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ALMOST CONTACT METRIC 3-SUBMERSIONS

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ABSTRACT. An almost contact metric 3-submersion is a Riemannian submersion, π, from an almost contact metric manifold $(M^{4m+3}, (\varphi_i, \xi_i, \eta_i)_{i=1}^{3}, g)$ onto an almost quaternionic manifold $(N^{4n}, (J_i)_{i=1}^{3}, h)$ which commutes with the structure tensors of type (1,1); i.e., $\pi_* \varphi_i = J_i \pi_*$, for $i = 1,2,3$. For various restrictions on $\nabla\varphi_i$, (e.g., M is 3-Sasakian), we show corresponding limitations on the second fundamental form of the fibres and on the complete integrability of the horizontal distribution. Concommitantly, relations are derived between the Betti numbers of a compact total space and the base space. For instance, if M is 3-quasi-Sasakian $(d\Phi = 0)$, then $b_1(N) \leq b_1(M)$. The respective φ_i-holomorphic sectional and bisectional curvature tensors are studied and several unexpected results are obtained. As an example, if X and Y are orthogonal horizontal vector fields on the 3-contact (a relatively weak structure) total space of such a submersion, then the respective holomorphic bisectional curvatures satisfy: $B_{\varphi_i}(X,Y) = B'_{J_i}(X_*,Y_*) - 2$. Applications to the real differential geometry of Yang-Mills field equations are indicated based on the fact that a principal SU(2)-bundle over a compactified realized space-time can be given the structure of an almost contact metric 3-submersion.

KEY WORDS AND PHRASES. *Riemannian submersions, almost contact metric manifolds, quaternionic Kähler manifolds, harmonic mappings, Betti numbers, almost contact metric manifolds with 3-structure.*
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1. INTRODUCTION.

We became interested in almost contact metric 3-submersions because of a possible application to the formulation and solution of gauge field equations in general relativity[1-2]. The separate entities of an almost contact metric 3-submersion are not unknown. Kashiwada[3], Konishi[4], Kuo[5], Sasaki[6], Tachibana and Yu[7] and Tanno[8], among others, have studied manifolds with K-contact or Sasakian 3-structures and have developed many of their properties. The associated 3-fibrations of certain of these spaces over quaternionic Kähler manifolds were investigated by Ishihara[9], Ishihara and Konishi[10], Konishi[11], Shibuya[12], Tanno[8], and others. In another direction, quaternionic Kähler manifolds have been taken up by Alekseevskii[13-14], Gray[15-16], Ishihara[17], Kraines[18], Sakamoto[19], Wolf[20] and others.

In this report, we propose to extend the known 3-fibrations to a general theory of Riemannian submersions from almost contact metric manifolds with 3-structure to almost quaternionic manifolds in which the fibre submanifolds are almost contact metric manifolds with 3-structure. In sections 2 and 3, we define the basic objects of interest and study the geometry of the immersion of the fibre submanifolds and the integrability of the horizontal distribution of the submersion. The fourth section is devoted to questions of the existence of almost contact metric 3-submersions. Several examples are given. In sec. 5, we consider relationships between the several curvature tensors of the three associated manifolds and in sec. 6, the relationships between the cohomologies when the total space is compact. These latter results should be compared with those of [21].

All manifolds considered in this report are smooth, connected and complete. All mappings and tensors are smooth.

2. FUNDAMENTAL CONCEPTS.

Recall[22] that an almost contact metric manifold is a quintuple $(M^m, \varphi, \xi, \eta, g)$ where (M,g) is a Riemannian manifold of dimension m equipped with a metric tensor, g, and with an almost contact structure, (φ, ξ, η) compatible with g; i.e., φ is a tensor field on M of type (1,1), ξ is a unit distinguished vector field on M and η is a differential 1-form on M such that

$$\eta(\xi) = 1, \tag{2.1a}$$

$$\varphi \circ \varphi = -\mathrm{id} + \eta \otimes \xi, \tag{2.1b}$$

$$g(\varphi E, \varphi F) = g(E,F) - \eta(E)\eta(F). \tag{2.1c}$$

for all $E, F \in \mathcal{X}(M)$, the Lie algebra of vector fields on M.

From these axioms, it follows easily that:

$$\varphi(\xi) = 0, \tag{2.2a}$$

$$\eta \circ \varphi = 0, \tag{2.2b}$$

$$\eta(E) = g(E, \xi) \quad \forall\, E \in \mathcal{X}(M). \tag{2.2c}$$

Every almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ possesses a local orthonormal basis of vector fields of the form $\{E_1, \ldots, E_n, \varphi E_1, \ldots, \varphi E_n, \xi\}$. It follows that an almost contact metric manifold is odd-dimensional and orientable. The structure group of an almost contact metric manifold, M^{2m+1}, is $U(m) \times \{I_1\}$.

An almost contact metric manifold $(M^{2m+1}, \varphi, \xi, \eta, g)$ possesses a canonically defined differential 2-form, Φ, given by

$$\Phi(E,F) = g(E, \varphi F), \quad \forall\ E,F \in \mathfrak{X}(M). \tag{2.3}$$

Φ is called the fundamental 2-form of $(M, \varphi, \xi, \eta, g)$. We utilize Φ to define several structures on M. We say that $(M, \varphi, \xi, \eta, g)$ is:

(a) Contact, if $\Phi = d\eta$,

(b) K-contact, if M is contact and ξ is Killing,

(c) Nearly cosymplectic, if $(\nabla_E \varphi)E = 0, \quad \forall\ E \in \mathfrak{X}(M)$,

(d) Closely cosymplectic, if M is nearly cosymplectic and $d\eta = 0$,

(e) Almost cosymplectic, if $d\Phi = 0$ and $d\eta = 0$,

(f) Nearly Sasakian, if $(\nabla_E \varphi)F + (\nabla_F \varphi)E = 2g(E,F) - \eta(E)F - \eta(F)E$, $\forall\ E,F \in \mathfrak{X}(M)$.

Almost contact manifolds may possess a normality property analogous to the integrability of the almost complex structure of almost complex manifolds. We say that (M, φ, ξ, η) is normal if:

$$N^{(1)}(E,F) = [\varphi, \varphi](E,F) + 2d\eta(E,F) = 0, \quad \forall\ E,F \in \mathfrak{X}(M) \tag{2.4}$$

This property allows the definition of other interesting structures on $(M, \varphi, \xi, \eta, g)$. We say that M is:

(g) Quasi-Sasakian, if $d\Phi = 0$ and M is normal,

(h) Sasakian, if M is contact and normal,

(i) Cosymplectic, if $d\Phi = 0$, $d\eta = 0$ and M is normal.

