

GENERALIZED RAMSEY NUMBERS FOR PATHS IN 2-CHROMATIC GRAPHS

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ABSTRACT. Chung and Liu have defined the d -chromatic Ramsey number as follows. Let $1 \leq d \leq c$ and let $t = \binom{c}{d}$. Let $1, 2, \dots, t$ be the ordered subsets of d colors chosen from c distinct colors. Let G_1, G_2, \dots, G_t be graphs. The d -chromatic Ramsey number denoted by $r_d^c(G_1, G_2, \dots, G_t)$ is defined as the least number p such that, if the edges of the complete graph K_p are colored in any fashion with c colors, then for some i , the subgraph whose edges are colored in the i th subset of colors contains a G_i . In this paper it is shown that $r_2^3(P_i, P_j, P_k) = [(4k+2j+i-2)/6]$ where $i \leq j \leq k < r(P_i, P_j)$, r_2^3 stands for a generalized Ramsey number on a 2-colored graph and P_i is a path of order i .

KEY WORDS AND PHRASES. Ramsey Number, Generalized Ramsey Number, d -Chromatic Ramsey number, Colored Graph.

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1. INTRODUCTION AND NOTATION.

Chung and Liu [1] have defined the d -chromatic Ramsey number as follows. Let $1 \leq d \leq c$ and let $t = \binom{c}{d}$. Let $1, 2, \dots, t$ be the ordered subsets of d colors chosen from c distinct colors. Let G_1, G_2, \dots, G_t be graphs. The d -chromatic Ramsey number denoted by $r_d^c(G_1, G_2, \dots, G_t)$ is defined as the least number p such that, if the edges of the complete graph K_p are colored in any fashion with c colors, then for some i , the subgraph whose edges are colored in the i th subset of colors contains a G_i . In this paper the value of $r_2^3(P_i, P_j, P_k)$ is found. Let $P_{i(r,s)}$ and $C_{i(r,s)}$ respectively denote a path or a cycle connecting i nodes whose edges are colored in color r or color s . Let $d_i(x)$ denote the degree of node x in color i . Let $|N_i(x) \cup N_i(y)|$ denote the number of vertices adjacent to x or y in color i , and $r(P_i, P_j)$ be the least number p such that when the edges of the full graph K_p are colored in colors 1 and 2 contains a $P_{i(1)}$ or $P_{j(2)}$. It is assumed throughout that $2 \leq i \leq j \leq k$. Let $[i]$ and $\{i\}$ respectively denote the largest integer less than or equal to i and the smallest integer greater than or equal to i . A colored graph G is a complete graph whose edges are colored in colors 1, 2, or 3. $V(G)$ and $E(G)$ denote the set of nodes and edges of G and E_i is the set of edges in color i .

2. MAIN RESULT.

First a series of Lemmas are presented which is followed by a bounding theorem for $r_2^3(P_i, P_j, P_k)$ and finally, an example shows that the bound is tight.

Lemma 1:

$$r_2^3(P_i, P_j, P_k) = r(P_i, P_j) \text{ when } i \leq 3. \quad (2.1)$$

Proof: It is well known that

$$r(P_i, P_j) = j + [i/2] \text{ when } j \geq i \geq 2. \quad (2.2)$$

In [1] it is shown that

$$r_2^3(G_i, G_j, G_k) \leq r(G_i, G_j) \text{ for } i \leq j \leq k, \quad (2.3)$$

and the equality holds if $k \geq r(G_i, G_j)$.

Lemma 2:

$$r_2^3(P_i, P_j, P_k) \leq [(4k+2j+i-2)/6] \quad (2.4)$$

when $i=4$ and $k < r(P_i, P_j)$.

Proof: From (2.2) and (2.3), $k=j$ and $[(4k+2j+i-2)/6] = j + [(i-2)/6]$. Let $j=4$ and G be a colored K_4 with no $P_4(1,2)$ as a subgraph which implies that \exists a $x \in V(G) \rightarrow d_3(x) \geq 2$.

If y and z are adjacent to x is color 3 then G has a $P_4(1,3)$ or $P_4(2,3)$. Assume that

$$r_2^3(P_4, P_{j-1}, P_{j-1}) = j-1 \quad \text{for all } j > 4. \quad (2.5)$$

Let G be a colored K_j with no $P_4(1,2)$ as a subgraph. If \exists a $x \in V(G) \rightarrow d_3(x) \geq j-3$, then by (2.5) $G-x$ contains a $P_{[j-1]}(1,3)$ or $P_{[j-1]}(2,3)$ and so G contains a $P_j(1,3)$ or $P_j(2,3)$. Hence, let $d(x) \geq 3 \forall x \in N(G)$, which implies that G contains a $P_4(1,2)$, a contradiction. $(1,2)$

Lemma 3: Let $k \geq 3$ and $j = k - 2[(k+4)/6]$. Then

$$r_2^3(P_j, P_j, P_k) \leq k-1.$$

Proof: Let s be the least non-negative integer $\rightarrow s \equiv k \pmod{6}$.

