Internat. J. Math & Math Sci.
Vol. 9 No. 1 (1986) 17-22

FUNCTORIAL PROPERTIES OF THE LATTICE OF FUNCTIONAL SEMI-NORMS

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(Received August 10, 1984)

ABSTRACT. Given a measureable transformation between measure spaces, we determine when such gives rise to a mapping between the corresponding lattice of function semi-norms. We further determine when this mappings preserves norms and observe that it does preserve certain other important properties. We next establish a functorial connection between measure spaces and lattice. Finally, we show that the above lattice mapping does not commute with the associate construction.

KEY WORDS AND PHRASES: Function semi-norm, associate semi-norm, lattice of semi-norms, measure-preserving transformation, semi-norm preserving, associate preserving, lattice subhomomorphism, category, functor.
1980 AMS SUBJECT CLASSIFICATION CODE. 18A20, 18B99, 28A65, 06A20, 18A99, 18D35.

1. INTRODUCTION.

Let (X, S, μ) be a sigma-finite measure space and $M^+(\mu)$ the space of $[0,\infty]$-valued μ-measurable functions on X. Contrary to conventional practice, it will not be convenient to identify two functions in $M^+(\mu)$ which are equal μ-a.e. Accordingly, let $Z(\mu)$ denote the μ-null function in $M^+(\mu)$. Thus, $Z(\mu)$ is the null equivalence class in $M^+(\mu)$ of the zero function on X. In this setting, a (function) semi-norm on $M^+(\mu)$ is a mapping $\rho:M^+(\mu)\to[0,\infty]$ having the following properties. Let $c>0$, and $f,g \in M^+(\mu)$. Then:

(1) $f-g \in Z(\mu)$ implies $\rho(f) = \rho(g)$.

(2) $f \in Z(\mu)$ implies $\rho(f) = 0$.

(3) $\rho(cf) = c\rho(f)$.

(4) $\rho(f+g) \leq \rho(f)+ \rho(g)$.

(5) $f\leq g$ μ-a.e. implies $\rho(f)\leq \rho(g)$.

The semi-norm $\rho(f) = 0$ implies $f\in Z(\mu)$. Let $P(\mu)$ denote the set of all semi-norms and $P_o(\mu)$ the subset of all norms (never empty).

Observe that $P(\mu)$ is canonically partially ordered by:

$$\rho_1 \leq \rho_2 \text{ if } \rho_1(f) \leq \rho_2(f),\ f\in M^+(\mu).$$

It is well-known that, relative to this ordering, $P(\mu)$ is a complete lattice with sup and inf given by

$$(\rho_1 \vee \rho_2)(f) = \sup(\rho_1(f), \rho_2(f)),$$

and

$$(\rho_1 \wedge \rho_2)(f) = \inf\{\rho_1(f_1) + \rho_2(f_2): f_1, f_2 \in M^+(\mu), f_1+f_2=f, \mu\text{-a.e.}\}$$

(See sections 3 and 4 of [3] for the sup and inf of arbitrary families in $P(\mu)$.)

Now let (Y,T,ν) be another sigma-finite measure space and $\phi:X\to Y$ a measurable transformation. For such ϕ, we obtain a mapping $\phi^o:M^+(\nu)\to M^+(\mu)$ defined by $\phi^o(g) = g\phi$. This in turn yields a mapping $\Phi:\rho\to\rho\phi^o$ from $P(\mu)$ into the $[0,\infty]$-valued functions on $M^+(\nu)$. In general, $\Phi(\rho) = \rho\phi^o$ is not a semi-norm. Moreover, if ρ is a norm, then $\Phi(\rho)$ may be a semi-norm which is not a norm. Thus, the first question we ask is: Under what conditions is Φ semi-norm-preserving? In section 2, we give necessary and sufficient conditions for this to be the case (2.2). The next question is: Under what additional conditions is Φ norm-preserving? In section 3, we give necessary and sufficient conditions for this to be the case (3.5). There are certain very important sublattices in the lattice of semi-norms which have been studied extensively (see [2,3]). Also in section 3, we observe that all of these sublattices are preserved by Φ (3.7) - when ϕ is semi-norm-preserving. The previous results suggest there is a functorial connection between measure spaces and lattices. However, when ϕ is semi-norm-preserving, Φ may not be a lattice homorphism. Specifically, in general, "Φ of an infimum does not equal the infimum of the Φ 's". Despite this failing, Φ is a lattice "subhomomorphism" (4.3). With this notion of lattice morphism, we are able (in section 4) to establish the desired functorial connection. Finally, in section 5, we see that the mapping Φ and the assocconstriuction $\rho\to\rho'$ are incompatible in general. For this purpose, recall that

