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A PÓLYA "SHIRE" THEOREM FOR FUNCTIONS WITH ALGEBRAIC SINGULARITIES

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ABSTRACT. The classical "shire" theorem of Pólya is proved for functions with algebraic poles, in the sense of L. V. Ahlfors. A function f(z) is said to have an algebraic pole at z_0 provided there is a representation $f(z) = \Sigma_{k=-N}^{\infty} a_k (z - z_0)^{k/p} + A(z)$, where p and N are positive integers and A(z) is analytic at z_0. For p = 1, the proof given reduces to an entirely new proof of the shire theorem. New quantitative results are given on how zeros of the successive derivatives migrate to the final set.

KEYWORDS AND PHRASES. *Final set, algebraic singularity, branch line, shire theorem.*

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1. INTRODUCTION.

Pólya [1,2] defined the "final set" S(f) of a meromorphic function f(z) as follows. A point $z_0 \varepsilon S(f)$ if every neighborhood of z_0 contains zeros of infinitely many derivatives of f(z). The final set determines, roughly speaking, the final position of the zeros of the derivatives of f(z).

For functions with two or more poles, Pólya characterized the final set by showing that $z_0 \varepsilon S(f)$ if and only if z_0 is equidistant from two or more poles. Whittaker's "shire" [3] descripition of Pólya's Theorem is quite illuminating. Let us say that the shire of a pole λ of $f(z)$ will consist of all points z in the plane which are nearer to λ than to any other pole of $f(z)$. Then the final set of a meromorphic function consists of the boundaries of the shires of its poles.

If $f(z)$ has but one pole, the final set is empty, and the single pole generates zero-free regions for the successive derivatives of $f(z)$. The asymptotic size of such zero-free regions has been obtained in [4]. Thus the poles of $f(z)$ may be thought of as repellers of equal strength of the zeros of the derivatives of $f(z)$. A refinement of the "equal strength" aspect of the Pólya shire theorem is one of the features of this paper.

The general questions posed in Pólya's classic paper [1] concern the effect of the singularity structure of an analytic function on its final set. The present paper takes up the problem of determining the final sets of functions with *algebraic singularities*. We take a countable set Λ of points in the plane $\mathbb{C}$, and for each $\lambda \varepsilon \Lambda$ we let $\ell(\lambda)$ denote the "principal branch line" $\ell(\lambda) = \{z = \lambda + t | t \le 0\}$, corresponding to λ. We assume in addition that each bounded region $D \subset \mathbb{C}$ intersects the branch lines of only finitely many $\lambda \varepsilon \Lambda$. The functions $f(z)$ we study are those which have "algebraic poles", in the sense of Ahlfors [5, p. 299], at the points of Λ. To say that $f(z)$ has an algebraic pole at z_0 means that there are integers $p \ge 1$ and $N \ge 1$ such that $f(z)$ has the representation

$$f(z) = \sum_{k=-N}^{\infty} c_k (z - z_0)^{k/p}, \quad c_{-N} \neq 0 , \tag{1.1}$$

in a neighborhood of z_0. For each $\lambda \varepsilon \Lambda$ we associate integers $p(\lambda) \ge 1$, $N(\lambda) \ge 1$ and coefficients $c_k(\lambda)$, $-N(\lambda) \le k < \infty$, with $c_{-N(\lambda)}(\lambda) \neq 0$. Given a bounded region D, let $\Lambda(D) = \{\lambda \varepsilon \Lambda | \ell(\lambda)$ intersects $D\}$; $\Lambda(D)$ is a finite set. We shall suppose that the function

$$f(z) - \sum_{\lambda \varepsilon \Lambda(D)} \sum_{k=-N(\lambda)}^{\infty} c_k(\lambda)(z - \lambda)^{k/p(\lambda)} = A_D(z) \tag{1.2}$$

is analytic in D. By $(z - \lambda)^{\alpha}$ we mean the branch which is analytic for $z \notin \ell(\lambda)$ and real if $\text{Im}(z) = \text{Im}(\lambda)$ and $\text{Re}(z) > \text{Re}(\lambda)$. The factors $(z - \lambda)^{k/p(\lambda)}$, $k \ge 0$, cannot be thrown over into the analytic part $A_D(z)$, as would happen if $f(z)$ were meromorphic. Thus the singularities of f(z) in D arise from the algebraic poles and branch lines there, and may be subtracted off in the manner of poles for meromorphic functions. Since the unique representations (1.2) for f(z) are to hold for arbitrary D, then the coefficients $\{c_k(\lambda)\}_{k=0}^{\infty}$ must be *entire* coefficients; i.e., for each $\lambda \varepsilon \Lambda$, the series $\sum_{k=0}^{\infty} c_k(\lambda)(z-\lambda)^{k/p(\lambda)}$ must converge for all complex numbers z.

Disregarding the branch lines, form the shires of the points $\lambda \varepsilon \Lambda$ by the classical definition and let $P(\Lambda)$ be the union of all the boundaries of the shires. Our principal result asserts that in spite of algebraic singularities, $P(\Lambda)$ is still the final set of f(z), with one exceptional case. Not surprisingly, difficulties arise when a horizontal line segment in $P(\Lambda)$ determined by two points $\lambda_1, \lambda_2 \varepsilon \Lambda$ coincides with the branch line of a third point $\lambda_3 \varepsilon \Lambda$. Should this occur, we will suppose that λ_1 and λ_2 are of *unequal strength*, in that either $(N_1/p_1) \neq (N_2/p_2)$ or $|c_{N_1}| \neq |c_{N_2}|$ if $(N_1/p_1) = (N_2/p_2)$, where $N_1 = N(\lambda_1)$, $p_1 = p(\lambda_1)$, etc.