These classes are related by the following lattice in which each inclusion is strict:

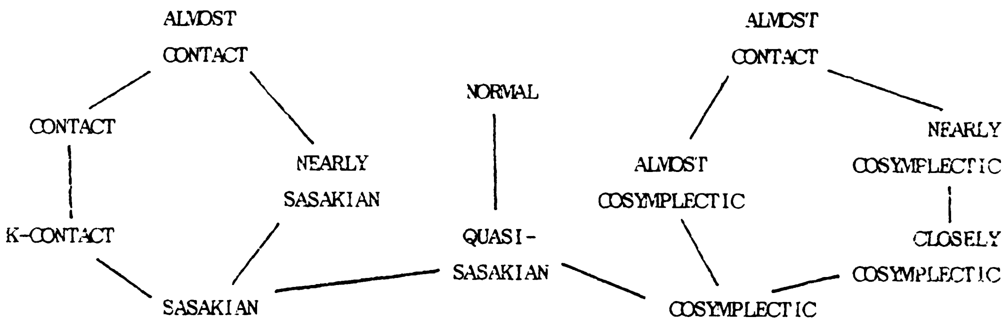

Fig. 2.1 Lattice of important almost contact metric structures.

Examples of almost contact metric manifolds are common. Odd-dimensional spheres may be given Sasakian structures. Compact, orientable 3-manifolds[23] and tangent sphere bundles[24] are other manifolds which may be given the structure of almost contact metric spaces.The product of a compact Kähler space and a circle, S^1, may be given a cosymplectic structure[25].

It will be important for our analysis to have a closed expression for the covariant derivative of the structure tensor, φ, of an almost contact metric manifold. We first define a partial integrability tensor:

$$N^{(2)}(E,F) = (\mathcal{L}_E \varphi)(F) - (\mathcal{L}_F \varphi)(E), \tag{2.5}$$

which vanishes on contact manifolds. Then,

LEMMA 2.1 [22]: For an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$,

(i) $$2g((\nabla_E \varphi)F,G) = 3d\Phi(E,\varphi F,\varphi G) - 3d\Phi(E,F,G) + g(N^{(1)}(F,G),\varphi E) + N^{(2)}(F,G)\,\eta(E) + 2d\eta(\varphi F,E)\,\eta(G) - 2d\eta(\varphi G,E)\,\eta(F)$$

(ii) If M is contact,

$$2g((\nabla_E \varphi)F,G) = g(N^{(1)}(F,G),\varphi E) + 2d\eta(\varphi F,E)\,\eta(G) - 2d\eta(\varphi G,E)\,\eta(F)$$

An almost contact metric 3-structure on (M,g) is a triple $(\varphi_i, \xi_i, \eta_i)_{i=1}^{3}$, of almost contact metric structures, each compatible with the Riemannian structure, g, and satisfying:

$$\eta_j(\xi_i) = 0, \quad i \neq j, \tag{2.6a}$$

$$\varphi_i o \varphi_j - \eta_j \otimes \xi_i = -\varphi_j o \varphi_i + \eta_i \otimes \xi_j = \varphi_k, \text{ for } i < j,\ k \neq i,\ k \neq j, \tag{2.6b}$$

$$g(\xi_i, \xi_j) = 0, \quad i \neq j, \tag{2.6c}$$

$$\eta_i o \varphi_j = -\eta_j o \varphi_i = \eta_k, \quad \text{for } i < j,\ k \neq i,\ k \neq j, \tag{2.6d}$$

$$\varphi_i(\xi_j) = -\varphi_j(\xi_i) = \xi_k, \quad \text{for } i < j,\ k \neq i,\ k \neq j. \tag{2.6e}$$

Every such almost contact metric manifold with 3-structure $(M,(\varphi_i,\xi_i,\eta_i)_{i=1}^{3},g)$ possesses a local orthonormal basis of vector fields of the form: $\{E_1,\ldots,E_n,\varphi_1E_1,\ldots,\varphi_1E_n,\varphi_2E_1,\ldots,\varphi_2E_n,\varphi_3E_1,\ldots,\varphi_3E_n,\xi_1,\xi_2,\xi_3\}$. It follows that an almost contact metric manifold with 3-structure is orientable and has real dimension of the form: $4m+3$. Each almost contact metric structure determines a fundamental 2-form, Φ_i, with respect to g. If each almost contact metric structure is of class $\mathscr{P}$ (see Fig. 2.1), we say that $(M^{4m+3},(\varphi_i,\xi_i,\eta_i)_{i=1}^{3},g)$ has a 3-$\mathscr{P}$ structure. For instance, if each (φ_i,ξ_i,η_i) is contact, then $(M,(\varphi_i,\xi_i,\eta_i)_{i=1}^{3},g)$ is 3-contact. The structure group of an almost contact metric manifold with 3-structure is $Sp(m)\times\{I_3\}$.

Examples of manifolds with almost contact metric 3-structure include S^7, the unit 7-sphere in R^8, and more generally, S^{4m+3} and $P_{4m+3}(R)$. The three almost contact metric structure on S^7, induced from the essentially unique quaternionic structure on R^8, are all Sasakian.

An invariant submanifold of an almost contact metric manifold with 3-structure is defined in the usual way. That is, if $f:M^{4m+3} \longrightarrow (N^{4n+3},(\varphi_i,\xi_i,\eta_i)_{i=1}^{3},g)$ is the immersion of M into N, we require that $\varphi_i f_{*,p}X_p \in T_{f(p)}(f(M))\ \forall\ p \in M$, $\forall\ X_p \in T_p(M)$ and $i = 1,2,3$. We shall investigate the inheritance of certain 3-structures onto invariant submanifolds when we study the fibre submanifolds of the 3-submersions which are the focus of this report.

An almost quaternionic metric manifold is a Riemannian manifold (M,g) which possesses in each coordinate neighborhood (U,ψ), a triple of locally defined almost Hermitian structures $\{J_1,J_2,J_3\}$ satisfying:

$$J_i o J_i = -\,id, \quad \text{for } i = 1,2,3, \tag{2.7a}$$

$$J_i o J_j = -J_j o J_i = J_k, \quad \text{for } i < j,\ k \neq i,\ k \neq j. \tag{2.7b}$$

An almost quaternionic metric manifold M possesses a local basis of its vector fields of the form: $\{E_1,\ldots,E_n,J_1E_1,\ldots,J_1E_n,J_2E_1,\ldots,J_2E_n,J_3E_1,\ldots,J_3E_n\}$. Thus, the dimension of such a manifold is a multiple of 4 and the manifold is orientable.

Each almost Hermitian structure J_i defines a local fundamental 2-form via:

$$\Phi_i(E,F) = g(E,J_iF), \quad \text{for } E,F \in \mathcal{X}(U) \text{ and } i = 1,2,3. \tag{2.3}$$

These allow the definition of a global fundamental 4-form on M via:

$$\Phi = \Phi_1 \wedge \Phi_1 + \Phi_2 \wedge \Phi_2 + \Phi_3 \wedge \Phi_3. \tag{2.9}$$

An almost quaternionic metric manifold for which $\nabla\Phi = 0$ is called quaternionic Kähler. The structure group of an almost quaternionic metric manifold $(M^{4m}, J_1, J_2, J_3, g)$ is $Sp(m)\cdot Sp(1) \cong Sp(m) \times \{Sp(1)/\pm 1\}$. A Riemannian manifold (M^{4m}, g) which is oriented is quaternionic Kähler if and only if its holonomy group is contained in $Sp(m)\cdot Sp(1)$.