It is easily shown that

$$r(P_j, P_j) = j + [j/2] - 1 = k - 1 - [s/2]. \quad (2.6)$$

From (2.3) and (2.6) the lemma follows.

Lemma 4: Let $k \geq 3$ and $\ell = k + [(k-2)/6]$. Let G be a colored K_k . If G contains a $C_{[k-1]}(1,2)$, $C_{[k-1]}(1,3)$, or $C_{[k-1]}(2,3)$, then G contains a $P_k(1,2)$, $P_k(1,3)$, or $P_k(2,3)$, respectively.

Proof: Without loss of generality assume that G contains a $C_{[k-1]}(2,3)$ but not $P_k(2,3)$ which implies that the $[(k+4)/6]$ vertices of G not in $C_{[k-1]}(2,3)$ are adjacent in color 1 to C_{k-1} . By Lemma 3, the subgraph generated by nodes of C_{k-1} contains a $P_j(1,2)$ or $P_j(1,3)$ where $j = k - 2[(k+4)/6]$. Without loss of generality assume that $P_j(1,2)$ is present and let x be one of its end vertices. Consider the remaining $2[(k+4)/6] - 1$ vertices of C_{k-1} . Since there exists $[(k+4)/6]$ vertices not in C_{k-1} , but adjacent to every vertex of C_{k-1} in color 1, there exists a path P with $2[(k+4)/6]$ vertices in color 1, vertex disjoint from $P_j(1,2)$ referred above.

This path P has an end vertex adjacent to x in color 1 and hence G contains $P_{k(1,2)}$ as a subgraph.

Theorem 1:

$$r_2^3(P_i, P_j, P_k) \leq [(4k+2j+i-2)/6] \tag{2.7}$$

when $k < j + [i/2] - 1 = r(P_i, P_j)$.

Proof: If $i \leq 4$, the theorem follows from Lemma 2. The rest of the proof is divided into three main cases. Assume that the theorem holds when $i' < i, j' < j$ or $k' < k$ where $i' \geq 5$. Define $\ell = [(4k+2j+i-2)/6]$. Let G be a colored K_ℓ .

Case 1: Let $i = j = k$. Without loss of generality let $x_1 \in V(G)$ be \rightarrow

$$n = d_1(x_1) \geq d_i(x) \tag{2.8}$$

For $i = 2, 3$ and $x \in V(G)$ and $2 \leq n$.

Consider $G-x_1$. By the induction hypothesis,

$$r_2^3(P_{i-2}, P_{i-2}, P_i) \leq i-2 + [(i+4)/6]$$

which implies that $G-x_1$ contains a $P_{[i-2](1,2)}$, $P_{[i-2](1,3)}$ or $P_{i(2,3)}$.

Without loss of generality assume that $G-x_1$ has $P_{[i-2](1,2)}$ and denote this path by $P = (y, \dots, z)$.

Case 1.1: Let $(x_1, y) \in E_1$ and $(x_1, z) \in E_3$, since otherwise the proof follows from Lemma 4. If $(x_1, u) \in E-E(P)$ and $(x_1, u) \in E_1$ then G has a $P_{i(1,2)}$. Thus x_1 is adjacent to n vertices of $V(P)$ in color 1. Let $v \neq y$ be $\rightarrow (x_1, v) \in E_1$ and $v \in V(P)$. Let u be adjacent to v in P on the segment from v to y . Let $(y, u) \notin E_3$. Then the existence of cycle $(x_1, y, \dots, u, z, \dots, v, x_1)$, by Lemma 4 implies the existence of $P_{i(1,2)}$ completing the proof.

Suppose we let $(z, u) \in E_3$. Let $f \in V(G)-V(P)$. If $(z, f) \in E_1 \cup E_2$, then G has a $P_{i(1,2)}$ and hence let $(z, f) \in E_3$ for all f . Since $V(G)-V(P) = [(i+4)/6]+1$, $d_3(z) \geq [(i+4)/6]+1+n-1$. Since $d_3(z) \leq n$, $[(i+4)/6] = 0$ contradicting $i \geq 5$.