THEOREM 1. If f(z) is defined and restricted as above, the final set of f(z) is $P(\Lambda)$.

Note that the theorem makes no mention of the branch lines apart from the exceptional case; the branch lines come into play only in the proof, and only in the exceptional case.

We prove Theorem 1 in the next section, after which we make a series of Remarks on how the proof can be modified to cover various situations. For example, it is reasonable to ask how the theorem would be altered if we allow zeros of derivatives of *all branches* to contribute, instead of a single, fixed branch.

There is no hope of treating the case where (1.2) is replaced by a doubly infinite series. Even if p = 1, essentially nothing is known about final sets for functions with isolated essential singularities.

Our proof of Theorem 1 covers the case where f(z) is meromorphic (p = 1) and without restriction on the location or strength of poles. As such, it is a new proof of the Pólya shire Theorem, independent of the two existing proofs in [6], [1], [3]. Moreover, the classical proofs do not extend to the case of algebraic singularities, and this is why it was necessary to devise a new proof.

The classical theory asserts that the location of the final set does not depend on the order of the poles or the size of their coefficients. Even so, our proof reveals the previously unobserved phenomenon that if adjacent poles have unequal strength, then all the zeros of sufficiently high order derivatives are pushed to one side of the common boundary of the two shires.

Various other features of the proof bearing on the migration of zeros to the fina. set are mentioned in the Remarks.

2. PROOFS AND REMARKS.

It is convenient to express the derivatives of the factors $(z-\lambda)^{k/p}$ in terms of the Gamma function, and in terms of certain factors $C_{k,n}$ which were studied in the paper [4]. For an arbitrary integer k, let us define

$$C_{k,n} = k(k+p)(k+2p)\cdots(k+(n-1)p) ,$$

where $p \geq 1$, so that the $n^{\underline{th}}$ derivative of $(z-\lambda)^{k/p}$ may be written as

$$\begin{aligned}\frac{d^n}{dz^n}(z-\lambda)^{k/p} &= (k/p)((k/p)-1)\cdots((k/p)-(n-1))(z-\lambda)^{(k/p)-n} \\ &= (-1)^n p^{-n}(-k)(-k+p)(-k+2p)\cdots(-k+(n-1)p)(z-\lambda)^{(k/p)-n} \\ &= (-1)^n p^{-n} C_{-k,n}(z-\lambda)^{(k/p)-n} \\ &= (-1)^n\{\Gamma(-kp^{-1}+n)/\Gamma(-kp^{-1})\}(z-\lambda)^{(k/p)-n}\end{aligned} \tag{2.1}$$

(see [7, p. 25?]). We require certain information about the asymptotic behavior of the terms in (2.1). To begin with, if a and b are arbitrary real numbers, then [7, p. 257]

$$\lim_{n\to\infty}\left\{n^{b-a}\Gamma(a+n)/\Gamma(b+n)\right\} = 1 .$$

Therefore, if N is a positive integer we have

$$\frac{C_{1,n}}{C_{N,n}} = \frac{\Gamma(p^{-1}+n)}{\Gamma(p^{-1})}\cdot\frac{\Gamma(Np^{-1})}{\Gamma(Np^{-1}+n)} = 0\,(n^{(1-N)/p}),\ n\to\infty,$$

using order-of-magnitude notation.

Let [x] denote the greatest integer not exceeding the real number x, and define

$$\delta_n = \max_{1\le k\le[(n-1)p]} |C_{-k,n}/C_{1n}| , \quad n = 1,2,3,\ldots$$

We shall need the result of Lemma 3.1 of [4] which asserts that there is a constant Δ, which depends only on p, such that

$$\delta_n \, n^{1/p} \le \Delta , \; n = 1, 2, 3, \ldots . \tag{2.2}$$

Finally, we note that if $k \ge [(n-1)p]+1$, then $p \ge 1$ implies $k > (n-1)p \ge n-1$, and therefore $k \ge n$. Also for $k \ge [(n-1)p]+1$,

$$\begin{aligned}\left|\frac{C_{-k,n}}{C_{1n}}\right| &= \frac{k!}{n!(k-n)!} \cdot \frac{k(k-p)\cdots(k-(n-1)p)}{k(k-1)\cdots(k-(n-1))} \cdot \frac{(1)(2)\cdots(n)}{(1)(1+p)\cdots(1+(n-1)p)} \\ &\le \frac{k!}{n!(k-n)!} = \binom{k}{n} ,\end{aligned} \tag{2.3}$$

the binomial coefficient.

Our first lemma is a result which shows that compact subsets of shires of points $\lambda\varepsilon\Lambda$ are zero-free for sufficiently high order derivatives of f(z). A similar result for meromorphic functions appears in Polya's original proof [1,3].

LEMMA 1. Let $a\varepsilon\Lambda$ and let f(z) be defined as in Theorem 1. Then

$$\lim_{n\to\infty} |z - a| \quad |f^{(n)}(z)/n!|^{1/n} = 1 ,$$

uniformly on compact subsets of a-shire.