$$\rho'(f) = \sup\{\textstyle\int_X fg\,d\mu:\rho(g) \leq 1\}, f\in M^+(\mu)$$

Also, let $N(\mu)$ denote the space of μ-null subsets of X (similarly for ν).

2. SEMI-NORM PRESERVATION.

Before investigating the conditions under which Φ preserves semi-norms, let us see first that it does not have this property in general.

2.1 Example. Let $X = Y = \{a,b\}$ with μ and ν defined as follows: $\mu(\{a\}) = \mu(\{b\}) = \nu(\{a\}) = 1$ and $\nu(\{b\}) = 0$. Let ϕ be the identity mapping. Then $\{b\} \in N(\nu)$, while $\{b\} = \phi^{-1}(\{b\}) \notin N(\mu)$. Let ρ be the L^1-norm in $P_o(\mu)$, i.e.,

$$\rho(f) = ||f||_1 = f(a) + f(b),\ f \in M^+(\mu)$$

The function g on Y defined by $g(a) = 0$, $g(b) = 1$, is ν-null. However, $g\phi$ is not μ-null, i.e. $\phi^o(Z(\nu)) \not\subseteq Z(\mu)$. Thus, $\Phi(\rho)(g) = \rho(g\phi) \neq 0$, i.e. $\Phi(\rho)$ is not constant on null equivalence classes in $M^+(\nu)$.

2.2 Theorem. The following are equivalent:

(i) $\Phi:P(\mu) \to P(\nu)$

(ii) $\phi^{-1}(N(\nu)) \subseteq N(\mu)$

(iii) $\phi^o(Z(\mu)) \subseteq Z(\mu)$

Proof. (ii) implies (i): Let $g_1, g_2 \in M^+(\nu)$ be such that $g_1 \leq g_2$, ν-a.e. Then

$$\{x \in X: g_1\phi(x) \nleq g_2\phi(x)\} \subseteq \phi^{-1}(\{y \in Y : g_1(y) \nleq g_2(y)\}).$$

Since the set in the right parentheses is ν-null, it follows from (ii) that its inverse image under ϕ is μ-null, i.e. $g_1\phi \leq g_2\phi$, μ-a.e. Hence, for $\rho \in P(\mu)$, we have

$$\Phi(\rho)(g_1) = \rho(g_1\phi) \leq \rho(g_2\phi) = \Phi(\rho)(g_2),$$

i.e. $\Phi(\rho)$ satisfies (5) of §1. This also proves (1) of §1. The remaining properties (2), (3), (4) of §1 are easily verified. Therefore, $\Phi(\rho) \in P(\nu)$, for all $\rho \in P(\mu)$.

(iii) implies (ii): Let E be an element of $N(\nu)$. The characteristic function χ_E is then in $Z(\nu)$, i.e. $\Phi^o(\chi_E) \in Z(\mu)$ by (iii). We then have

$$\chi_E\phi = \phi^o(\chi_E) = \chi_{\phi^{-1}(E)}$$

so that $\chi_{\phi^{-1}(E)} \in Z(\mu)$, i.e. $\phi^{-1}(E) \in N(\mu)$.

(i) implies (iii): Suppose (iii) is false. Then there exists f in $\phi^o(Z(\nu))$ such that f is not in $Z(\mu)$. Let $g \in Z(\nu)$ be such that $f = g\phi$. If $\rho \in P_o(\mu)$, then by (i) we must have $\Phi(\rho)(g) = \rho(f) = 0$. This contradicts the fact that ρ is a norm.

