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A TEMPERATURE CONTROL PROBLEM

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ABSTRACT. In an open bounded region of n-space occupied by a homogeneous and isotropic medium, we control the temperature through the boundary. The normal derivative of the temperature (which measures the appropriate heat flux) is restricted to be nonnegative. This gives rise to a free boundary in space-time separating the areas of positive and zero heat flux. Under a natural monotonicity condition, the free boundary is the graph of a function of space. This function is shown to be locally Lipschitz. Moreover for n=2 the time derivative of the temperature is proven to be continuous across the free boundary.

KEY WORDS AND PHRASES. *Variational inequalities, free boundaries.*
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1. INTRODUCTION

Let $\Omega \subset \mathbf{R}^n$ be open, bounded and have smooth boundary $\partial\Omega$. Suppose that Γ, a subset of the boundary, lies on the hyperplane $\mathbf{R}^{n-1}$. Let u be a solution to

$$\begin{cases} Hu \equiv \Delta u - \dfrac{\partial u}{\partial t} = 0 \quad \text{in} \quad \Omega \times (0,\infty) \\ u(X,t) = f(X,t) \quad \text{on} \quad (\partial\Omega - \Gamma) \times (0,\infty) \\ u \geq 0\,, \quad \dfrac{\partial u}{\partial \nu} \geq 0 \quad u \cdot \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma \times (0,\infty) \\ u(X,0) = u_o(X) \end{cases} \qquad (*)$$

where $X \in \mathbf{R}^n$, ν is the unit outward normal and f, u_o given.

J.L. Lions and G. Stampacchia showed in [5] that (*) can be formulated as a variational inequality and obtained a unique solution in the appropriate function space. Regularity of the solution was considered by the author in [2]. He proved there that $u \in C_X^{1,\alpha}(\bar{\Omega} \times (0,\infty))$ $(0 < \alpha < 1)$ and $u \in C_t^{o,1}(\bar{\Omega} \times (0,\infty))$ under the assumptions $u_o \in C^2$ and f is Lipschitz in time and twice continuously differentiable in space with $f > 0$ on $\partial\Gamma \times (0,\infty)$.

Throughout this note we shall assume, together with the above, that $\partial f/\partial t$ and Δu_o are nonnegative.

2. FREE BOUNDARY.

Let Ω^* be the reflection of Ω about $\mathbf{R}^{n-1}$. Put $\bar{D} = \bar{\Omega} \cup \overline{\Omega^*}$. As in [2] the solution u to (*) can be extended across $\mathbf{R}^{n-1}$ in such a way that $Hu = 0$ in $D-\{u|_\Gamma = 0\}$ and u satisfying the relevant conditions.

A careful examination of the proof of Lemma 1(i) in [2] proves the following

LEMMA 1. $\partial u/\partial t$ is nonnegative and subcaloric (i.e., $Hu_t \geq 0$) in $D \times (0,\infty)$.

Let $X = (x,y)$ where $x \in \mathbf{R}^{n-1}$ and $y \in \mathbf{R}$. Set

$$\phi(x) = \sup\{t\colon u(x,0,t) = 0,\ x \in \Gamma\}$$

then $\{t = \phi(x)\}$ is a well-defined function which we call the free boundary.

THEOREM 1. $t = \phi(x)$ is locally Lipschitz.

PROOF. Let $B'_\kappa(x_o)$ denote an (n-1)-dimensional ball of radius κ centered at x_o. Set

$$\eta(x,y,t) = \frac{2}{\mu}\left(t + \frac{1}{2}\mu - t_o\right)^- + \zeta(x) - \frac{1}{\lambda^2}y^2$$

where

$$\zeta(x) = \frac{4}{\kappa^2}\left[\left(|x-x_o| - \frac{1}{2}\kappa\right)^+\right]^2$$

with

$$(g)^+ = \begin{cases} g & \text{if} \quad g > 0 \\ 0 & \text{if} \quad g < 0 \end{cases}, \qquad (g)^- = |g| - (g)^+ .$$

Choose $\lambda < ((\kappa^2\mu)/4\mu + \kappa^2))^{1/2}$ so that

$$H\eta \leq 0 \quad \text{in} \quad C(\kappa,\lambda,\mu)$$

where $C(\kappa,\lambda,\mu) := B'_\kappa(x_o)\times(-\lambda,\lambda)\times(t_o-\mu,t_o]$. For $\tau \geq 0$ and $\xi \in \mathbf{R}^{n-1}$, consider the function

$$w(x,y,t) = \tau u_t + \xi \cdot u_x - u + \varepsilon\eta$$

where $u_x = (u_{x_1},\ldots,u_{x_{n-1}},0)$ and $0 < \varepsilon << \lambda$. Since

$$Hw = \varepsilon H\eta \quad \text{in} \quad C(\kappa,\lambda,\mu) - \{u(x,0,t) = 0\}$$

by minimum principle, the minimum of w is attained on $\partial'(C(\kappa,\lambda,\mu)-\{u(x,0,t) = 0\})$ (∂'-parabolic boundary). First suppose it occurs on $(\partial'C(\kappa,\lambda,\mu)-\{u(x,0,t) = 0\}) \cap N_\delta$ where N_δ denotes a δ-neighborhood of $\{u(x,0,t) = 0\}$ in $\mathbf{R}^{n-1}$ (δ to be chosen), then using the regularity of u and Lemma 1,

$$w = \tau u_t + \xi \cdot u_x - u + \varepsilon\eta \geq -C|\xi|\kappa^\alpha - C\delta^{1+\alpha} + \varepsilon\left(1 - \frac{\delta^2}{\lambda^2}\right) .$$

