = m(\Theta(G),\overline{G}) - 2$.

THEOREM 6. (DISCONNECTED SUBSEMI-EULERIAN GRAPHS) Every disconnected graph is subsemi-eulerian. If we set $m(\Theta(G),\overline{G}) = 0$ when $\Theta(G) = \emptyset$, then $s^+(G) = \max\left(\frac{1}{2}p_0 + k_e - 1,\ m(\Theta(G),\overline{G}) - 2\right)$.

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