Examples of almost quaternionic metric manifolds are legion. Any 4-dimensional oriented Riemannian manifold is almost quaternionic metric by virtue of the group isomorphism: $SO(4) \cong Sp(1)\cdot Sp(1)$. Obviously, R^{4m} and quaternionic projective space $P_m(H)$ are quaternionic Kähler.

Invariant submanifolds of almost quaternionic metric manifolds are defined as before in the case of invariant submanifolds of almost contact metric 3-manifolds.

3. ALMOST CONTACT METRIC 3-SUBMERSIONS.

A Riemannian submersion is a surjective mapping $\pi: M \to N$ between Riemannian manifolds of maximal rank such that the restriction of its pointwise differential to the orthogonal complement of the kernel of that differential is a linear isometry. The basic reference is O'Neill[26]. See also [15] and [21]. Vector fields on M which belong to the kernel of π_* are called vertical, while those orthogonal to the vertical distribution are horizontal. We shall follow the usual conventions and denote vertical vector fields by $U,V,W,\ldots$, horizontal vector fields by $X,Y,Z,\ldots$, and general vector fields on M by $E,F,G,\ldots$. If the horizontal vector field X is π-related to a vector field X_* on N, then X is said to be basic. It is easy to see that the vertical distribution is completely integrable. The completeness of M implies that the Riemannian submersion $\pi: M \to N$ is a locally trivial Riemannian fibre space in the usual sense[20]. We denote the fibre $\pi^{-1}(y)$, $y \in N$, by F and denote tensors and operators on F by a caret, $\hat{}$. We denote those on the base manifold N by a prime, $'$;e.g., the Levi-Civita connection on F is $\hat{\nabla}$ and on N is ∇'. Examples of Riemannian submersions include Riemannian covering mappings, principal fibre bundles, tangent and cotangent bundle projection mappings, Hopf mappings, etc.,[15,21,26].

A particularly interesting example of a Riemannian submersion is the Boothby-Wang fibration[27] of a compact, contact manifold $(M^{2m+1}, \varphi, \xi, \eta, g)$ whose distinguished vector field ξ is regular. The quotient space M/ξ is symplectic (and so, almost Kähler) with integral fundamental 2-form. In fact, if M is Sasakian, then M/ξ is Kähler and therefore it is a Hodge manifold.

We define an <u>almost contact metric 3-submersion</u> to be a Riemannian submersion $\pi:(M^{4m+3},(\varphi_i, \xi_i, \eta_i)_{i=1}^{3}, g) \longrightarrow (N^{4n},(J_i')_{i=1}^{3}, g')$ from an almost contact metric manifold with 3-structure onto an almost quaternionic metric manifold which satsifies:

$$\pi_* \varphi_i E = J_i' \pi_* E, \quad \text{for all } E \in \mathcal{X}(M) \text{ and for } i = 1,2,3. \tag{3.1}$$

We capture several fundamental properties of almost contact metric 3-submersions in the following:

PROPOSITION 3.1: Let $F \to M \xrightarrow{\pi} N$ be an almost contact metric 3-submersion. Then,

(i) The vertical and horizontal distributions induced by π are invariant by each of the three almost contact structure tensors: $\varphi_1, \varphi_2, \varphi_3$,

(ii) The fibre submanifolds of M are invariant almost contact metric submanifolds with 3-structure,

(iii) ξ_1, ξ_2, and ξ_3 are vertical vector fields,

(iv) $\eta_i(X) = 0$, for all horizontal X and for $i = 1,2,3$,

(v) $d\eta_i(X,Y) = -\frac{1}{2}\eta_i(\mathcal{V}[X,Y])$, for all horizontal X and Y and for $i = 1,2,3$.

PROOF: (i) Let $V \in V(M)$, the vertical distribution. Then, $\pi_*\varphi_i V = J_i' \pi_* V = 0$. Therefore, $\varphi_i V \in V(M)$. Let $X \in H(M)$, the horizontal distribution. Then, $g(\varphi_i X, V) = -g(X, \varphi_i V) = 0$. Thus, $\varphi_i X \perp V(M)$ and $\varphi_i X \in H(M)$. (ii) follows immediately from (i). Consider $J_i' \pi_* \xi_i = \pi_* \varphi_i \xi_i = 0$. Thus, (iii) is shown. To see assertion (iv), we need only recall that $\eta_i(X) = g(\xi_i, X) = 0$. (v) follows from the identity:

$$2d\eta_i(X,Y) = X\eta_i(Y) - Y\eta_i(X) - \eta_i([X,Y]). \tag{3.2}$$

While almost contact metric manifolds have been extensively studied in the last two decades(see refs. in [22]), as have been almost Hermitian manifolds(see refs. in [28]), very little has been reported on almost quaternionic metric manifolds, their integrability and the various classes of structures inducible by representations of the structure group. Perhaps their newfound applicability to the study of Yang-Mills equations[29,30], σ-models[31-33] and supergravity[34] will stimulate interest in them.

In general, the existence of a particular structure on the total space of a Riemannian submersion which commutes with certain G-structures will induce a structure on the base space. For example, if $\pi:M \to N$ is an almost Hermitian submersion[21] and M is almost Kähler(resp., Kähler), then N must be almost Kähler(resp.,Kähler). Such a transference does not always obtain;e.g., the almost semi-Kähler case[35]. In the almost contact metric 3-submersion situation, we are limited by the paucity of distinguishable structures on the almost quaternionic metric base space. Nevertheless, THEOREM 3.2: Let $F \to M \to N$ be an almost contact metric 3-submersion with M, 3-quasi-Sasakian(in particular, M may be 3-cosymplectic or 3-Sasakian). Then N is quaternionic Kähler.

PROOF: Gray[36] has shown that the quaternionic Kähler property on N may be characterized by the vanishing of

$$(\nabla'_{X_*}(J'_1)Y_*)\wedge(J'_2Y_*)\wedge(J'_3Y_*) + (J'_1Y_*)\wedge(\nabla'_{X_*}(J'_2)Y_*)\wedge(J'_3Y_*) + (J'_1Y_*)\wedge(J'_2Y_*)\wedge(\nabla'_{X_*}(J'_3)Y_*).$$

For a 3-quasi-Sasakian structure, the relation on $\nabla\varphi_i$ given in Lemma 2.1(i) becomes:

$$g((\nabla_X\varphi_i)Y,Z) = d\eta_i(\varphi_iY,X)\eta_i(Z) - d\eta_i(\varphi_iZ,X)\eta_i(Y) = 0$$

because $\eta_i = 0$ on horizontal vector fields. Therefore, $\mathcal{H}(\nabla_X\varphi_i)Y = 0$, implying that $(\nabla'_{X_*}(J'_i))Y_* = 0$. Thus, N is quaternionic Kähler.