PROOF. Let K be a compact subset of a-shire and let $R > 0$ be such that K is contained in $|z - a| < (R/2)$. Let D be the disc $|z - a| < R$ and put $\Lambda(D) = \{\lambda\varepsilon\Lambda | \ell(\lambda)$ intersects D, $\lambda \neq a\}$. Then for $z\varepsilon D - \bigcup_{\lambda\varepsilon\Lambda(D)} \ell(\lambda)$ and $z \notin \ell(a)$, f(z) can be represented as

$$f(z) = \sum_{k=-N}^{\infty} a_k(z-a)^{k/p} + \sum_{\lambda\varepsilon\Lambda(D)}^{\infty} \; \sum_{j=-N(\lambda)}^{\infty} c_j(\lambda)(z-\lambda)^{j/p(\lambda)} + A_D(z) \tag{2.4}$$

where $A_D(z)$ is analytic in D and where we have written $a_i = c_i(a)$ and $N = N(a)$.

Computing derivatives in (2.4) and using (2.1), we obtain

$$(-1)^n f^{(n)}(z) = \sum_{k=-N}^{\infty} \left\{a_k \Gamma(-kp^{-1}+n)/\Gamma(-kp^{-1})\right\}(z-a)^{(k/p)-n}$$

$$+ \sum_{\lambda\varepsilon\Lambda(D)}^{\infty} \sum_{j=-N(\lambda)}^{\infty} \left\{c_j(\lambda)\Gamma(-jp(\lambda)^{-1}+n)/\Gamma(-jp(\lambda)^{-1})\right\}(z-\lambda)^{(j/p(\lambda))-n}$$

$$+ (-1)^n A_D^{(n)}(z) \tag{2.5}$$

$$= \frac{a_{-N}\Gamma(Np^{-1}+n)}{\Gamma(Np^{-1})(z-a)^{(N/p)+n}} \left\{1 + F_n(z) + G_n(z) + H_n(z)\right\} ,$$

where

$$F_n(z) = \sum_{k=-N+1}^{\infty} \frac{a_k\Gamma(Np^{-1})\Gamma(-kp^{-1}+n)}{a_{-N}\Gamma(-kp^{-1})\Gamma(Np^{-1}+n)} (z - a)^{(k+N)/p} ,$$

$$G_n(z) = \sum_{\lambda\varepsilon\Lambda(D)}^{\infty} \sum_{j=-N(\lambda)}^{\infty} \frac{c_j(\lambda)\Gamma(Np^{-1})\Gamma(-jp(\lambda)^{-1}+n)}{a_{-N}\Gamma(-jp(\lambda)^{-1})\Gamma(Np^{-1}+n)} \frac{(z-\lambda)^{(j/p(\lambda))-n}}{(z-a)^{-(N/p)-n}} ,$$

$$H_n(z) = \frac{\Gamma(Np^{-1})(z-a)^{(N/p)+n}}{a_{-N}\Gamma(Np^{-1}+n)} (-1)^n A_D^{(n)}(z) .$$

For $n > 0$ the terms in (2.5) corresponding to $k = 0$ and $j = 0$ vanish since $1/\Gamma(0) = 0$.

We are going to show that the sequences $F_n(z)$, $G_n(z)$ and $H_n(z)$ all converge to 0 as $n \to \infty$, uniformly on D. Starting with $F_n(z)$, we first break its defining sum into three parts, with k ranging successively over $-N + 1 \le k \le -1$, $1 \le k \le [(n-1)p]$ ($[x]$ = greatest integer function) and $[(n-1)p] + 1 \le k < \infty$. The k^{th} term of $F_n(z)$ in the first range has order of magnitude, for large n, $\Gamma(-kp^{-1}+n)/\Gamma(Np^{-1}+n) = O(n^{-k+N)/p})$ $= o(1)$, $n \to \infty$, since $k \ge -N + 1$. Since the sum over the first range has only finitely many terms, it follows that this part of $F_n(z)$ is uniformly small for large n, $z\varepsilon D$. The sum over the second range is, in view of (2.1) and the definition of δ_n, bounded by

$$|a_{-N}|^{-1} \sum_{k=1}^{[(n-1)p]} |a_k| \ |C_{-k,n}/C_{1,n}| \ |C_{1,n}/C_{N,n}| \ |z - a|^{(k+N)/p}$$

$$\le J\ |a_{-N}|^{-1} \sum_{k=1}^{[(n-1)p]} |a_k| \ \delta_n\ n^{(1-N)/p}\ |z - a|^{(k+N)/p}$$

where J is constant. By (2.2), the above does not exceed

$J\Delta|a_{-N}|^{-1}n^{-(N/p)}\sum_{k=1}^{\infty}|a_k|\ |z-a|^{(k+N)/p}$.