2.3 Remarks. Observe that (iii) of the theorem says that ϕ^o essentially sends the zero-class in $M^+(\nu)$ to the zero-class in $M^+(\mu)$ because, modulo nullity, $\phi^o(Z(\nu)) \supseteq Z(\mu)$. Specifically, if $f \in Z(\mu)$, then $\phi^o(g) = f$, μ-a.e., for g the zero function on Y.

2.4 Definition. The measurable transformation $\phi: X \to Y$ is semi-norm-preserving if the conditions of 2.2 hold.

3. PROPERTY PRESERVATION.

The natural next question to ask about ϕ is the following: Under what conditions does it preserve norms? The answer to this question is somewhat complicated because of some measure - theoretic technicalities. These (together with some additional notation) are necessitated by the fact that ϕ need not preserve measurable sets, i.e. it may not be bimeasurable.

Let $\bar{\nu}$ denote the completion of ν and $\bar{T}$ its domain [1]. Let ν^* (resp. ν_*) denote the outer (resp. inner) measure derived from ν. Also let

$$N_\phi(\mu) = \{E \in N(\mu): \phi^{-1}(\phi(E)) = E\}.$$

In general, $N_\phi(\mu)$ is a proper subset of $N(\mu)$. However:

3.1 Lemma. The transformation ϕ is semi-norm-preserving, (i.e. $\phi^{-1}(N(\nu)) \subseteq N(\mu)$) if and only if $\phi^{-1}(N(\nu)) \subseteq N_\phi(\mu)$.

Proof. The elements of $\phi^{-1}(N(\nu))$ automatically have the extra property.

For any semi-norm ρ, define

$$K(\rho) = \{f \in M^+(\mu): \rho(f) = 0\}.$$

Of course, $K(\rho) \supseteq Z(\mu)$ in general.

3.2 Lemma. Suppose ϕ is semi-norm preserving. If $\rho \in P(\mu)$, then

$$K(\Phi(\rho)) = (\phi^o)^{-1}(K(\rho)).$$

Proof. Straightforward.

We then have the following answer to our question:

3.3 Proposition. Suppose ϕ is semi-norm preserving and ρ is a norm in $P(\mu)$. Then $\Phi(\rho)$ is a norm if and only if $(\phi^o)^{-1}(Z(\mu)) = Z(\nu)$.

Proof. Apply 2.2 and 3.2.

In order to obtain an answer analogous to 2.2 in terms of ϕ itself, we first require the following.

3.4 Lemma. Let $C = Y - \phi(X)$ (set difference). If $\rho \in P(\mu)$ and $\Phi(\rho)$ is a norm, then $\nu_*(C) = 0$, i.e. ϕ is ν_*-essentially onto.

Proof. If not, there exists E in T such that $E \subseteq C$ and $\nu(E) > 0$. Then for $g = \chi_E$, we have $g\phi \in Z(\mu)$, so that $\Phi(\rho)(g) = 0$, while $g \notin Z(\nu)$.

3.5 Theorem. Suppose ϕ is semi-norm preserving, ρ is a norm in $P(\mu)$ and $\nu_*(C) = 0$. If $\phi(X) \in \bar{T}$, then the following are equivalent:

(i) $\Phi(\rho)$ is a norm.

(ii) $N_\phi(\mu) \subseteq \phi^{-1}(N(\nu_*))$.

(iii) $(\phi^o)^{-1}(Z(\mu)) = Z(\nu)$ (recall 2.2, 2.3).

Proof. (i) is equivalent to (iii) by 3.3.

