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A PROPERTY OF L-L INTEGRAL TRANSFORMATIONS

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ABSTRACT. The main result of this paper is the result that the collection of all integral transformations of the form $F(x) = \int_0^\infty G(x,y)f(y)dy$ for all $x \geq 0$, where $f(y)$ is defined on $[0,\infty)$ and $G(x,y)$ defined on $D = \{(x,y): x \geq 0, y \geq 0\}$ has no identity transformation on L, where L is the space of functions that are Lebesgue integrable on $[0,\infty)$ with norm $\| f \| = \int_0^\infty |f(x)| dx$. That is to say, there is no $G(x,y)$ defined on D such that for every $f \in L$, $f(x) = \int_0^\infty G(x,y)f(y)dy$ for almost all $x \geq 0$. In addition, this paper gives a theorem that is an improvement of a theorem that is proved by J. B. Tatchell (1953) and Sunonchi and Tsuchikura (1952).

KEY WORDS AND PHRASES. *L-L Integral Transformation. Absolutely continuity of integrals. Lebesgue measurable. Lebesgue points.*
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1. INTRODUCTION.

The well known property of sequence to sequence transformations of the form $(Ax)_n = \sum_{k\geq 1} a_{nk}x_k$ for which there is an identity mapping such that $(Ax)_n = x_n$, does not carry over to the function to function transformations of the form $F(x) = \int_0^\infty G(x,y)f(y)dy$, $x \geq 0$. There is no identity mapping on L of the collection of all transformations of the form $F(x) = \int_0^\infty G(x,y)f(y)dy$. This is the main theorem of this paper. We list it as Theorem 2.

A result by Knopp and Lorentz [4] on sequence to sequence transformations of the form $(Ax)_n = \sum_{k\geq 1} a_{nk}x_k$ gives necessary and sufficient condition for the sequence $(Ax)_n$ to be defined and $\sum_{k\geq 1} |(Ax)_k|$ convergent whenever $\sum_{k\geq 1} |x_k|$ is convergent. Knopp and Lorentz proved that A is an ℓ-ℓ matrix if and only if there is a number M such that for each K, $\sum_{n\geq 1} |a_{nk}| < M$. Sunonchi and Tsuchikura [2] gives a similar result on function to function transformations of the form $F(x) = \int_0^\infty G(x,y)f(y)dy$, where x is a real variable and the kernel function $G(x,y)$ is assumed to be a measurable function on the plane $x \geq 0$, $y \geq 0$. Then $\int_0^\infty |F(x)| dx < \infty$ whenever $\int_0^\infty |f(y)| dy < \infty$ if and only if $\text{L.u.b.}_{y \geq 0} \int_0^\infty |G(x,y)| dx < \infty$. There are, however, nonmeasurable functions $G(x,y)$ which define summable function $F(x)$. J. B. Tatchell has recently found the conditions for $F(x)$ to be defined and $F(x) \in L$ whenever

$f(y) \in L$. We will first establish Theorem 1 and then use it in the proof of Theorem 2. Theorem 1 is an improvement of Tatchell's theorem.

2. NOTATION.

Although some of the symbols are the standard ones that are familiar to the reader, others are introduced here for the specific purpose of this paper.

The statement that f is integrable on $[0,\infty)$ in some conditionally convergent sense means that for every $u > 0$, f is integrable on $[0,u]$ and that $\int_0^u f(x)\,dx$ tends to a finite limit as $u \to \infty$.

L - the space of functions that are Lebesgue integrable on $[0,\infty)$ with norm $\| f \| = \int_0^\infty |f(x)|\,dx$.

D - the first quadrant of the plane, i.e., $D = \{(x,y): x \geq 0,\ y \geq 0\}$.

G - an integral transformation, $G: f \to F$, of the form

$$(*) \qquad F(x) = \int_0^\infty G(x,y)f(y)\,dy, \quad \text{for all} \quad x \geq 0,$$

where f is defined on $[0,\infty)$ and $G(x,y)$ defined on D.

L_G - the inverse image of L under the integral transformation G of the form (*).

$\mathcal{G}$ - the collection of all G of the form (*).

$\mathcal{G}L$ - the subcollection of $\mathcal{G}$ such that $F \in L$ whenever $f \in L$, i.e., $\mathcal{G}L = \{G \in \mathcal{G}:\ L \subseteq L_G\}$.