Since $\sum_{k=1}^{\infty}|a_k|\ \ |z-a|^{k/p}$ is uniformly bounded over D, it follows that the component of $F_n(z)$ coming from the sum over the range $1 \le k \le [(n-1)p]$ also converges uniformly to 0 on D. Looking at the third range, $k \ge [(n-1)p] + 1$, we use (2.1) and (2.3) and bound the sum by

$$J\ |a_{-N}|^{-1}\ n^{(1-N)/p}\sum_{k=n}^{\infty}|a_k|\ \ \binom{k}{n}\ |z-a|^{(k+N)/p}$$

$$= J\ |a_{-N}|^{-1}\ n^{(1-N)/p}|z-a|^{N/p}\sum_{k=n}^{\infty}\binom{k}{n}(\ |a_k|^{1/k}|z-a|^{1/p})^k\ .$$

Since $|a_k|^{1/k} \to 0$, then $|a_k|^{1/k}\ |z-a|^{1/p} \le \gamma < (1/2)$ for all $z\varepsilon D$, if k is sufficiently large. This means that the infinite sum in the above expression does not exceed

$$\sum_{k=n}^{\infty}\binom{k}{n}\gamma^k = \gamma^{-1}\left(\frac{\gamma}{1-\gamma}\right)^{n+1}\ ,$$

for n sufficiently large. Since $(\gamma/(1-\gamma)) < 1$, then the component of $F_n(z)$ taken from the range $k \ge [(n-1)p] + 1$ is asymptotically smaller than $n^{(1-N)/p}(\lambda/(1-\lambda))^{n+1} \to 0$, $n \to \infty$. This completes the proof that $F_n(z) \to 0$, $n \to \infty$, uniformly in D.

As for $G_n(z)$, we need the fact that K is compact and in a-shire so that there assuredly is a number τ, $0 < \tau < 1$, such that $|z-a| < \tau\ |z-\lambda|$ for all $z\varepsilon K$ and $\lambda\varepsilon\Lambda(D)$. We pick one $\lambda\varepsilon\Lambda(D)$ and break the corresponding sum over j into three parts exactly as above. A typical term in $G_n(z)$ lying in the range $-N(\lambda) \le j \le -1$ has order of magnitude $\tau^n\ n^{-(j+N)/p(\lambda)} \to 0$, $n \to \infty$. Hence this component of $G_n(z)$ tends uniformly to 0 over K. The arguments for the ranges $1 \le j \le [(n-1)p(\lambda)]$ and $[(n-1)p(\lambda)] + 1 \le j < \infty$ run exactly as the corresponding ones for $F_n(z)$ except that we now have $z\varepsilon K$ and the additional factor τ^n before each sum. We omit the details but conclude that $G_n(z) \to 0$, $n \to \infty$, uniformly on K.

Working with $H_n(z)$, we first have

$$\frac{(z-a)^n}{n!} A_D^{(n)}(z) = \frac{(z-a)^n}{n!} \frac{d^n}{dz^n}\left(\sum_{k=0}^{\infty} \alpha_k (z-a)^k \right)$$

$$= \frac{(z-a)^n}{n!} \sum_{k=n}^{\infty} \frac{k!\alpha_k}{(k-n)!} (a-a)^{k-n} = \sum_{k=n}^{\infty} \binom{k}{n} \alpha_k (z-a)^k$$

$$= \sum_{j=0}^{\infty} \binom{j+n}{n} \alpha_{j+n} (z-a)^{j+n} .$$

Now $|\alpha_{j+n}|\rho^{j+n} = O(1)$, $n+j \to \infty$, if $\rho < R$, since $A_D(z)$ is analytic in D. Therefore

$$\left| \frac{(z-a)^n}{n!} A_D^{(n)}(z) \right| = O\left\{ \left|\frac{z-a}{\rho}\right|^n \sum_{j=0}^{\infty} \binom{j+n}{n} \left|\frac{z-a}{\rho}\right|^j \right\} = O\left\{ \left[\frac{(|z-a|/\rho)}{1-(|z-a|/\rho)}\right]^n \right\}. \tag{2.6}$$

If $z \varepsilon K$ then $(|z-a|/\rho) \leq \gamma < (1/2)$, for some $\rho < R$ and a constant γ, and so the O-term in (2.6) is dominated by $O(\gamma/(1-\gamma))^n$. Then since $n! = \Gamma(n+1)$, we have

$$H_n(z) = O\left\{ n^{1-(N/p)} \left(\frac{\gamma}{1-\gamma}\right)^n \right\} = o(1), \ n \to \infty, \ z\varepsilon K .$$

Going back to (2.5) we now have

$$\left| \frac{(z-a)^{(N/p)+n} f^{(n)}(z)}{n!} \right|^{1/n} = \left| \frac{a_{-N}\Gamma(Np^{-1}+n)}{\Gamma(Np^{-1})\Gamma(n+1)} (1+o(1)) \right|^{1/n}$$

and since $[\Gamma(Np^{-1}+n)/\Gamma(1+n)] \sim n^{(N/p)-1}$, the desired result follows.

Lemma 1 shows that no point of any shire can be in the final set. The next lemma establishes the existence of zeros of derivatives near points on the boundaries of shires, and thus Theorem 1 will follow.

LEMMA 2. Let $\xi \varepsilon P(\Lambda)$ and let $f(z)$ be defined as in Theorem 1. Then every neighborhood of ξ contains zeros of $f^{(n)}(z)$ for n sufficiently large.

PROOF. First we note some reductions. If ξ lies on the boundaries of a-shire, b-shire and other shires, then we can consider a point ξ' near ξ which lies on the boundaries of a-shire and b-shire only. If we produce zeros in arbitrary neighborhoods of ξ', then zeros in neighborhoods of ξ are implied by elementary arguments. We note that in this reduction the point ξ' can also be taken not to lie on any branch line with the single exception that a and b have the same real part. In this case, recall our hypothesis that a and b are of unequal strength. Our proof, then, splits into the two cases where ξ either does or does not lie on a branch line, but in any case is on the common boundary of exactly two shires.