Each of the almost contact metric 3-structures we have defined for the total space(see Fig. 2.1) is inherited by the fibre submanifolds. To begin with, we note that ξ_1, ξ_2, and ξ_3 are vertical vector fields and are the characteristic vectors associated to the restrictions $\hat{\varphi}_1$, $\hat{\varphi}_2$, and $\hat{\varphi}_3$, respectively. Since the metric structure, $\hat{g}$, on F is the restriction of g from M, the fundamental 2- and 1-forms, Φ_i and η_i, are inherited. The embedding mapping of a fibre commutes with the exterior differential operator. Therefore, contact($\Phi = d\eta$) and almost cosymplectic($d\Phi = 0$ and $d\eta = 0$) are inherited properties. To see the inheritance of the K-contact structure, we recall[22] that a contact manifold $(F, \hat{\varphi}, \hat{\xi}, \hat{\eta}, g)$ is K-contact if and only if $\hat{\nabla}_U\xi = -\hat{\varphi}U$. But this equation restricts directly from the same statement on M. Normality and the relation, $(\nabla_U\varphi)U = 0$, inherit as has been shown in [36] for the corresponding assertions for an almost Hermitian structure, J. Thus, Sasakian, quasi-Sasakian, normal, nearly cosymplectic, closely cosymplectic and cosymplectic are inherited. The inheritance of the nearly Sasakian property is shown by direct calculation emulating that for $(\nabla_U\varphi)U$.

O'Neill[26] defined two configuration tensors, T and A, associated to a Riemannian submersion, $\pi: M \to N$, which will be particularly useful in our investigations of almost contact metric 3-submersions. Let $\mathcal{V}: \mathcal{X}(M) \to V(M)$ and $\mathcal{H}: \mathcal{X}(M) \to H(M)$ be the obvious projections. For $E,F \in \mathcal{X}(M)$,

$$T_EF = \mathcal{H}\nabla_{\mathcal{V}E}(\mathcal{V}F) + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F), \tag{3.3}$$

and

$$A_EF = \mathcal{V}\nabla_{\mathcal{H}E}(\mathcal{H}F) + \mathcal{H}\nabla_{\mathcal{H}E}(\mathcal{V}F). \tag{3.4}$$

T and A are skew-symmetric tensor fields(with respect to g) of type (1,2) which reverse the horizontal and vertical distributions induced on M by π. For U and V, vertical, T_UV coincides with the <u>second fundamental form</u> of the immersion of the fibre submanifolds. For X and Y, horizontal, $A_XY = \frac{1}{2}\mathcal{V}[X,Y]$, reflecting the complete integrability of the horizontal distribution. Several useful properties of T and A are to be found in O'Neill[26], Gray[15] and Vilms[37]. For now, it suffices to recall:

LEMMA 3.3: Let $F \to M \to N$ be a Riemannian submersion. Let $X,Y \in H(M)$ and $U,V \in V(M)$. Then,

(i) $\nabla_UV = T_UV + \hat{\nabla}_UV$,

(ii) $\nabla_UX = \mathcal{H}\nabla_UX + T_UX$,

(iii) $\nabla_XU = A_XU + \mathcal{V}\nabla_XU$,

(iv) $\nabla_XY = \mathcal{H}\nabla_XY + A_XY$,

(v) if X is basic, then $\mathcal{H}\nabla_UX = A_XU$,

(vi) if X is basic, then $\mathcal{H}[X,U] = 0$,

(vii) $T_UV = 0$ for all $U,V \in V(M)$ iff $T_UX = 0$ for all $U \in V(M)$ and for all $X \in H(M)$ iff $T = 0$.

(viii) $A_XY = 0$ for all $X,Y \in H(M)$ iff $A_XU = 0$ for all $X \in H(M)$ and for all $U \in V(M)$ iff $A = 0$,

(ix) $T = 0$ and $A = 0$ iff π is a totally geodesic mapping iff M is covered by a Riemannian product space, one of whose factors is isometric to N [37].

The intertwining of the three tensors $\{\varphi_1,\varphi_2,\varphi_3\}$ restricts the configuration tensors, T and A. This limitations is fundamental to our analysis of these submersions. For example,

LEMMA 3.4: Let $F \to M \to N$ be an almost contact metric 3-submersion which satisfies $T_UV + T_{\varphi_i U}(\varphi_iV) = 0$ for all U,V and for $i = 1,2,3$. Then,

$$T = 0.$$

PROOF: We calculate:

$$0 = T_UV + T_{\varphi_1U}(\varphi_1V) = T_UV + T_{\varphi_3U}(\varphi_3V).$$

Thus, $$T_{\varphi_1U}(\varphi_1V) = T_{\varphi_3U}(\varphi_3V).$$

On the other hand, $T_{\xi_i} V = 0$ implies:

$$0 = T_{\varphi_1 U}(\varphi_1 V) + T_{\varphi_2 \varphi_1 U}(\varphi_2 \varphi_1 V) = T_{\varphi_1 U}(\varphi_1 V) + T_{\varphi_3 U}(\varphi_3 V).$$

Therefore. $T = 0$.

THEOREM 3.5: Let $F \to M \to N$ be an almost contact metric 3-submersion with M, either 3-Sasakian or 3-cosymplectic. Then the fibre submanifolds are totally geodesically immersed.

PROOF: Clearly, 3-cosymplectic($\nabla \varphi_i = 0$) implies that $T_U V + T_{\varphi_i U}(\varphi_i V) = 0$. Lemma 3.4 then applies. The Sasakian defining identity on φ_i can be written as

$$\nabla_U(\varphi_i V) - \varphi_i(\nabla_U V) = g(U,V)\,\xi_i - \eta_i(V)U.$$

Taking horizontal projections yields

$$T_U(\varphi_i V) - \varphi_i T_U V = 0,$$

from which $T_U V + T_{\varphi_i U}(\varphi_i V) = 0$ is immediate.

THEOREM 3.6: Let $F \to M \to N$ be an almost contact metric 3-submersion with 3-dimensional fibres. Suppose that M is one of the following:

(i) 3-contact,
(ii) 3-nearly Sasakian,
(iii) 3-quasi-Sasakian,
(iv) 3-almost cosymplectic,
(v) 3-nearly cosymplectic.

Then, $T = 0$.

PROOF: We may choose $\{\xi_1, \xi_2, \xi_3\}$ as our orthonormal local basis for the vector fields on the fibres, F. From Lemma 2.1(i), a contact structure satisfies: $g((\nabla_E \varphi_i)F,G) = \frac{1}{2}g(N^{(1)}(F,G),\varphi_i E) + d\eta_i(\varphi_i F,E)\,\eta_i(G) - d\eta_i(\varphi_i G,E)\eta_i(F)$. Thus, $g(\varphi_i T_{\xi_i} V,X) = 0$, because $d\eta_i(\varphi_i X, \xi_i) = 0$. Hence, $T_{\xi_k}(\xi_i) = 0$ for all i and for all k. Therefore, $T = 0$. A 3-almost cosymplectic structure satisfies $g((\nabla_E \varphi_i)F,G) = \frac{1}{2}g(N^{(1)}(F,G),\varphi_i E)$, from which a similar analysis yields $T = 0$. On a nearly Sasakian manifold, $T_U(\varphi_i V) + T_{\varphi_i U}V = 2\varphi_i T_U V$, which implies that $T_{\xi_i}(\xi_j) = 0$. Analogously, the relation $g((\nabla_E \varphi_i)F,G) = = d\eta_i(\varphi_i E,F)\,\eta_i(G) - d\eta_i(\varphi_i G,E)\,\eta_i(F)$, arising from the identities $d\Phi = 0$ and $N^{(1)} = 0$ on a quasi-Sasakian manifold gives $T_{\xi_i}(\xi_j) = 0$. On a nearly cosymplectic manifold, $(\nabla_E \varphi_i)E = 0$ implies $T_U(\varphi_i V) + T_{\varphi_i U}V = 2\varphi_i T_U V$, as in the case of a nearly Sasakian structure. Again, $T = 0$.

