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ANOTHER NOTE ON ALMOST CONTINUOUS MAPPINGS AND BAIRE SPACES

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ABSTRACT. The following result is proved:

Let Y be a second countable, infinite topological space with an ascending chain of regular open sets. Then a topological space X is a Baire space if and only if every mapping f: $X \to Y$ is almost continuous on a dense subset of X.

It is another improvement of a theorem of Lin and Lin [2].

KEY WORDS AND PHRASES. Regular open set, almost continuous mapping, Baire space.
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1. INTRODUCTION.

In [1], the present author established a lemma by replacing Hausdorff space with R_0-space with an ascending chain of open sets. In this paper, a lemma is established which has the same conclusion under independent conditions without any assumption on separation, and it is used to give another improvement to a theorem of Lin and Lin [2].

2. MAIN RESULT.

An open set U in a topological space is a regular open set [3, p. 92] if $\mathrm{Int}(\bar{U}) = U$. Countably many regular open sets $0_1, 0_2, \ldots, 0_n, \ldots$ is called an ascending chain of regular open sets if $0_1 \subsetneq 0_2 \subsetneq \cdots \subsetneq 0_n \subsetneq \cdots$.

LEMMA 1. An infinite Hausdorff space has an ascending chain of regular open sets.

PROOF. By [4, Prob. 14, p. 147], we have a countably infinite subspace $\{y_1, y_2, \ldots, y_n, \ldots\}$ and disjoint open sets $U_1, U_2, \ldots, U_n, \ldots$ such that $y_n \in U_n$. Let $0_n = \mathrm{Int}(\overline{\bigcup_{i=1}^{n} U_i})$ $(n = 1,2,\ldots)$. Then from [2, p. 92] we know that 0_n are regular open sets. It is easily seen that $y_n \in 0_n$. Since U_i are disjoint, $y_n \notin \bar{U}_{n-k}$ $(k = 1,2,\ldots,n-1)$; hence, $y_n \notin 0_{n-1}$. Thus, $0_{n-1} \subsetneq 0_n$ where $\{0_n, n = 1,2,\ldots\}$ is an ascending chain of regular open sets.

The converse of Lemma 1 is not true.

EXAMPLE 1. Let $D = \{d_1, d_2, \ldots, d_n, \ldots\}$ be an infinite set of distinct points. a, b, c are distinct points not in D. Let $X = \{a,b,c\} \cup D$ with topology $\tau = \{N, \{a\} \cup N, \{a,b,c\} \cup N;$ N is a subset of D$\}$. Then $0_i = \{d_1, d_2, \ldots, d_i\}$ $(i = 1,2,\ldots)$ is an ascending chain of regular open sets. X is not T_0 since neither b nor c can be separated by open sets from the other. X is not R_0 since $\overline{\{a\}} = \{a,b,c\}$ doe not belong to any $\{a\} \cup N$.

In Example 1 of [1], X is the only regular open set. This shows that an R_0-space with an ascending chain of open sets does not imply the existence of an ascending chain of regular open sets; thus, the two conditions are independent.

LEMMA 2. Let X be an infinite space with an ascending chain of regular open sets. Then X contains a countably infinite discrete subspace.

PROOF. Let 0_i $(i = 1,2,\ldots)$ be an ascending chain of regular open sets. Then $V_n = 0_{n+1}/\bar{0}_n$ is a nonempty open set, otherwise $0_{n+1}/\bar{0}_n = \emptyset$ implies $0_{n+1} \subset \bar{0}_n$; hence, $0_{n+1} = \mathrm{Int}(0_{n+1}) \subset \mathrm{Int}(\bar{0}_n) = 0_n$, contradicting $0_n \subsetneq 0_{n+1}$. Now we prove that $\{V_n\}$ are disjoint. If $m > n$, then $V_m = 0_{m+1}/\bar{0}_m$, $V_m \cap \bar{0}_m = \emptyset$, but $0_{n+1} \subseteq 0_m$; hence, $V_m \cap \bar{0}_{n+1} = \emptyset$, $V_n \subset 0_{n+1}/\bar{0}_n \subset 0_{n+1}$. Therefore, $V_m \cap V_n = \emptyset$, $\{V_n;\ n = 1,2,\ldots\}$ are disjoint. Select a point $y_n \in V_n$ for $n = 1,2,\ldots$; then, $S = \{y_n;\ n = 1,2,\ldots\}$ is a countably infinite discrete subspace.

Now, Theorems 2 and 3 in [2] can be written as follows:

THEOREM 1. Let Y be an infinite space with an ascending chain of regular open sets. If X is a topological space such that every mapping $f\colon X \to Y$ is almost continuous on a dense subset of X, then X is a Baire space.

THEOREM 2. Let Y be a second countable infinite space with an ascending chain of regular open sets. Then a topological space X is a Baire space if and only if every mapping $f\colon X \longrightarrow Y$ is almost continuous on a dense subset of X.

REMARK 1. It is worth mentioning that, in Theorems 1 and 2, no separation property is required.

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