CASE I: *ξ does not lie on a branch line.* Suppose ξ lies on the common boundary of a-shire and b-shire, let $R > 0$ so that $|\xi - a| < (R/2)$ and $|\xi - b| < (R/2)$, let Ω be the union of the discs $|z - a| < R$ and $|z - b| < R$, and put $\Lambda(\Omega) = \{\lambda\varepsilon\Lambda | \ell(\lambda)$ intersects Ω, $\lambda \neq a$, $\lambda \neq b\}$. Then if $z\varepsilon\Omega - \bigcup_{\lambda\varepsilon\Lambda(\Omega)} \ell(\lambda)$, $f(z)$ may be expressed as

$$f(z) = \sum_{k=-N}^{\infty} a_k (z-a)^{k/p} + \sum_{k=-M}^{\infty} b_k (z-b)^{k/q}$$

$$+ \sum_{\lambda\varepsilon\Lambda(\Omega)} \sum_{j=-N(\lambda)}^{\infty} c_j(\lambda)\ (z-\lambda)^{j/p(\lambda)} + A_\Omega(z)\ ,$$

where $A_\Omega(z)$ is analytic in Ω and p, a_k, N, q, b_k, M have obvious meaning. If we differentiate this representation, we get

$$f^{(n)}(z) = \sum_{k=-N}^{\infty} (-1)^n [a_k \Gamma(-kp^{-1}+n)/\Gamma(-kp^{-1})](z-a)^{(k/p)-n}$$

$$+ \sum_{k=-M}^{\infty} (-1)^n [b_k \Gamma(-kq^{-1}+n)/\Gamma(-kq^{-1})](z-b)^{(k/q)-n}$$

$$+ \sum_{\lambda\varepsilon\Lambda(\Omega)} \sum_{j=-N(\lambda)} (-1)^n [c_j(\lambda)\Gamma(-jp(\lambda)^{-1}+n)/\Gamma(-jp(\lambda)^{-1})](z-\lambda)^{(j/p(\lambda))-n}$$

$$+ A_\Omega^{(n)}(z)$$

$$= \frac{(-1)^n a_{-N}\Gamma(Np^{-1}+n)}{\Gamma(Np^{-1})(z-a)^{(N/p)+n}} \left\{1 + F_n(z) + G_n(z) + H_n(z) + I_n(z)\right\}\ ,$$

where

$$F_n(z) = \sum_{k=-N+1}^{\infty} \frac{a_k \Gamma(-kp^{-1}+n)\Gamma(Np^{-1})(z-a)^{(k+N)/p}}{a_{-N}\Gamma(-kp^{-1})\Gamma(Np^{-1}+n)}\ ,$$

$$G_n(z) = \sum_{k=-M}^{\infty} \frac{b_k \Gamma(-kq^{-1}+n)\Gamma(Np^{-1})(z-a)^{(N/p)+n}}{a_{-N}\Gamma(-kq^{-1})\Gamma(Np^{-1}+n)(z-b)^{-(k/q)+n}}\ , \tag{2.7}$$

$$H_n(z) = \sum_{\lambda\varepsilon\Lambda(\Omega)} \sum_{j=-N(\lambda)}^{\infty} \frac{c_j(\lambda)\Gamma(-jp(\lambda)^{-1}+n)\Gamma(Np^{-1})(z-a)^{(N/p)+n}}{a_{-N}\Gamma(-jp(\lambda)^{-1})\Gamma(Np^{-1}+n)(z-\lambda)^{-(j/p(\lambda))+n}}$$

$$I_n(z) = \frac{(-1)^n\Gamma(Np^{-1})(z-a)^{(N/p)+n}}{a_{-N}\Gamma(Np^{-1}+n)} A_\Omega^{(n)}(z) .$$

If we group $G_n(z)$ with $I_n(z)$ and factor out the term $[(z-a)/(z-b)]^n$, we get the expression

$$G_n(z) + I_n(z) = \left[\sum_{k=-M}^{\infty} \frac{b_k\Gamma(-kq^{-1}+n)\Gamma(Np^{-1})}{a_{-N}\Gamma(-kq^{-1})\Gamma(Np^{-1}+n)} (z-a)^{N/p}(z-b)^{k/q} \right.$$

$$\left. + \frac{(-1)^n\Gamma(Np^{-1})(z-a)^{N/p}(z-b)^n}{a_{-N}\Gamma(Np^{-1}+n)} A^{(n)}(z) \right] \left(\frac{z-a}{z-b}\right)^n$$

$$= [U_n(z) + V_n(z)] \left(\frac{z-a}{z-b}\right)^n ,$$

where $U_n(z)$ and $V_n(z)$ denote the respective terms in the brackets. Let T be a compact neighborhood of ξ, with ξ in its interior and

$$T \subset \{|z-a| < (R/2)\} \cup \{|z-b| < (R/2)\} - \bigcup_{\lambda\varepsilon\Lambda(\Omega)} \ell(\lambda) .$$

We calculate the asymptotic form of $U_n(z)$ as $n \to \infty$, $z\varepsilon T$. Write $U_n(z)$ in the form

$$U_n(z) = \left[\frac{b_{-M}\Gamma(Mq^{-1}+n)}{a_{-N}\Gamma(Np^{-1}+n)} \frac{\Gamma(Np^{-1})(z-a)^{N/p}}{\Gamma(Mq^{-1})(z-b)^{M/q}} + \right.$$

$$\left. + \sum_{k=-M+1}^{\infty} \frac{b_k\Gamma(Np^{-1})\Gamma(-kq^{-1}+n)}{b_{-M}\Gamma(-kq^{-1})\Gamma(Mq^{-1}+n)} (z-a)^{N/p}(z-b)^{k/q} \right].$$