In the cases described in Thms. 3.5 and 3.6, Hermann's Theorem[38] implies that the submersion is a principal fibre bundle with structure group, $G = I(F)$, the isometry group of the fibres.

LEMMA 3.7: Let $F \quad M \quad N$ be an almost contact metric 3-submersion which satisfies

$$T_U U + T_{\varphi_i U}(\varphi_i U) = 0$$

for all U and for $i = 1,2,3$. Then, $H = 0$.

PROOF: Let $\{E_1,\ldots,E_{m-n},\varphi_1 E_1,\ldots,\varphi_1 E_{m-n},\varphi_2 E_1,\ldots,\varphi_2 E_{m-n},\varphi_3 E_1,\ldots$ $\ldots,\varphi_3 E_{m-n},\xi_1,\xi_2,\xi_3\}$ be a local orthonormal basis for the vector fields on the fibre submanifolds, F. The mean curvature vector field, H, of the fibres is given by:

$$H = \sum_{i=1}^{m-n}\left\{T_{E_i}(E_i) + T_{\varphi_1 E_i}(\varphi_1 E_i) + T_{\varphi_2 E_i}(\varphi_2 E_i) + T_{\varphi_3 E_i}(\varphi_3 E_i)\right\} + \sum_{j=1}^{3}\{T_{\xi_j}(\xi_j)\}.$$

By hypothesis,

$$T_{E_i}(E_i) + T_{\varphi_1 E_i}(\varphi_1 E_i) = 0,$$

and
$$T_{\varphi_2 E_i}(\varphi_2 E_i) + T_{\varphi_1\varphi_2 E_i}(\varphi_1\varphi_2 E_i) = 0.$$

Thus,
$$T_{\varphi_2 E_i}(\varphi_2 E_i) + T_{\varphi_3 E_i}(\varphi_3 E_i) = 0.$$

Clearly,
$$T_U U + T_{\varphi_j U}(\varphi_j U) = 0$$

implies
$$T_{\xi_j}(\xi_j) = 0,$$

and the Lemma is demonstrated.

THEOREM 3.8: Let $F \to M \to N$ be an almost contact metric 3-submersion with M, one of the following:

(i) 3-quasi-Sasakian,
(ii) 3-nearly Sasakian,
(iii) 3-nearly cosymplectic,
(iv) 3-closely cosymplectic.

Then the mean curvature vector field, H, vanishes and the fibre submanifolds are therefore minimally immersed.

PROOF: For (iii) and (iv), the defining relation, $(\nabla_U \varphi_i)U = 0$ immediately implies that

$$T_U U + T_{\varphi_i U}(\varphi_i U) = 0,$$

and Lemma 3.7 applies. For nearly Sasakian,

$$T_U(\varphi_i V) + T_V(\varphi_i U) = 2\varphi_i T_U V$$

implies

$$T_U U + T_{\varphi_i U}(\varphi_i U) = 0.$$

If M is 3-quasi-Sasakian, then,

$$g(T_U(\varphi_k U) - \varphi_k T_U U, X) = d\eta_k(X, \varphi_k U)\,\eta_k(U).$$

Since $\eta_k(E_i) = g(E_i, \xi_k) = 0$ for the orthonormal φ_k - basis chosen in the proof of Lemma 3.7, we have

$$T_{E_i}(E_i) + T_{\varphi_k E}(\varphi_k E_i) = 0.$$

Exactly as in the proof of Lemma 3.7, this result implies that $H = 0$.

THEOREM 3.9: Let $F \to M \to N$ be an almost contact metric 3-submersion with 3-dimensional fibres and with M, 3-normal. Then the fibre submanifolds are minimally immersed.

PROOF: The fundamental relation of Lemma 2.1(i) implies that

$$\begin{aligned} 2g((\nabla_{\xi_i}\varphi_i)\xi_i, X) &= 3d\Phi_i(\xi_i, \varphi_i\xi_i, \varphi_i X) - 3d\Phi_i(\xi_i, \xi_i, X) + \\ &+ 2d\eta_i(\varphi_i\xi_i, \xi_i)\,\eta_i(X) - 2d\eta_i(\varphi_i X, \xi_i)\,\eta_i(\xi_i) = \\ &= 0. \end{aligned}$$

Thus, $\varphi_i T_{\xi_i}(\xi_i) = 0$ and $T_{\xi_i}(\xi_i) = 0$. Hence, $H = 0$.

Eells and Sampson[39] showed that a Riemannian submersion is a harmonic mapping if and only if its fibres are minimal. Thus the 3-submersions specified in Thms. 3.5, 3.6, 3.8 and 3.9 are all harmonic. As we shall see, this property has interesting implications for the possible cohomologies of the total space and base space.

An almost contact metric submersion $\pi : M^{2m+1} \to N^{2n}$ of type II with M, cosymplectic, has completely integrable horizontal distribution. Otherwise, among the structures we have considered, A does not vanish identically. However, when the three almost contact metric structures on the total space of an almost contact metric 3-submersion are nearly or almost cosymplectic, they intertwine to cause A to vanish. This produces (sec. 6) an even richer cohomology relationship.

THEOREM 3.10: Let $F \to M \to N$ be an almost contact metric 3-submersion with M, nearly cosymplectic. Then, $A = 0$.

PROOF: Since η_i vanishes on the horizontal distribution, it suffices to consider

$$A_X(\varphi_3 Y) = A_X(\varphi_1 \varphi_2 Y) = A_{\varphi_1 X}(\varphi_2 Y) =$$

$$= A_{\varphi_2 \varphi_1 X} Y = - A_{\varphi_3 X} Y =$$

$$= - A_X(\varphi_3 Y).$$

Therefore, $A = 0$.

THEOREM 3.11: Let $F \longrightarrow M \longrightarrow N$ be an almost contact metric 3-submersion with M, almost cosymplectic. Then, $A = 0$.

PROOF: On an almost cosymplectic 3-submersion,

$$A_{\varphi_i X} Y = - A_Y(\varphi_i X) = - \varphi_i A_Y X = \varphi_i A_X Y = A_X(\varphi_i).$$

This implies that $A = 0$, exactly as in the proof of the preceding theorem.

Recalling that $T = 0$ for a cosymplectic submersion(Thm. 3.5) and that $A = 0$ (Thm. 3.11), we see that a cosymplectic 3-submersion is a totally geodesic mapping. Vilm's characterization of such a fibration, cited in Lemma(ix), forces its total space to be covered by a Riemannian product manifold, one of whose factors is a quaternionic Kähler manifold, the other a 3-cosymplectic manifold and the 3-submersion mapping to be covered by the Riemannian product projection onto the quaternionic Kähler factor.