L^∞ - the space of functions which are measurable and essentially bounded on $[0,\infty)$ with norm

$$\| f \| = \text{ess} - \sup_{x \geq 0} |f(x)| \quad .$$

Q - the set of nonnegative rational numbers.

3. MAIN THEOREM.

The first theorem is an improvement of Theorem 2 of J. B. Tatchell [1] and Theorem 1 of Sunonchi and Tsuchikura [2]. We will refer to that Theorem (T.S.T.). Next a lemma is used to justify an inversion in an order of integration in the proof of Theorem (T.S.T.).

LEMMA. If $G(x,y)$ is a function of y summable on every finite interval in $[0,\infty)$ whenever $x \geq 0$, and if

$$g(x,t) = \int_0^t G(x,y)\,dy$$

is a function of x measurable on $[0,\infty)$ whenever $t \geq 0$, then

$$G_*(x,t) = \lim_{h \to 0} \inf \frac{1}{h} \int_t^{t+h} G(x,y)\,dy$$

is measurable on $D = \{(x,t) : x \geq 0,\ t \geq 0\}$.

PROOF. $g(x,t)$ is a continuous function of t whenever $x \geq 0$, and, by hypotheses, for each $t \geq 0$ $g(x,t)$ is a measurable function of x on $[0,\infty)$. It follows from a theorem by Ursell [3] that $g(x,t)$ is measurable on $D = \{(x,t): x \geq 0,\ t \geq 0\}$, and this is sufficient to ensure that $G_*(x,t)$ is measurable on D.

THEOREM 1. (T.S.T.). Necessary and sufficient conditions for $F(x) = \int_0^\infty G(x,y)f(y)\,dy$ to be defined and summable on $[0,\infty)$ whenever $f(y)$ is summable on $[0,\infty)$ are

i) for each $x \geq 0$, $G(x,y)$ is a function of y measurable and essentially bounded on $[0,\infty)$;

ii) for each $t \geq 0$, $g(x,t) = \int_0^t G(x,y)dy$ is a function of x measurable on $[0,\infty)$;

iii) there is a real number H such that for almost all $t \geq 0$,

$$\int_0^\infty |G_*(x,t)|dx \leq H \quad ,$$

where

$$G_*(x,t) = \liminf_{h\to 0} \frac{1}{h}\int_t^{t+h} G(x,y)dy \quad .$$

PROOF. It follows from Theorem 2 of J. B. Tatchell [1] that i) and ii) are true since they are the same as the i) and ii) of the Theorem 2 of J. B. Tatchell [1].

We now prove that condition iii) is a necessary and sufficient condition for $F(x) = \int_0^\infty G(x,y)f(y)dy$ summable on $[0,\infty)$ whenever $f(y)$ is summable on $[0,\infty)$. The proceeding lemma shows us that $G_*(x,t)$ is measurable on D. It follows from Theorem 1 of G. I. Sunonchi and T. Tsuchikura [2] that the transformation $\int_0^\infty G_*(x,t)f(t)dt$ is defined and summable on $[0,\infty)$ whenever $f(y)$ is summable on $[0,\infty)$ if and only if there is a $H > 0$ such that for almost all $t \geq 0$

$$\int_0^\infty |G_*(x,t)|dx \leq H \quad .$$

If $x \geq 0$, then $G_*(x,t) = G(x,t)$ for almost all $t \geq 0$ and so

$$F(x) = \int_0^\infty G(x,y)f(y)dy = \int_0^\infty G_*(x,y)f(y)dy \quad .$$

Therefore, for any $f(y) \in L$ the transformation $F(x) = \int_0^\infty G(x,y)f(y)dy$ is defined and $F(x) \in L$ if and only if there is a $H > 0$, such that for almost all $t \geq 0$

$$\int_0^\infty |G_*(x,t)|dx \leq H \quad .$$

The proof is completed. Next is the main theorem.

THEOREM 2. The collection $\mathcal{G}$ of all transformations of the form (*) has no identity transformation on L; i.e., there is no transformation G in $\mathcal{G}$ such that for every $f \in L$

$$f(x) = \int_0^\infty G(x,y)f(y)dy$$

for almost all $x \geq 0$.