We proceed to treat the above infinite sum inside the brackets just as in Lemma 1; i.e., we break it up into summands with ranges $-M \le k \le -1$, $1 \le k \le [(n-1)q]$ and $[(n-1)q] + 1 \le k < \infty$. The $k^{\underline{th}}$ term in the first range has dominant asymptotic form $n^{-(k+M)/q} \to 0$, $n \to \infty$, since $k \ge -M+1$. In the middle range, take note of (2.1) and (2.2) to see that the significant terms asymptotically are

$$\sum_{k=1}^{[(n-1)q]} |b_k| \; |C_{-k,n}/C_{1,n}| \; |C_{1,n}/C_{M,n}| \; |z - b|^{k/q}$$

$$\leq J \sum_{k=1}^{[(n-1)q]} |b_k| \; \delta_n \; n^{(1-M)/q} \; |z - b|^{k/q}$$

$$\leq J\Delta(q) \; n^{-M/q} \sum_{k=1}^{\infty} |b_k| \; |z - b|^{k/q} \; .$$

where J is a constant and the terms of (2.1) now depend on q instead of p. This term tends to 0 with $n^{-M/q}$, in view of the fact that $\Sigma_{k=1}^{\infty} |b_k| \; |z-b|^{k/q}$ is uniformly bounded on T. The sum in $U_n(z)$ over the range $[(n-1)q] + 1 \leq k < \infty$ behaves like

$$\sum_{k=[(n-1)q]+1}^{\infty} |b_k| \; |C_{-k,n}/C_{1,n}| \; |C_{1,n}/C_{M,n}| \; |z - b|^{k/q}$$

$$\leq J \, n^{(1-M)/q} \sum_{k=n}^{\infty} \binom{k}{n} \; |b_k| \; |z - a|^{k/q} \; ,$$

and the rest of the proof runs as in Lemma 1. Consequently, we may say that

$$U_n(z) = \frac{b_{-M}\Gamma(Mq^{-1}+n)}{a_{-N}\Gamma(Np^{-1}+n)} \left[\frac{\Gamma(Np^{-1})(z-a)^{N/p}}{\Gamma(Mq^{-1})(z-b)^{M/q}} + o(1) \right] \; , \quad n \to \infty \; ,$$

where o(1) denotes a term which converges uniformly to 0 in T.

Since T lies in $|z - b| < (R/2)$, we may treat the term $V_n(z)$ as we did the function $A_D(z)$ in (2.6). That is,

$$|V_n(z)| = \left| \frac{\Gamma(Np^{-1})\Gamma(1+n)}{a_{-N}\Gamma(Np^{-1}+n)} \cdot \frac{A_\Omega^{(n)}(z)(z - b)^n}{n!} (z - a)^{N/p} \right|$$

$$= 0 \left\{ n^{1-(N/p)} \left(\frac{\gamma}{1 - \gamma} \right)^n \right\} \; , \; n \to \infty \; , \; z \varepsilon T \; ,$$

which means that $V_n(z) \to 0$ uniformly in T. Moreover, if we let $W_n(z) = U_n(z) + V_n(z)$, it then follows that

$$G_n(z) + I_n(z) = W_n(z) \; \left(\frac{z - a}{z - b}\right)^n$$

where

$$\lim_{n\to\infty} \frac{W_n(z)}{n^{(M/q)-(N/p)}} = W \frac{(z-a)^{N/p}}{(z-b)^{M/q}} \quad (W = \text{constant}) \tag{2.8}$$

and the convergence is uniform in T.

The functions $F_n(z)$ and $H_n(z)$ in (2.7) have the same form as $F_n(z)$ and $G_n(z)$ in (2.5). It is easy to see that the arguments of Lemma 1 carry over to show that $F_n(z) \to 0$ and $H_n(z) \to 0$ in (2.7), uniformly in T.

If we let

$$f_n(z) = \frac{(-1)^n \Gamma(Np^{-1})(z-a)^{(N/p)+n}}{a_{-N}\Gamma(Np^{-1}+n)} f^{(n)}(z) \ , \quad z\varepsilon T$$

then $f_n(z) = 0$ if and only if $f^{(n)}(z) = 0$, and the foregoing discussion yields the representation

$$f_n(z) = 1 + F_n(z) + H_n(z) + W_n(z) \left(\frac{z-a}{z-b} \right)^n , \tag{2.9}$$

where (2.8) holds, and moreover,

$$F_n(z) + H_n(z) = o(1), \quad n \to \infty , \quad z\varepsilon T \ . \tag{2.10}$$

All terms in (2.9) are analytic in T.