(iii) implies (ii): Let $E \in N_\phi(\mu)$, so that $\mu(E) = 0$ and $\phi^{-1}(\phi(E)) = E$. Then $\chi_E \in Z(\mu)$ and $\chi_{\phi(E)}\phi = \chi_E$. Let $F \in T$ be such that $F \subseteq \phi(E)$. Then $\chi_F \leq \chi_{\phi(E)}$ and $\chi_F\phi \leq \chi_{\phi(E)}\phi = \chi_E$. Hence, $\chi_F\phi \in Z(\mu)$, i.e. $\phi^o(\chi_F) \in Z(\mu)$. This implies that $\chi_F \in (\phi^o)^{-1}(Z(\mu)) = Z(\nu)$, i.e. $\nu(F) = 0$. Consequently, $\nu_*(\phi(E)) = 0$, so that $\phi(E) \in N(\nu_*)$. (ii) implies (i): Let $g \in M^+(\nu)$ be such that $\Phi(\rho)(g) = 0$. Then $\rho(g\phi) = 0$, so that $g\phi \in Z(\mu)$ (ρ is a norm). Since

$$\phi^{-1}(\mathrm{supp}(g)) = \mathrm{supp}(g\phi),$$

it follows that $\mu(\phi^{-1}(\mathrm{supp}(g))) = 0$, i.e. $\phi^{-1}(\mathrm{supp}(g)) \in N_\phi(\mu)$. Let

$$G_r = \mathrm{supp}(g) \cap \phi(X),\quad G_c = \mathrm{supp}(g) \cap C,$$

observing that $\phi(X)$ and C belong to $\bar{T}$. Then $G_r, G_c \in \bar{T}$, $\mathrm{supp}(g) = G_r \cup G_c$ (disjoint) and

$$\phi^{-1}(\mathrm{supp}(g)) = \phi^{-1}(G_r) \in \phi^{-1}(N(\nu_*)),$$

by condition (ii), so that $\nu_*(G_r) = 0$. On the other hand,

$$\nu^*(G_c) = \bar{\nu}(G_c) = \nu_*(G_c) = 0 \qquad [1, p. 60].$$

Therefore,

$$\nu(\mathrm{supp}(g)) \leq \nu_*(G_r) + \nu^*(G_c) = 0 \qquad [1, p. 61],$$

so that $g \in Z(\nu)$, i.e. $\Phi(\rho)$ is a norm.

3.6 Corollary. Suppose ϕ is semi-norm-preserving and ρ is an norm in $P(\mu)$. If ϕ is bimeasurable and maps X ν-essentially onto Y, then (i) and (iii) of the theorem are equivalent to (ii'): $N_\phi(\mu) = \phi^{-1}(N(\nu))$.

We next consider our question in the context of the subsets of $P(\mu)$ introduced in section 2 of [3]. Here the answers are the best possible. The subsets consist of those norms having the Riesz-Fisher (R), weak (W) or strong (S) Fatou property, those satisfying the infinite triangle inequality (I) and those which are of absolutely continuous norm (A) (see[2,3,4]).

3.7 Theorem. For the following, let B denote either R,I,W, S or A. If ϕ is semi-norm-preserving, then Φ preserves the property defining B, i.e. $\Phi: B(\mu) \to B(\nu)$.

Proof. The proof for each choice of B is more-or-less straightforward. Therefore, we leave the details to the interested reader - after remarking that 3.2 is required in proving the theorem for the case B = R.

4. THE FUNCTOR.

In this section we investigate the categorical connection between measurable transformations and lattices of semi-norms. As the next example shows, if ϕ is semi-norm-preserving, the corresponding morphism Φ may not be a lettice homomorphism.

4.1 Example. Let $X = \mathbb{N}$ the set of all positive integers, and $Y = \{y\}$ with $\nu(Y) = 1$. Define $\phi(x) = y$, $x \in X$. Then ϕ is a semi-norm-preserving measurable transformation and we have $\Phi: P(\mu) \to P(\nu)$ as in Section 2. Define

$$\rho_1(f) = \Sigma_1^\infty f(n)/2^n$$

and

$$\rho_2(f) = \lim_n \sup(f(n)) + \tfrac{1}{4} \sup_n(f(n)), \quad f \in M^+(\mu).$$

Let g be the function equal to 1 on Y, so that $g \in M^+(\nu)$. We leave to the reader the verification that

$$[\Phi(\rho_1) \wedge \Phi(\rho_2)]\,(g) = 1,$$

While

$$[\Phi(\rho_1 \wedge \rho_2)]\,(g) \leq \tfrac{1}{4}.$$

Thus, $\Phi(\rho_1 \wedge \rho_2) \neq \Phi(\rho_1) \wedge \Phi(\rho_2)$ in general.