PROOF. Suppose that there is a $\overline{G}(x,y)$ which defines an integral transformation G such that for every $f \in L$

$$f(x) = \int_0^\infty \overline{G}(x,y)f(y)dy$$

for almost all $x \geq 0$. Then $\overline{G} \in \mathcal{G}L$. It follows from Theorem (T.S.T.) that for each $x \geq 0$, $\overline{G}(x,y) \in L^\infty$. Thus for any measurable set E with finite measure $\int_E \overline{G}(x,y)dy < \infty$. Hence for each $x \in [0,1]$, $\int_0^1 \overline{G}(x,y)dy < \infty$. It follows from the absolute continuity of integrals that, given $\varepsilon = 1/2$, there is a $\delta_x > 0$, such that for every measurable set $e_x \subset [0,1]$ with $me_x < \delta_x$

$$|\int_{e_x} \overline{G}(x,y)dy| < 1/2 \quad .$$

Now for each $x \in [0,1]$, we choose an interval $[a,b] = e_x$, $a \in Q \cap [0,1]$, $b \in Q \cap [0,1]$, containing $y_0(=x)$ and $0 < b - a < \delta_x$ so that

$$|\int_a^b \overline{G}(x,y)dy| < 1/2 \quad .$$

Let $F = \{e_x : x \in [0,1]\}$,

$H_1 = \{[0,a] : [a,b] \in F\}$,

$H_2 = \{[b,1] : [a,b] \in F\}$,

$H = H_1 \cup H_2$

and $\chi_\beta(y) = \begin{cases} 1, & \text{if } y \in \beta \\ 0, & \text{otherwise.} \end{cases}$

Hence for each $\beta \in H$, $\chi_\beta(y) \in L$ and

$$\chi_\beta(x) = \int_0^\infty \bar{G}(x,y)\, \chi_\beta(y)\,dy = \int_\beta \bar{G}(x,y)\,dy$$

for all $x \geq 0$, except a set $E_\beta \subset [0,\infty)$ with $mE_\beta = 0$. Since H is a countable set so $m \sum_{\beta\in H} E_\beta = 0$. Let $K = [0,1]/ \sum_{\beta\in H} E_\beta$, then $mK = 1 - m \sum_{\beta\in H} E_\beta = 1$. Therefore for each $x \in K$, $\chi_\beta(x) = \int_0^\infty \bar{G}(x,y)\chi_\beta(y)\,dy$

$$= \int_\beta \bar{G}(x,y)\,dy$$

for all $\beta \in H$. It follows that for each $x \in K \subset [0,1]$, there is a measurable set $e_x = [a,b] \in F$ with $me_x < \delta_x$ so that

$$\left|\int_{e_x} \bar{G}(x,y)\,dy\right| < 1/2 \quad ,$$

and there are $[0,a] \in H$ and $[b,1] \in H$ so that

$$\begin{aligned}
\left|\int_0^1 \bar{G}(x,y)\,dy\right| &= \left|\int_0^a \bar{G}(x,y)\,dy + \int_a^b \bar{G}(x,y)\,dy + \int_b^1 \bar{G}(x,y)\,dy\right| \\
&\leq \left|\int_0^a \bar{G}(x,y)\,dy\right| + \left|\int_a^b \bar{G}(x,y)\,dy\right| + \left|\int_b^1 \bar{G}(x,y)\,dy\right| \\
&= |\chi_{[0,a]}(x)| + \left|\int_{e_x} \bar{G}(x,y)\,dy\right| + |\chi_{[b,1]}(x)| \\
&= 0 + \left|\int_{e_x} \bar{G}(x,y)\,dy\right| + 0 \\
&< 1/2 \quad .
\end{aligned}$$

Thus for each $x \in K \subset [0,1]$, $mK = 1$ and

$$\begin{aligned}
|\chi_{[0,1]}(x)| &= \left|\int_0^\infty \bar{G}(x,y)\ \chi_{[0,1]}(y)\,dy\right| \\
&= \left|\int_0^1 \bar{G}(x,y)\,dy\right| \\
&< 1/2 \quad .
\end{aligned}$$

This is a contradiction which completes the proof.

COROLLARY. If f, g are measurable functions on $[0,\infty)$ such that $f(y)g(x-y) \in L$, $x \in [0,\infty)$, the convolution $f*g$ of f and g at point x is defined by

$$(f * g)(x) = \int_0^\infty f(y)g(x-y)\,dy \quad .$$

$(L,*)$ is a Banach algebra [5]. Then Banach algebra $(L,*)$ has no unit element, i.e. there is no $g \in L$ such that $f * g = g * f = f$, for all $f \in L$.

PROOF. It is clear this is a special case of preceding theorem where $G(x,y) = g(x-y) \in L$.

REMARK: This corollary is a well-known theorem, see [5], here is a new proof.

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