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COMMUTATIVITY THEOREMS FOR RINGS AND GROUPS WITH CONSTRAINTS ON COMMUTATORS

EVAGELOS PSOMOPOULOS

Department of Mathematics
University of Thessaloniki
Thessaloniki, Greece

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ABSTRACT. Let $n > 1$, m, t, s be any positive integers, and let R be an associative ring with identity. Suppose $x^t[x^n,y] = [x,y^m]y^s$ for all x, y in R. If, further, R is n-torsion free, then R is commutativite. If n-torsion freeness of R is replaced by "m, n are relatively prime," then R is still commutative. Moreover, example is given to show that the group theoretic analogue of this theorem is not true in general. However, it is true when $t=s=0$ and $m=n+1$.

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1. INTRODUCTION.

Throughout this note, R will be an associative ring with identity, Z the center R, N the set of all nilpotent elements of R, and $C(R)$ the commutator ideal of R. We set $[x,y] = xy-yx$.

Our objective is to prove the following

THEOREM 1. Let $n(> 1)$, m be positive integers and let t, s be any non-negative integers. Let R be an associative ring with identity. Suppose $x^t[x^n,y] = [x,y^m]y^s$ for all x, y in R. If, further, R is n-torsion free, then R is commutative.

In preparation for the proof of this theorem, we first establish the following lemmas.

LEMMA 1. Let R be a ring with 1, k any positive integer, and let x, y be in R.

(i) If $[x,[x,y]] = 0$, then $[x^k,y] = kx^{k-1}[x,y]$.

(ii) If $x^ky = 0 = (x+1)^ky$, then $y = 0$.

(iii) If $(m,n) = 1$ and $[x^n,y] = [x^m,y] = 0$, for all x in R, then $[x,y] = 0$.

This lemma is very well-known.

LEMMA 2. Under the hypotheses of the above theorem, every nilpotent element of R is central.

PROOF. It is a triviality to prove that hypothesis

$$x^t[x^n,y] = [x,y^m]y^s \quad \text{for all } x, y \text{ in } R \tag{1.1}$$

implies

$$x^{t'}[x^{n^2},y] = [x,y^{m^2}]y^{s'}, \text{ for all } x,y \text{ in } R \quad t'=nt+t,\ s'=2s \tag{1.2}$$

Let $x \in N$; then there exists a positive integer p, such that

$$a^k \in Z, \quad \text{for all } k \geq p, \ p \text{ minimal}. \tag{1.3}$$

Suppose $p > 1$. In (1.1), replace x by a^{p-1} to get

$$(a^{p-1})^t[(a^{p-1})^n,y] = [a^{p-1},y^m]y^s$$

which implies, in view of (1.3),

$$[a^{p-1},y^m]y^s = 0 \tag{1.4}$$

Now, in (1.1) replace x by $1+a^{p-1}$, to obtain

$$(1+a^{p-1})^t[(1+a^{p-1})^n,y] = [a^{p-1},y^m]y^s.$$

In view of (1.4), and the fact that $1+a^{p-1}$ is invertible, the last equation implies

$$[(1+a^{p-1})^n,y] = 0. \tag{1.5}$$

Combining (1.5) and 1.3), we see that

$$0 = [(1+a^{p-1})^n,y] = [1+na^{p-1},y] = n[a^{p-1},y].$$

Since R is n-torsion free, the last identity implies $[a^{p-1},y] = 0$, for all y in R, which contradicts the minimality of p. This contradiction shows that $p = 1$. Therefore, $N \subseteq Z$.

Now, observe that by [1, Theorem 1], C(R) is a nil ideal, since $x=e_{22}$ and $y=e_{21} + e_{22}$ fail to satisfy (1.1). Hence in view of Lemma 2, we obtain