We recapitulate our findings in Fig. 3.1:

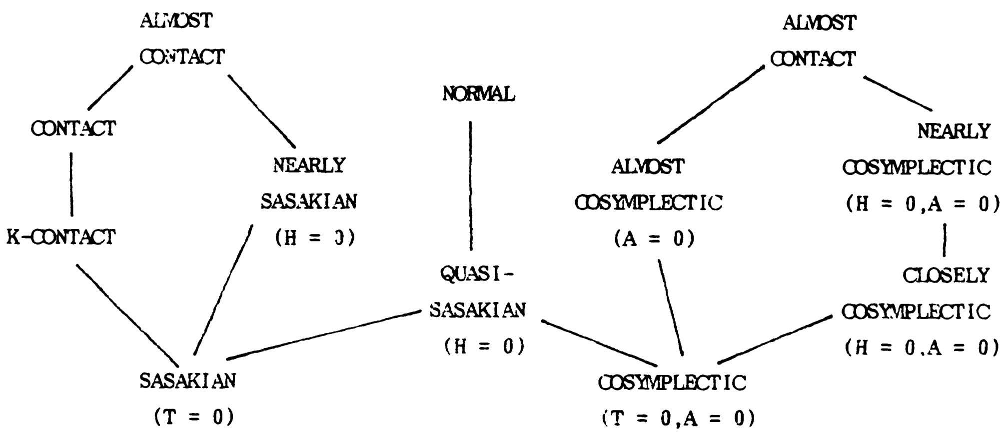

Fig. 3.1. Almost contact metric 3-submersions with totally geodesic or minimal fibres and/or completely integrable horizontal distribution.

Since a most important application of almost contact metric 3-submersions is in the form of SU(2)-bundles over a compactification of a realized space-time, we also recapitulate in Fig. 3.2, the vanishing of the T and A tensors for 3-submersions with 3-dimensional fibres. It is important when examining the left-hand side of Fig. 3.2 to recall Hermann's Theorem on principal I(F)-bundles[38].

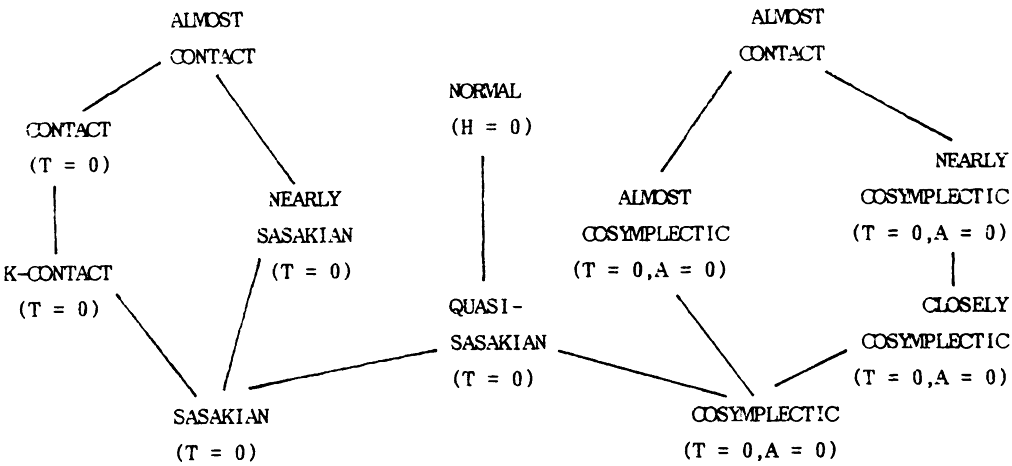

Fig. 3.2. Almost contact metric 3-submersions 3-dimensional fibres which have totally goedesic or minimal fibres.

4. EXISTENCE.

As we have mentioned, the canonical mapping, $S^3 \to S^{4m+3} \to P_m(H)$, is the standard example of an almost contact metric 3-submersion. More generally, Konishi[4], and others, have studied an almost contact metric manifold $(M^{4m+3}, (\varphi_i, \xi_i, \eta_i)^3_{i=1}, g)$ with 3-K-contact structure. In fact, they only require that the distribution spanned by the three characteristic vectors, $V_0 = \mathrm{span}\{\xi_1, \xi_2, \xi_3\}$ satisfy the basic K-contact identity: $\nabla_U \xi_i = -\varphi_i U$. In that case, V_0 is easily seen to be involutive(for $i \neq j$, $[\xi_i, \xi_j] = \nabla_{\xi_i}(\xi_j) - \nabla_{\xi_j}(\xi_j) = -\varphi_j \xi_i + \varphi_i \xi_j = 2\xi_k$, $k \neq i,j$). V_0 is now assumed to be a regular distribution so that M/V_0 is meaningful. Actually, one need only assume that ξ_1 is a regular vector field[8]. Then M/V_0 is an almost quaternionic metric manifold and the canonical projection $\pi : M \to M/V_0$ is an almost contact metric 3-submersion. It can be shown that if M/V_0 is quaternionic Kähler, then the total space M must be Sasakian[4]. Note the 3-dimensionality of the fibres.

Konishi[11] constructed a 3-Sasakian submersion over an arbitrary quaternionic Kähler manifold M of positive scalar curvature by defining a pseudo-Riemannian submersion structure on the standard $P_3(R)$-bundle over M. Again, the fibres are 3-dimensional.

The inherent geometry of 3-Sasakian manifolds necessarily limits our search for 3-Sasakian submersions. Kashiwada[3] has shown that manifolds with 3-Sasakian structure are Einstein. A 3-Sasakian submersion has a quaternionic Kähler manifold as base space. Alekseevskii[13] and Ishihara[17] have shown that any quaternionic Kähler manifold of dimension 8 or greater is Einstein.

On the other hand, the total spaces of 3-cosymplectic submersions are covered by Riemannian product manifolds. Essentially, we have only the product of S^3 and a quaternionic Kähler manifold.

5. CURVATURE.

It is straightforward to prove that a Riemannian submersion is Riemannian sectional curvature increasing on horizontal 2-planes[26]. In fact, if X and Y span the horizontal plane Π_{XY}, then

$$K(\Pi_{XY}) = K'(\Pi_{X_*Y_*}) - 3|A_XY|^2/|X\wedge Y|^2. \quad (5.1)$$

If the Riemannian submersion is required to commute with the structure tensors of some more restrictive G-structure, then certain curvature tensors associated to these G-structures may be affected. For instance[21], if $\pi:(M,J,g) \rightarrow (N,J',g')$ is an almost Hermitian submersion from the quasi-Kähler total space M to the necessarily quasi-Kähler base space N, then the respective holomorphic sectional curvatures[36] are equal. If M is Kähler, then the holomorphic bisectional curvatures are equal. The intertwining of the three almost contact metric structures on an almost contact metric 3-submersion similarly restricts the various φ_i-holomorphic sectional and bisectional curvatures.