Let us now consider the covering properties of the mapping $w = (z-a)/(z-b)$ for z near ξ. Recall that ξ lies on the perpendicular bisector of the line segment from a to b. Moreover, we may assume without loss of generality that $\xi \neq (a+b)/2$. The image $\omega = e^{i\theta}$ of ξ under the map lies on the unit circle in the w plane. In fact, the image of the perpendicular bisector is the circle $|w| = 1$. Note that $\omega \neq 1$ and $\omega \neq -1$ (the image of $(a + b)/2$) so that we can surround ω with a sectorial neighborhood S_ω: $r < |w| < r^{-1}$, $\theta - \delta < \arg w < \theta + \delta$, where $0 < r < 1$ and $\delta > 0$, whose image omits the points $w = 1$ and $w = -1$. The inverse image of S_ω under the mapping $w = (z-a)/(z-b)$ is the region S_ξ in the z-plane which contains ξ and is bounded by the four circles which are the inverse images of the circle $|w| = r$, $|w| = r^{-1}$ and lines $\arg w = \theta + \delta$,

$\arg w = \theta - \delta$; see Figure 1. Let $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ denote the boundaries of S_ω formed respectively by appropriate portions of $|w| = r$, $\arg w = \theta - \delta$, $|w| = r^{-1}$ and $\arg w = \theta + \delta$, and let $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ be the corresponding inverse images forming the boundaries of S_ξ. Finally, suppose that δ and $(1 - r)$ are sufficiently small that $S_\xi \subset T$.

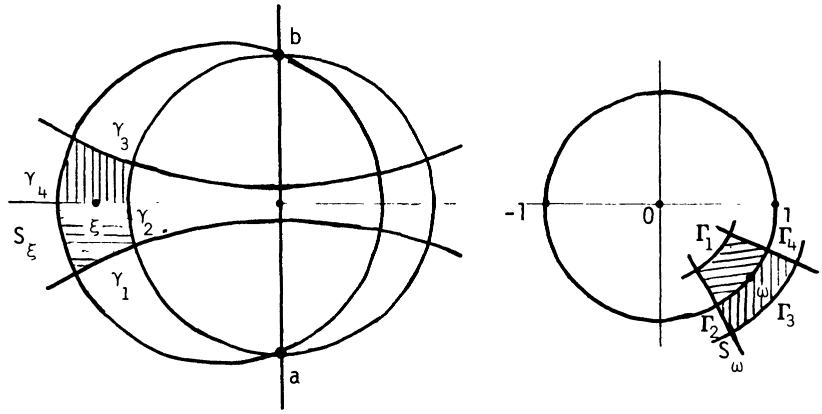

Figure 1

Let $\varepsilon > 0$. By (2.8) we can make the argument of $W_n(z)$, as z ranges over a small T, vary by less than ε by taking n sufficiently large; i.e., $\Delta_T \arg W_n(z) < \varepsilon$ for all large n. Now as z ranges over γ_1, $w = (z-a)/(z-b)$ traces out the circular arc Γ_1 on $|w| = r$. Thus we can make $[(z-a)/(z-b)]^n$ wrap around $|w| = r$ as many times as desired by taking n large. Thus for suitably large n, the variation of the argument of $W_n(z)[(z-a)/(z-b)]^n$ for $z \varepsilon \gamma_1$ can be made large. Specifically, we require $\Delta_{\gamma_1} W_n(z)[(z-a)/(z-b)]^n > 4\pi + 4\varepsilon$, for all large n.

On γ_2 and γ_4, $(z-a)/(z-b)$ has constant argument, so we may impose $\Delta_{\gamma_2} \arg W_n(z)[(z-a)/(z-b)]^n < \varepsilon$ and $\Delta_{\gamma_4} \arg W_n(z)[(z-a)/(z-b)]^n < \varepsilon$ simply by taking n large enough.

On γ_1, $|(z-a)/(z-b)| = r < 1$, whereas $W_n(z)$ has order of magnitude $n^{(M/q)-(N/p)}$ by (2.8). Thus if n is large enough, $|W_n(z)[(z-a)/(z-b)]^n| < (3/4)$ for all $z \varepsilon \gamma_1$. Similarly, $|W_n(z)[(z-a)/(z-b)]^n| > (5/4)$ for all $z \varepsilon \gamma_3$ when n is large.

Taking into account the arguments and moduli restricted as above, we see that the image of S_ξ under the map $W_n(z)[(z-a)/(z-b)]^n$ contains the doubly-covered annulus $(3/4) < |w| < (5/4)$. Hence the image of T under $W_n(z)[(z-a)/(z-b)]^n$ contains a disc centered at -1 of radius (1/4) (in fact, the image contains a disc of radius (1/4) centered at *each* point of $w = 1$).

All that remains is to make n large enough that, say $|F_n(z)| < (1/16)$ and $|H_n(z)| < (1/16)$ for $z \varepsilon T$. Then by (2.9) and Rouché's Theorem, $f_n(z)$ maps T onto a region which contains a disc centered at 0 of radius (1/8). Thus $f_n(z)$ has zeros in T for all n sufficiently large, and this finishes Case I.

CASE II. *ξ lies on a branch line.* We suppose that ξ lies on the common boundary of a-shire and b-shire only, where Re(a) = Re(b). We represent f(z) exactly as in Case I, where this time at least one $\lambda \varepsilon \Lambda(\Omega)$ has imaginary part (a + b)/2. By hypothesis, a and b have unequal strength, and we assume without loss of generality that b is stronger than a; i.e., either $Mq^{-1} > Np^{-1}$, or $|b_{-M}| > |a_{-N}|$ if $Mq^{-1} = Np^{-1}$.