$$C(R) \subseteq Z \tag{1.6}$$

PROOF OF THEOREM 1. In (1.1), replace x by $2x$ to get

$$2^{n+t}x^t[x^n,y] = 2[x,y^m]y^s.$$

Combining the last identity with (1.1), we obtain

$$2^{n+t}[x,y^m]y^s = 2[x,y^m]y^s. \tag{1.7}$$

In view of (1.6) and Lemma 1, (1.7) yields

$$2^{n+t}my^{m+s-1}[x,y] = 2my^{m+s-1}[x,y]$$

$$(2^{n+t}-2)my^{m+s-1}[x,y] = 0.$$

Then, if $k = (2^{n+t}-2)m(1+s)$, $[x,y^k] = ky^{k-1}[x,y] = 0$. Therefore,

$$x^k \in Z, \quad \text{for all } x \in R; \ k = (2^{n+t}-2)m(1+s). \tag{1.8}$$

Next, by (1.1) we obtain

$$x^t[x^n,y] = my^{m+s-1}[x,y].$$

Replace y by y^m in the above equation to get

$$x^t[x^n,y^m] = my^{m(m+s-1)}[x,y^m]$$

$$mx^t[x^n,y]y^{m-1} = my^{m(m+s-1)}[x,y^m].$$

Combining the last identity with (1.1) and (1.6), we obtain

$$mx[x,y^m]y^{m+s-1}(1-y^{(m-1)(m+s-1)}) = 0. \tag{1.9}$$

Multiply (1.9) by $y^{(m-1)(m+s-1)}$ to obtain
$$m[x,y^m]y^{m+s-1}(y^{(m-1)(m+s-1)} - y^{2(m-1)(m+s-1)}) = 0. \tag{1.10}$$
Adding together (1.9) and (1.10), we see that
$$m[x,y^m]y^{m+s-1}(1-y^{2(m-1)(M+s-1)}) = 0.$$
Continue this process k times (k being as in (1.8)) to obtain
$$m[x,y^m]y^{m+s-1}(1-y^{k(m-1)(M+s-1)}) = 0. \tag{1.11}$$

It is well known that R is isomorphic to a subdirect sum of subdirectly irreducible rings R_i $(i\varepsilon\gamma)$. Each R_i satisfies (1.2), (1.6), (1.8), and (1.11), but R_i is <u>not necessarily n-torsion free</u>.

We consider the ring R_i $(i\varepsilon\gamma)$. Let S be the intersection of all non-zero ideals of R_i. Then, it can be easily verified
$$Sd = 0, \text{ for all } \underline{\text{central}} \text{ zero divisors } d \tag{1.12}$$

If a is any zero divisor of R_i, then
$$m[x,a^m]a^{m+s-1}(1-a^{k(m-1)(m+s-1)}) = 0.$$
Thus,
$$m[x,a^m]a^{m+s-1} = 0 \tag{1.13}$$
For if $m[x,a^m]a^{m+s-1} \neq 0$, $1-a^{k(m-1)(m+s-1)}$ will be a central (see (1.8)) zero divisor and by (1.12), $0 = S(1-a^{k(m-1)(m+s-1)}) = S$, a contradiction. Combining (1.2) and (1.13), we see that
$$x^{t'}[x^{n^2},a] = [x,a^{m^2}]a^{s'} = m[x,a^m]a^{m(m-1)+s'} = 0.$$
Hence by Lemma 1,
$$n^2x^{n^2+t'-1}[x,a] = x^{t'}[x^{n^2},a] = 0.$$
Replacing x by $x+1$ in the last identity and using Lemma 1, we obtain $n^2[x,a] = 0$, which yields $[x^{n^2},a] = n^2x^{n^2-1}[x,a] = 0$. Therefore,
$$[x^{n^2},a] = 0, \text{ for all } x \text{ in } R_i, \text{ and all zero divisors } a \text{ of } R_i. \tag{1.14}$$

Next, let c be any central element of R_i. In (1.1), replace x by cx to get
$$c^{n+t}x^t[x^n,y] = c[x,y^m]y^s = cx^t[x^n,y]$$
$$(c^{n+t}-c)x^t[x^n,y] = 0.$$
Apply once more Lemma 1 to obtain
$$n(c^{n+t}-c)x^{n+t-1}[x,y] = 0.$$
If we replace x by $x+1$, and apply Lemma 1, we finally get $n(c^{n+t}-c)[x,y] = 0$, which implies
$$(c^{n+t}-c)[x^n,y] = 0, \text{ for all } x,y\varepsilon R_i, \text{ and any central element } c \text{ of } R_i. \tag{1.15}$$
In particular,
$$(y^{k(n+t)}-y^k)[x^n,y] = 0 \text{ for all } x,y\varepsilon R_i. \tag{1.16}$$

Now, let $y \varepsilon R_i$. If $[y, x^{n^2}] = 0$, then clearly $[y^q - y, x^{n^2}] = 0$ for all positive integers q. If $[y, x^{n^2}] \neq 0$, then $[y, x^n] \neq 0$. For $[x^n, y] = 0$ implies $[y, x^{n^2}] = 0$, a contradiction. Since $[x^n, y] \neq 0$, (1.16) implies that $y^{k(n+t)} - y^k$ is a zero divisor. Therefore, $y^{k(n+t-1)+1} - y$ is also a zero divisor. Hence, (1.14) implies

$$[y^p - y, x^{n^2}] = 0 \text{ for all } x, y \varepsilon R_i;\ p = k(n+t-1)+1 \tag{1.17}$$

Since each R_i $(i \varepsilon \gamma)$ satisfies (1.17), the original ring R also satisfies (1.17). But R is n-torsion free. Thus, combining (1.17) and Lemma 1, we finally obtain

$$[y^p - y, x] = 0, \text{ for all } x, y \varepsilon R,$$

which implies commutativity of R by Herstein's theorem [3].