Despite this failing, Φ does have suitable lattice morphism properties.

4.2 Lemma. If $\rho_1, \rho_2 \in P(\mu)$, then $\Phi(\rho_1 \vee \rho_2) = \Phi(\rho_1) \vee \Phi(\rho_2)$ and $\Phi(\rho_1 \wedge \rho_2) \leq \Phi(\rho_1) \wedge \Phi(\rho_2)$, in general.

4.3 Definition. Any mapping between lattices having the properties exhibited by Φ in 4.2 will be called a lattice subhomomorphism.

We are now ready to define a functor. On the one hand, consider all sigma-finite measure spaces as the objects and semi-norm-preserving, measurable transformations as the morphisms. These form a category which we denote by X. On the other hand, consider all lattices as the objects and lattice subhomomorphisms as the morphisms. These form a category which we denote by P. By the results of section 2, we obtain a "mapping"

$$F : X \to P$$

determined by

$$F(X, S, \mu) = P(\mu), \quad (X, S, \mu) \in \mathrm{Obj}\,(X),$$

and

$$F(\phi) = \Phi, \quad \phi \in \mathrm{Mor}((X, S, \mu), (Y, T, \nu)).$$

We leave to the reader the task of verifying the F is in fact a functor.

5. ASSOCIATE PRESERVATION.

Our final concern is the question of whether Φ preserves associates. We shall see in the next examples that $\Phi(\rho')$ and $\Phi(\rho)'$ are not even comparable in general.

5.1 Example. Let X, Y, ν, ϕ be as in 4.1. Denote the respective characteristic functions of X,Y by f,g. Let ρ be the norm in $P(\mu)$ given by

$$\rho(h) = \Sigma_1^\infty h(n)/2^n, \qquad h \in M^+(\mu).$$

Then $\rho(f) = 1$ and

$$\begin{aligned}\Phi(\rho)'(g) &= \sup\{|h(y)| : \rho(h\phi) \leq 1\} \\ &= \sup\{|h(y)| : h(y)\rho(f) \leq 1\} \\ &= \rho(f)^{-1} \\ &= 1.\end{aligned}$$

On the other hand,

$$\begin{aligned}\Phi(\rho')(g) &= \sup\,\{\Sigma_1^\infty |h(n)| : \rho(h) \leq 1\} \\ &\leq \Sigma_1^\infty |f(n)|\end{aligned}$$

Thus, $\Phi(\rho') \not\leq \Phi(\rho)'$, in general.

5.2 Example. Now let $X = \{x\}$, $Y = \mathbb{N}$ with $\mu(X) = 1$. Define $\phi(x) = 1$, so that ϕ is semi-norm-preserving. Let h denote the characteristic function of Y and define

$$g(y) = 0, \; y = 1$$
$$= 1, \; y \neq 1.$$

Let ρ be the norm in $P(\mu)$ given by $\rho(f) = f(x)$. Then

$$\Phi(\rho')(g) = g(1) \sup\{|f(1)| : \rho(f) \leq 1\}$$
$$= 0.$$

On the other hand,

$$\Phi(\rho)'(g) = \sup\{\Sigma_1^\infty |f(n)| g(n) : \rho(f\phi) \leq 1\}$$
$$\leq \Sigma_2^\infty |f(n)|$$
$$= \infty.$$

Thus, $\Phi(\rho') \not\geq \Phi(\rho)'$, in general

5.3 Remarks. It is possible to find non-trivial conditions on ϕ which will at least guarantee a comparison of $\Phi(\rho')$ and $\Phi(\rho)'$. However, the conditions we have in mind are not far from requiring that Φ be an essential measure isomorphism (need not be essentially one-one). Thus, the strength of the hypothesis, combined with the weakness of the conclusion (namely, $\Phi(\rho') \geq \Phi(\rho)'$), provide little motivation for presenting the details here.

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