Case 1.2: Let (x_1, y) and (x_1, z) be in E_3 . Let x_1 be adjacent to n_1 vertices of $V(P)$ and n_2 vertices of $V(G)-V(P)$ in color 1 where $n_1, n_2 \geq 0$. Let $v \in V(P)$ be such that $(x_1, v) \in E_1$. Let u be a vertex adjacent to v on the segment (y, \dots, v) . By an argument similar to that used in Case 1.1, it can be shown that if $(z, u) \notin E_3$ the proof follows from Lemma 3. For the other case, let z be adjacent to at least n_1 vertices of $V(P)$ in color 3. For $w \in V(G)-V(P)$ if $(x, w) \in E_1$ and $(z, w) \in E_1 \cup E_2$ the theorem follows. So z is adjacent to at least n_2 vertices of $V(G)-V(P)$ in color 3. So $d_3(z) \geq n_1 + n_2 + 1$, contradicting (2.8).

Case 2: Let $i = j < k$. If x_1 is $\rightarrow d_1(x_1) \geq d_i(x)$ for $i = 1, 2, 3$ and $x \in V(G)$ then $r_2^3(P_{i-2}, P_{i-2}, P_k) \leq \ell - 1$ and Case 1 applies. Hence, without loss of generality assume that $d_2(x_2) \geq d_i(x)$ for $i = 1, 2, 3$ and $x \in V(G)$. Consider $G-x_2$. By induction hypothesis $r_2^3(P_{i-2}, P_j, P_{k-1}) \leq \ell - 1$ and hence $G-x_2$ contains $P_{[i-2](1,2)}$, $P_j(1,3)$ or $P_{[k-1](2,3)}$. If $P_{[i-2](1,2)}$ is present the proof is similar to Case 1. Let $G-x_2$ contain $P_{[k-1](2,3)}$. If z is an end vertex of this path then by arguments similar to Case 1

a contradiction, $d_1(z) = d_2(x_2)$ is derived thus proving the theorem. The theorem again follows if $P_j(1,3)$ is a subgraph of $G-x_2$.

Case 3: Let $i < j \leq k < j + \lfloor i/2 \rfloor - 1$. Let $x \in V(G)$ be such that $d_1(x) + d_2(x) \leq d_1(y) + d_2(y)$ for $y \in V(G)$. By induction hypothesis $r_2^3(P_i, P_{j-1}, P_{k-1}) \leq \ell - 1$, $G-x$ contains a $P_i(1,2)$, $P_{\lfloor j-1 \rfloor}(1,3)$ or $P_{\lfloor k-1 \rfloor}(2,3)$. The case is not obvious, if one of the latter two paths is present. If $d_1(x) + d_2(x) \leq \lfloor j/2 \rfloor$, then $d_3(x) \geq \lfloor k/2 \rfloor$ so that x is adjacent to more than half the vertices of the graph and hence of the path under consideration in color 3. Therefore, G contains a $P_j(1,3)$ or $P_k(2,3)$. If $d_1(x) + d_2(x) \geq \lfloor j/2 \rfloor$ and if $\langle E_1 \cup E_2 \rangle$ is connected, it is a standard result that G contains a $P_\ell(1,2)$, $\ell \geq 2\lfloor j/2 \rfloor$ and hence G has a $P_i(1,2)$. However, if $\langle E_1 \cup E_2 \rangle$ is disconnected, it contains at least two components, each of which is of order $\lfloor j/2 \rfloor$ or greater and hence G contains a $P_j(3)$.

Theorem 2:

$$r_2^3(P_i, P_j, P_k) > \lfloor (4k+2j+i-2)/6 \rfloor - 1.$$

Proof: Let $G = K_{\ell-1}$, where $\ell = \lfloor (4k+2j+i-2)/6 \rfloor$. Let X, Y, Z be pairwise disjoint subgraphs of G such that $|X| = \lfloor (2k+j-i-1)/3 \rfloor$, $|Y| = \lfloor (2j-2k+i-2)/6 \rfloor$ and $|Z| = \lfloor (k-j+i-2)/3 \rfloor$. It can be verified that $V(G) = |X| + |Y| + |Z|$. Color the edges of G as follows. Color the edges of X using color 3, edges of Y using color 1, edges of Z using color 2, edges between X and Y using color 1, edges between X and Z using color 2, and edges between Y and Z using color 1. It can be shown that $|X| + |Z| = k-1$ which rules out the existence of $P_k(2,3)$. Similarly $2|Y| + |X| \leq j-1$ ruling out $P_j(1,3)$ and $2|Y| + 2|Z| + 1 \leq i-1$ ruling out $P_i(1,2)$.

Theorem 3:

$$r_2^3(P_i, P_j, P_k) = \lfloor (4k+2j+i-2)/6 \rfloor \text{ when } k < r(P_i, P_j) = j + \lfloor \frac{i}{2} \rfloor - 1$$

and

$$r_2^3(P_i, P_j, P_k) = r(P_i, P_j) \text{ when } k \geq r(P_i, P_j) = j + \lfloor \frac{i}{2} \rfloor - 1.$$

Proof: Follows from (2.3) and Theorems 1 and 2.

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