Let (M,φ,ξ,η,g) be an almost contact metric manifold and let E and F be vector fields on M. The <u>φ-holomorphic bisectional</u> curvature tensor, is defined by:

$$B_\varphi(E,F) = |E|^{-2}|F|^{-2}g(R(E,\varphi E)F,\varphi F). \quad (5.2)$$

Letting $F = E$ with $E \perp \xi$, the <u>φ-holomorphic sectional curvature</u> is:

$$H_\varphi(E) = |E|^{-4}g(R(E,\varphi E)E,\varphi E). \quad (5.3)$$

The properties of these two curvature tensors are well-known(see e.g.,[22]). For each of the three structure tensors φ_i on an almost contact metric manifold with 3-structure, we define three such φ-holomorphic bisectional curvature tensors and three φ-holomorphic sectional curvature tensors, with obvious notation. For each of the three almost Hermitian structures J_i on an almost quaternionic manifold, we define the holomorphic bisectional and holomorphic sectional curvature tensors in the usual way for almost Hermitian manifolds. We denote them, for instance, by $B'_{J'_i}(X_*,Y_*)$ and $H'_{J'_i}(X_*)$.

The following is a translation of the results of Gray[15] and O'Neill[26] to the present situation:

THEOREM 5.1: Let $F \rightarrow M \rightarrow N$ be an almost contact metric submersion. Then the holomorphic sectional and bisectional curvature tensors take the forms:

(i) $B_{\varphi_i}(U,V) = \hat{B}_{\hat{\varphi}_i}(U,V) + |U|^{-2}|V|^{-2}\{g(T_U(\varphi_i V), T_{\varphi_i} U V) - g(T_U V, T_{\varphi_i} U(\varphi_i V))\},$

(ii) $B_{\varphi_i}(X,V) = |X|^{-2}|V|^{-2}\{g((\nabla_V A)_X(\varphi_i X), \varphi_i V) + g(A_X V, A_{\varphi_i X}(\varphi_i V)) +$

$- g(A_X(\varphi_i V), A_{\varphi_i X} V) - g((\nabla_{\varphi_i V} A)_X(\varphi_i X), V) +$

$+ g(T_{\varphi_i V} X, T_V(\varphi_i X)) - g(T_V X, T_{\varphi_i V}(\varphi_i X))\},$

(iii) $B_{\varphi_i}(X,Y) = B'_{J'_i}(X_*,Y_*) - |X|^{-2}|Y|^{-2}\{2g(A_X(\varphi_i X), A_Y(\varphi_i Y)) +$

$- g(A_{\varphi_i X} Y, A_X(\varphi_i Y)) - g(A_X Y, A_{\varphi_i X}(\varphi_i Y))\},$

(iv) $H_{\varphi_i}(V) = \hat{H}_{\hat{\varphi}_i}(V) + |V|^{-4}\{|T_V(\varphi_i V)|^2 - g(T_V V, T_{\varphi_i V}(\varphi_i V))\},$

(v) $H_{\varphi_i}(X) = H'_{J'_i}(X_*) - 3|X|^{-4}|A_X(\varphi_i X)|^2.$

Different structural restrictions on the $\{\varphi_i\}$ limit these relationships on the sectional and bisectional holomorphic curvature tensors.

THEOREM 5.2: Let $F \longrightarrow M \longrightarrow N$ be an almost contact metric 3-submersion with M, 3-contact. Then,

$$B_{\varphi_i}(X,Y) = B'_{J'_i}(X_*,Y_*) - 2 - |X|^{-2}|Y|^{-2}g(X,Y)^2.$$

PROOF: Consider the terms in brackets in Thm. 5.1(iii):

$$\{2g(A_X(\varphi_i X), A_Y(\varphi_i Y)) - g(A_{\varphi_i X} Y, A_X(\varphi_i Y)) - g(A_X Y, A_{\varphi_i X}(\varphi_i Y))\}.$$

Since M is 3-contact, we have $A_X(\varphi_i X) = |X|^2 \xi_i$ which implies that

$$g(A_X(\varphi_i X), A_Y(\varphi_i Y)) = |X|^2|Y|^2.$$

For the third term,

$$g(A_X Y, A_{\varphi_i X}(\varphi_i Y)) = g(A_X Y, \varphi_i A_{\varphi_i X} Y) + g(X,Y)g(A_X Y, \xi_i) =$$

$$= -|A_X Y|^2 + \eta_i(A_X Y)^2.$$

Similarly, $g(A_{\varphi_i X} Y, A_X(\varphi_i Y)) = |A_X Y|^2 - g(X,Y) - \eta_i(A_X Y)^2.$

COROLLARY 5.2.1: $B_{\varphi_i}(X,Y) < B'_{J'_i}(X_*,Y_*).$

COROLLARY 5.2.2: If $X \perp Y$, then $B_{\varphi_i}(X,Y) = B'_{J'_i}(X_*,Y_*) - 2.$

THEOREM 5.3: Let $F \to M \to N$ be an almost contact metric 3-submersion with M, 3-contact. Then,

$$H_{\varphi_i}(X) = H'_{J'_i}(X_*) = 3.$$

THEOREM 5.4: Let $F^3 \to M^{4m+3} \to N^{4m}$ be an almost contact metric submersion with 3-dimensional fibres and with M, 3-contact. Then,

$$B_{\varphi_i}(U,V) = \hat{B}_{\hat{\varphi}_i}(U,V).$$

PROOF: For a 3-contact submersion, $\dim(F) = 3$ implies that T vanishes.

THEOREM 5.5: Under the same hypotheses as in Thm. 5.4 for a 3-contact submersion,

$$H_{\varphi_i}(U) = \hat{H}_{\hat{\varphi}_i}(U).$$

THEOREM 5.6: Let $F \to M \to N$ be an almost contact metric 3-submersion with M, 3-Sasakian. Then,

$$B_{\varphi_i}(X,V) = 2|X|^{-2}|V|^{-2}\{ \eta_i(V)^2|X|^2 - |A_XV|^2\}.$$

PROOF: It is straightforward to show that $A_{\varphi_i X}V = \varphi_i A_X V$ and that $A_X(\varphi_i V) = \varphi_i A_X V - \eta_i(V)X$ for a Sasakian structure, φ_i. From this, the basic definition of the covariant derivative of A allows the derivation of

$$g((\nabla_V A)_X(\varphi_i X), \varphi_i V) = g((\nabla_{\varphi_i V} A)_X(\varphi_i X), V)$$

from which

$$g(A_XV, A_{\varphi_i X}(\varphi_i V)) - g(A_X(\varphi_i V), A_{\varphi_i X}V) = -2|A_XV|^2 + 2|X|^2\eta_i(V)^2.$$

Since $T = 0$ on a 3-Sasakian submersion, the assertion follows.

THEOREM 5.7: Let $F^3 \to M^{4m+3} \to N^{4m}$ be an almost contact metric 3-submersion with 3-dimensional fibres and with M, 3-Sasakian. Then

(i) $H_{\varphi_1}(X) + H_{\varphi_2}(X) + H_{\varphi_3}(X) = 3,$

(ii) $H'_{J'_1}(X_*) + H'_{J'_2}(X_*) + H'_{J'_3}(X_*) = 0.$

PROOF: Tanno[8] proved (i) for vector fields orthogonal to a 3-dimensional vertical distribution on a 3-Sasakian manifold. (ii) follows from Thm. 5.3.