In (2.9) it is no longer true that each term is analytic in a neighborhood of ξ, for at least one term in the sum $H_n(z)$ has a branch line through ξ. However, all terms in (2.9) are analytic in the lower half S^*_ξ of S_ξ, and it is easy to see that (2.8) and (2.10) hold uniformly over S^*_ξ.

We now proceed with a covering type argument nearly identical to Case I, but for the half neighborhood S^*_ξ. Let $\gamma^*_1 = \gamma_1$, let γ^*_2 and γ^*_4 be the lower halves of the arcs γ_2 and γ_4, and let γ^*_3 be the segment of the perpendicular bisector joining γ^*_2 and γ^*_4. Thus, γ^*_1, γ^*_2, γ^*_3, γ^*_4 form the boundary of S^*_ξ. Let Γ^*_1, Γ^*_2, Γ^*_3, Γ^*_4 be the corresponding parts of the boundary of the image S^*_ω of S^*_ξ under $w = (z-a)/(z-b)$.

The function $W_n(z)$ is analytic on S^*_ξ and continuous on its closure. As before, we make $\Delta_{S^*_\xi} \arg W_n(z) < \varepsilon$ and $\Delta_{\gamma^*_1} \arg W_n(z)[(z-a)/(z-b)]^n > 4\pi + 4\varepsilon$ by choosing n large enough. Moreover, $(z-a)/(z-b)$ still has constant argument on γ^*_2 and γ^*_4, and therefore $\Delta_{\gamma^*_2} \arg W_n(z)[(z-a)/(z-b)]^n < \varepsilon$, $\Delta_{\gamma^*_4} \arg W_n(z)[(z-a)/(z-b)]^n < \varepsilon$ if n is taken large enough.

As for what happens on γ^*_3, we consult (2.8) and note the constant W which comes from $U_n(z)$. If $Mq^{-1} > Np^{-1}$, then $W_n(z) \to \infty$, $n \to \infty$, everywhere on the closure of S^*_ξ.

If $Mq^{-1} = Np^{-1}$ then $|b_{-M}| > |a_{-N}|$ and it follows that $|W| > 1$. On γ_3^*, $|z-a| = |z-b|$ and so all we can say is that for some constant $\eta > 0$, $W_n(z)[(z-a)/(z-b)]^n > 1 + \eta$ for all $z \varepsilon \gamma_3^*$ and n sufficiently large. As before, $|W_n(z)[(z-a)/(z-b)]^n| < 1 - \eta$ for all $z \varepsilon \gamma_1^*$ when n is large. Consequently, the image of S_ξ^* under the mapping $W_n(z)\ [(z-a)/(z-b)]^n$ contains a doubly covered annulus $1 - \eta < |w| < 1 + \eta$. In particular, the image contains a disc centered at -1 of radius η.

Now we make $|F_n(z)| < (\eta/4)$ and $|H_n(z)| < (\eta/4)$ for $z \varepsilon S_\xi^*$ and n sufficiently large. Then $f_n(z) = 1 + F_n(z) + H_n(z) + W_n(z)[(z-a)/(z-b)]^n$ maps S_ξ^* over a disc centered at 0 of radius $(\eta/2)$ which completes the proof in Case II.

This completes the proof of theorem 1.

REMARKS.

1. If f(z) is meromorphic then a trivial modification of Case I gives a proof of the Polya shire theorem, without restriction on the location or strength of poles. The argument shows, moreover, that we can make the image S_ξ under $W_n(z)[(z-a)/(z-b)]^n$ wrap around the annulus $(3/4) < |w| < (5/4)$, any given number of times by taking n large enough. Having specified a positive integer k, then, all the derivatives beyond a certain point will have at least k zeros in S_ξ. Note that the further away ξ is from $(a-b)/2$, the further ω is away from -1. Loosely speaking, this means n will have to be taken relatively larger in order that the image of S_ξ under the map $W_n(z)[(z-a)/(z-b)]^n$ contain the point -1. In otherwords, the further out a point is from the poles, the longer one has to wait for the appearance of zeros of derivatives.

Consider a meromorphic function with poles a and b, where b is stronger than a. Arguing as in Case II, we see that the image of the upper half S_ξ^{**} of S_ξ is pushed out away from the disc $|w| \le 1$ by $W_n(z)[(z-a)/(z-b)]^n$. Therefore S_ξ^{**} will be free of zeros of $f^{(n)}(z)$ for n large enough. Put another way, we may say that pole b, being stronger than pole a, pushes the zeros of $f^{(n)}(z)$ past the perpendicular bisector, even though the cluster points of zeros remain on the bisector. Note that meromorphic functions real on the real axis, whose derivatives have zeros occurring in conjugate pairs, have poles which occur in conjugate pairs, with the same orders, and with conjugate Laurent coefficients.

2. The choice of the principal branch in Theorem 1 was only for convenience. Any fixed branch of f(z) would serve as well, with minor modifications in the proof. We would have to retain the hypothesis that any bounded region intersects only finitely many branch lines.

3. If f(z) has a single algebraic pole, the final set is empty. This is a special instance of Case I. Moreover, the results of [4] regarding the asymptotic size of zero-free regions created by a single singularity hold with minor modifications.

4. If we alter the definition of the final set so as to allow all zeros of all derivatives of all branches, then the conclusion of Theorem 1 holds without restriction on the location or strength of singularities. Even if infinitely many branch lines coincide with a perpendicular bisector, we may say that *there exists* a branch of f(z) whose branch lines are elsewhere. Hence Case II is entirely avoided.

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