2. If we replace, in Theorem 1, hypothesis "R is n-torsion free" by the condition "n and m are relatively prime," the ring R is still commutative.

THEOREM 2. Let n, m be relatively prime positive integers, and let t,s be any non-negative integers. Suppose R is an associative ring with identity satisfying $x^t[x^n, y] = x, y^m]y^s$ for all x,y in R. Then R is commutative.

PROOF. Here, without loss of generality, we assume that R is subdirectly irreducible.

Let $a \varepsilon N$. Following the same argument as in Theorem 1, we prove (see (1.5)) that $n[a^{p-1}, y] = 0$ for all $y \varepsilon R$; similarly, we can prove that $m[a^{p-1}, y] = 0$ for all $y \varepsilon R$. Since $(m,n) = 1$, we obtain

$$C(R) \subseteq N \subseteq Z. \tag{2.1}$$

Note that the proof of (1.8) also works in the present situation, so that there exists k for which

$$x^k \varepsilon Z \text{ for all } x \varepsilon R. \tag{2.2}$$

Furthermore, as in the proof of Theorem 1 we obtain $[x^{n^2}, a] = 0$ for all $x \varepsilon R$ and all zero divisors a (see (1.14)); similarly $[x^{m^2}, a] = 0$. Thus, the last part of Lemma 1 yields

$$[x, a] = 0 \text{ for all } x \varepsilon R \text{ and all zero divisors } a. \tag{2.3}$$

As we observed in the paragraph following (1.14), we have $n(c^{n+t} - c)[x, y] = 0$ for all $x, y \varepsilon R$ and all $c \varepsilon Z$; and a variation of the argument yields $m(c^{n+t} - c)[x, y] = 0$ as well. Thus

$$(c^{n+t} - c)[x, y] = 0 \text{ for all } x, y \varepsilon R \text{ and all } c \varepsilon Z. \tag{2.4}$$

Using (2.2) to substitute y^k for c, we complete the proof by arguing as in the previous proof that $y^{k(n+t-1)+1} - y \varepsilon Z$ for all $y \varepsilon R$. Hence, R is commutative by Herstein's theorem [3].

3. A close look at the symmetric group S_3 with $t=s=6$, $n=7$ and $m=1$ shows that S_3 satisfies the identity $x^t[x^n,y] = [x,y^m]y^s$, but, as it is well known, S_3 is not abelian. Hence, Theorem 2 is not true for groups in general. However, we prove the following:

THEOREM 3. Let G be a multiplicate group, n an arbitrary positive integer, and suppose $[x^n,y] = [x,y^{n+1}]$ for all x,y in G. Then G is abelian.

PROOF: In hypothesis, replace x by xy to obtain

$$[(xy)^n,y] = [xy,y^{n+1}]. \tag{3.1}$$

A direct calculation shows that $[xy,y^{n+1}] = [x,y^{n+1}]$. Combining this with hypothesis and (3.1) we see that $[(xy)^n,y] = [x^n,y]$. Replace y by $x^{-1}y$, in the last equation to get

$$[y^n,x^{-1}y] = [x^n,x^{-1}y]. \tag{3.2}$$

A direct calculation shows that $[y^n,x^{-1}y] = [x^n,x^{-1}]$, and $[y^n,x^{-1}y] = x^{-1}[x^n,y]x$. Thus (3.2) yields $[y^n,x^{-1}] = x^{-1}[x^n,y]x$, which yields

$$x[y^n,x^{-1}] = [x^n,y]x = [x,y^{n+1}]x.$$

Hence,

$$xy^{n+1}x^{-1}y^{-n-1}x = xy^nx^{-1}y^{-n}x$$

and after cancellations $yx^{-1}y^{-1} = x^{-1}$, which implies $xy = yx$. Hence, G is abelian.

4. We conclude with the following

REMARK. As a corollary to Theorem 1, with $t=s=0$ and $m=n$, we obtain the following result of Bell [2,Theorem 5]:

COROLLARY. Let R be a ring with 1 and $n>1$ a fixed positive integer. If R is n-torsion free and R satisfies the identity $x^ny-yx^n = xy^n-y^nx$, then R is commutative.

Also, Theorem 1 generalizes a result of E. Psomopoulos, H. Tominaga, and A. Yaqub [4, Theorem 2].

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