Choosing $|\xi| < (C\kappa^\alpha)^{-1}(\varepsilon/4)$ and $\delta < (C^{-1}(\varepsilon/4))^{1/(1+\alpha)}$ we have $w \geq 0$. Now, suppose it occurs on $\partial'C(\kappa,\lambda,\mu)-N_\delta$, since $u_t > 0$ in $C(\kappa,\lambda,\mu)-N_\delta$,

$$w \geq \tau u_t - \frac{\varepsilon}{4} - u - \varepsilon$$

$$\geq \tau \cdot \inf\{u_t\colon (x,y,t) \in C(\kappa,\lambda,\mu) - N_\delta\} - \sup\{u\colon (x,y,t) \in C(\kappa,\lambda,\mu)\} - \frac{5}{4}\varepsilon$$

choosing τ large enough we see that $w \geq 0$. Finally, if it occurs on $\partial\{u(x,0,t) > 0\} \cap C(\kappa,\lambda,\mu)$,

$$w = \varepsilon\left[\left(t + \frac{1}{2}\mu - t_o\right)^- + \zeta(x)\right] > 0 .$$

In any case $w \geq 0$ in $C(\kappa,\lambda,\mu) - \{u(x,0,t) = 0\}$. Therefore,

$$\tau u_t + \xi \cdot u_x \geq u \geq 0 \quad \text{in} \quad C(\kappa/2,0,\mu/2) - \{u(x,0,t) = 0\} .$$

Hence if $(x,0,t)$ is a free boundary point there is a cone K of rays emanating from $(x,0,t)$ such that u is increasing.

qed.

The idea of using the solution as a barrier to the directional derivative was introduced by H.W. Alt in the "Dam Problem" (see [1]).

3. CONTINUITY.

In this section we assume that n=2.

THEOREM 2. u_t is continuous across the free boundary.

PROOF. Let $(x_o,0,t_o)$ be a free boundary point. By Theorem 1, there is a cone $K = \{(x,0,t)\colon t_o - t \geq c|x-x_o|\}$ for some $c > 0$ such that

$$u_t \equiv 0 \quad \text{in} \quad K \cap (B_\kappa(x_o,0) \times (t_o-\lambda,t_o])$$

for $\kappa,\lambda > 0$ and $B_\kappa(x_o,0)$ denotes a two-dimensional ball of radius κ centered at $(x_o,0)$. With no loss of generality let $\kappa = 1$ and $\lambda < c$. Let h be a harmonic function in $B_1(x_o,0) - \{x\colon |x-x_o| \leq \lambda/c\}$ with h=0 on $\partial B_1(x_o,0)$ and on $\{x\colon |x-x_o| \leq \lambda/c\}$. Also let Cap (L) denote the newtonian capacity of L. Then

$$\inf\{h(x,y)\colon (x,y) \in B_r(x_o,0)\} \geq C \operatorname{Cap}\{|x-x_o| \leq \lambda/c\}$$

for $\lambda/c < r < 1$.

Set $v = (1-u_t)/M_1$ where $M_1 = \sup_{C(1,\lambda)} u_t$ and $C(1,\lambda) = B_1(x_o,0)\times(t_o-\lambda,t_o]$. Observe that

$$v \geq h - h*F \quad \text{in} \quad C_1(1,\lambda)$$

where $h*F$ denotes the convolution of h with the fundamental solution F of the heat operator

$$v \geq Ch \quad \text{on} \quad B_{r_o}(x_o,0)\times\{t = t_o - \lambda/2\}$$

for $\lambda/c < r_o < r$. Since $Hv \leq 0$, $0 \leq v \leq 1$, by Green's representation formula,

$$v \geq Ch \quad \text{in} \quad B_{r_o/2}(x_o,0) \times (t_o - 1/4\lambda, t_o] .$$

i.e.,

$$\inf\{v(x,y,t):\ (x,y,t)\in C(r_o/2,\lambda/4)\} = -\frac{C}{\log\left(\frac{\lambda}{2c}\right)} .$$

Therefore suppose inductively we have proved that

$$\inf\{v(x,y,t):\ (x,y,t)\in C(r_o/2^{k+1},\lambda/(4(2)^{2k}))\} \geq C\ \mathrm{Cap}\left\{x:\ |x-x_o| \leq \frac{\lambda}{c}\,2^{-2k}\right\}$$

$$= \frac{C}{\log\left(\frac{\lambda}{2c}\cdot 2^{-2k}\right)} = \frac{C}{k}$$

$$\Rightarrow\quad M_{k+1} \leq M_k\left(1-\frac{C}{k}\right)$$

where $M_{k+1} := \sup\{u_t(x,y,t):\ (x,y,t)\in C(r_o/2^{k+1},\lambda/4\cdot 2^{2k})\}$. Therefore

$$M_{k_o} = \prod_{k=1}^{k_o}\left(1-\frac{C}{k}\right) .$$

Hence $M_k \to 0$ if $\prod_{k=1}^{\infty}(1-(C/k))$ goes to zero. But $\prod_{k=1}^{\infty}(1-(C/k))$ goes to zero if and only if $\sum_{k=1}^{\infty} C/k$ diverges.

qed.

The above proof fails for $n \geq 3$. The situation here is similar to the one-phase Stefan Problem considered by Caffarelli in [3] (see also [4]).

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