THEOREM 5.8: Let $F^3 \to M \to N$ be an almost contact metric 3-submersion with M, 3-nearly Sasakian and with 3-dimensional fibres. Then

$$B_{\varphi_i}(U,V) = \hat{B}_{\hat{\varphi}_i}(U,V).$$

PROOF: On a 3-nearly Sasakian submersion with 3-dimensional fibres, O'Neill's T tensor vanishes (Thm. 3.6(ii)).

THEOREM 5.9: Let $F \longrightarrow M \longrightarrow N$ be an almost contact metric 3-submersion with M, 3-nearly Sasakian. Then

(i) $H_{\varphi_i}(V) = \hat{H}_{\hat{\varphi}_i}(V) + 2|V|^{-4}|T_V V|^2$,

(ii) $H_{\varphi_i}(X) = H'_{J'_i}(X_*) - 3$.

THEOREM 5.10: Let $F^3 \rightarrow M \rightarrow N$ be an almost contact metric 3-submersion with M, 3-nearly Sasakian and with 3-dimensional fibres. Then

$$H_{\varphi_i}(V) = \hat{H}_{\hat{\varphi}_i}(V).$$

PROOF: See Thm. 5.8.

THEOREM 5.11: Let $F^3 \rightarrow M \rightarrow N$ be an almost contact metric 3-submersion with M, 3-quasi-Sasakian and with 3-dimensional fibres. Then,

$$B_{\varphi_i}(U,V) = \hat{B}_{\hat{\varphi}_i}(U,V).$$

PROOF: Again, Thm. 3.6.

THEOREM 5.12: Let $F^3 \rightarrow M \rightarrow N$ be an almost contact metric 3-submersion with M, 3-quasi-Sasakian and with 3-dimensional fibres. Then,

$$H_{\varphi_i}(U) = \hat{H}_{\hat{\varphi}_i}(U).$$

6. COHOMOLOGY.

The classic Hodge Theorem relates the space $\mathcal{H}^r(M)$ of real harmonic r-forms on an oriented, compact Riemannian manifold, M, to the classical de Rham cohomology space, $H^r(M,R)$, and, thereby, to a standard, say sheaf, cohomology space, $H^r(M,R)$. In fact, an isomorphism is established: $\mathcal{H}^r(M) \cong H^r(M,R) \cong H^r(M,R)$. This isomorphism plays an important part in the theory of Riemannian submersions. We recall first the easily established fact that for any r-form ω on a compact, oriented Riemannian manifold M, $\Delta\omega = 0$ if and only if both $d\omega = 0$ and $\delta\omega = 0$. The author proved[40] that a smooth manifold surjection commutes with the codifferential on all 1-forms if and only if (a) it is a Riemannian submersion and (b) the fibre submanifolds are minimally immersed. Goldberg and Ishihara[41] extended the work of the author[42] by showing that a Riemannian submersion commutes with the Laplacian on forms of fixed degree $r (r \geq 2)$ if and only if the fibre submanifolds are minimally immersed and the horizontal distribution is completely integrable. The same conditions are necessary and sufficient for commutation with the codifferential on forms of fixed degree $r(r \geq 2)$. Our results in sec. 3 now permit us to establish necessary conditions for the existence of almost contact metric 3-submersions with specified structures. We assume for the remainder of this report that all manifolds are compact, even though we shall often reemphasize this point.

LEMMA 6.1: Let $F \to M \xrightarrow{\pi} N$ be a Riemannian submersion with M, compact.

(i) If the fibre submanifolds are minimal, then $b_1(N) \leq b_1(M)$.

(ii) If the fibre submanifolds are minimal and the horizontal distribution is completely integrable, then $b_r(N) \leq b_r(M)$, for $r = 0,1,2,\ldots,\dim(N)$.

PROOF: Under the stated hypotheses, $\pi^*: \mathcal{H}^r(N) \longrightarrow \mathcal{H}^r(M)$ is a linear isometry between finite dimensional real vector spaces.

As an example of such necessary conditions, note that the dim(N)-th Betti number of the total space must be positive when $H = 0$ and $A = 0$. Moreover, Poincaré duality then implies that the (dim(M) - dim(N))-th Betti number of M is also positive. As a specific example, if $\pi: M^{15} \to N^8$ is a 3-nearly cosymplectic(see Thms. 3.8(iii) and 3.10), then $b_8(M) \geq 1$ and $b_7(M) \geq 1$. If N is quaternionic Kähler, then the non-zero $\Lambda^k \Phi = \Phi \wedge \ldots \wedge \Phi$ (k times) is harmonic [18] on N, so $b_{4k}(M) \geq b_{4k}(N) \geq 1$. Poincaré duality, acting on such $\mathcal{H}^{4k}(M)$, further restricts the possibilities for π.

THEOREM 6.2: Let $F \to M \to N$ be an almost contact metric 3-submersion with M, compact and

(i) 3-nearly Sasakian, or

(ii) 3-quasi-Sasakian(including 3-Sasakian). Then,

$$b_1(N) \leq b_1(M).$$

PROOF: In each case, the fibre submanifolds are minimal.

THEOREM 6.3: Let $F \to M \to N$ be an almost contact metric 3-submersion with M, compact and nearly cosymplectic(including closely cosymplectic and cosymplectic). Then, for all $r = 0,1,2,\ldots, \dim(N)$,

$$b_r(N) \leq b_r(M).$$

PROOF: $H = 0$ and $A = 0$.

We recall that if $\pi: M \to N^{4n}$ is a 3-Sasakian submersion, then $b_{4k}(N) \geq 1$ for $k = 0,1,\ldots,n$ by [18]. However, this data cannot translate to the total space M, because the O'Neill tensor A can never vanish identically on a 3-Sasakian submersion.

In [10] Ishihara and Konishi proved that the first Betti numbers of the total space and the base space of a 3-Sasakian submersion with 3-dimensional fibres are equal. We hasten to note that most investigators of the properties of 3-Sasakian or 3-K-contact fibre spaces over quaternionic manifolds have limited themselves to the 3-dimensional fibre case. A glance at Fig. 3.2 illustrates how this hypothesis reduces the analysis, while a glance at Fig. 3.1 indicates the more general situation. In the 3-dimensional fibre case, the 3-almost cosymplectic and 3-nearly cosymplectic(including 3-closely cosymplectic and 3-cosymplectic) fibrations are trivial in that they are covered by Riemannian products[37]. The 3-contact (including 3-K-contact and 3-Sasakian) as well as the 3-nearly Sasakian and 3-quasi-Sasakian fibrations are principal fibre bundles by Hermann's Theorem[38]. The cosymplectic side of Fig. 3.2 being trivial, we capture the Betti number inequalities for the Sasakian side in:

THEOREM 6.4: Let $F^3 \to M \to N$ be an almost contact metric 3-submersion with M, compact and with $\dim(F) = 3$. Suppose M is:

(i) 3-contact,

(ii) 3-K-contact,

(iii) 3-Sasakian,

(iv) 3-nearly Sasakian,

(v) 3-quasi-Sasakian, or

(vi) 3-normal.

Then
$$b_1(N) \leq b_1(M).$$

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