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THE PACKING AND COVERING OF THE COMPLETE GRAPH I: THE FORESTS OF ORDER FIVE

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ABSTRACT. The maximum number of pairwise edge disjoint forests of order five in the complete graph K_n, and the minimum number of forests of order five whose union is K_n, are determined.

KEY WORDS AND PHRASES. Packing, Covering, Decomposition.
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1. INTRODUCTION

Graphs in this paper are finite with no multiple edges or loops. Beineke [1] defined the general covering (respectively, packing) problem as follows:

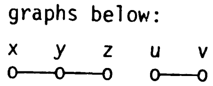
For a given graph G find the minimum (maximum) number of edge disjoint subgraphs of G such that each subgraph has a specified property P and the union of the subgraphs is G.

Solutions of these problems are known only for a few properties P, when G is arbitrary. In most cases G is taken to be the complete graph K_n or the complete bipartite graph $K_{m,n}$ (for particular references one may look at Roditty [2]).

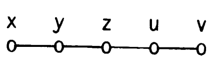
DEFINITION: The complete graph K_n is said to have a G-decomposition if it is the union of edge disjoint subgraphs each isomorphic to G. We denote such a decomposition by $G|K_n$.

The G-decomposition problem is to determine the set $N(G)$ of natural numbers such that K_n has a G-decomposition if and only if $n \in N(G)$. Note that G-decomposition is actually an exact packing and covering. In the proof of our problems of packing and covering, we make great use of the results obtained for the G-decomposition problem in cases when G has five vertices. As usual $[x]$ will denote the largest integer not exceeding x and $\{x\}$ the least integer not less than x. We will let $e(G)$ denote the number of edges of the graph G and $H = \bigcup_{i=1}^{t} G_i$ will show that the graph H is the union of t edge disjoint graphs G_i, $i=1,2,\ldots,t$.

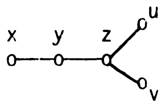
The Theorem of this paper solves variations of the covering and packing problems for the four graphs below:

(i) F_1 : 

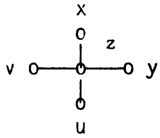
denoted $[(x,y,z)(u,v)]$

(ii) F_2 : 

denoted $[(x,y,z,u,v)]$

(iii) F_3 : 

denoted $(x,y,z;\ u,v)$

(iv) F_4 : 

denoted $(z;\ x,y,u,v)$

Our theorem may now be states as THEOREM (Packing and Covering).

Let F be F_1, F_2 or F_3 and $n \geq 5$ or F be F_4 and $n \geq 7$ then

(i) The maximum number of edge disjoint graphs F which are subgraphs of the complete graph K_n is

$$[e(K_n)/e(F)].$$

(ii) The minimum number of graphs F whose union is the complete graph K_n is

$$\{e(K_n)/e(F)\}.$$

2. PROOF OF THE THEOREM

We give a separate proof for each choice of F.

F_1: Proving the Theorem true for $n \geq 5$ is a straightforward exercise. Bermond et al. [3] show that

$$N(F_1) = \{n \mid n \equiv 0,1 (\mathrm{mod}\ 3)\ ,\ n \geq 6\}\ . \tag{2.1}$$

Thus we have to consider only $n = 3m + 2$, $m \geq 2$. Observe that

$$K_{3m+2} = K_{3m} \cup K_{2,3m} \cup K_2\ ,\ m \geq 2\ . \tag{2.2}$$

by (2.1) K_{3m} has an F_1-decomposition. Since $K_{2,3m} = mK_{2,3}$ and $K_{2,3}$ can be decomposed easily into two graphs F_1, it follows that $K_{2,3m}$ has an F_1-decomposition. Only K_2 in (2.2) is left non-packed. Hence, the Theorem is proved in this case.

F_2: The proof will examine several cases depending on the value of n. The following table summarizes the cases $n = 5,6,7,8m$, and $8m + 1$ for $m \geq 1$.

n	packing	remains for covering
5	(0,1,2,3,4);(1,3,0,4,2)	(0,2);(1,4)
6	(0,1,2,3,4);(0,5,4,1,3);(0,4,2,5,3)	(0,3),(0,2),(1,5)
7	(0,1,2,3,4);(0,2,4,6,1);(1,3,5,0,4)	
	(1,4,5,6,0);(1,5,2,6,3)	(0,3)
8m,8m+1	F_2 - decomposition [4]	

Table 1

We still have to prove the theorem for the cases:

$$n = 8m + k \quad , \quad k = 2,\dots,7$$

<u>k = 2</u>.

Let

$$K_{8m+2} = K_{8m} \cup K_{2,8m} \cup K_2 \ . \tag{2.3}$$

The graph K_{8m} has an F_2-decomposition. since $K_{2,8m} = 2mK_{2,4}$ and $K_{2,4}$ can be decomposed easily into two graphs F_2, it follows that $K_{2,8m}$ has an F_2-decomposition. Only K_2 in (2.3) is left non-packed.

<u>K = 3</u>.

Let

$$K_{8m+3} = K_{8m+1} \cup K_{2,8m+1} \cup K_2 \ . \tag{2.4}$$

The graph K_{8m+1} has an F_2-decomposition $K_{2,8m+1} = K_{2,8m} \cup K_{2,1}$ and $K_{2,8m}$ has an F_2-decomposition as we saw above. This decomposition of $K_{2,8m}$ can be done in such a way that the edge $(8m-1,\ 8m+2)$ is at one end of the F_2 which includes it and the point $8m-1$ is an end-point of that F_2. Thus we can replace the edge $(8m-1,8m+2)$ with the edge $(8m,\ 8m+2)$. Only the edges $(8m,8m+1)$, $(8m+1,8m+2)$, $(8m-1,8m+2)$ now remain non-packed, and they can be included in one more F_2.

<u>k = 4</u>.

Note that

$$K_{8m+4} = K_{8m} \cup K_{4,8m} \cup K_4 . \tag{2.5}$$

The graph K_{8m} has an F_2-decomposition. Now

$$K_{4,8m} \cup K_4 = 2(2m-1)K_{2,4} \cup 2K_{2,4} \cup K_4 \tag{2.6}$$

and the $2K_{2,4}$'s can be selected to be vertex disjoint. Since $K_{2,4}$ has an F_2-decomposition, so does $2(2m-1)K_{2,4}$. We need only to show that $2K_{2,4} \cup K_4$ can be packed by 5 F_2 graphs, leaving two non-packed edges.

Let $V(2K_{2,4}) = \{1,2,\dots 8,a,b,c,d\}$, $V(K_4) = \{a,b,c,d\}$.

Then, the 5 graphs of the packing of $2K_{2,4} \cup K_4$ are:

$$(a,3,b,c,8);(1,a,d,7,c);(4,b,d,c,5);(a,c,6,d,8);(1,b,2,a,4).$$

The edges $(d,5)$ and (a,b) are left non-packed.

<u>k = 5</u>.

Let

$$K_{8m+5} = K_{8m+1} \cup K_{4,8m} \cup K_{4,1} \cup K_4 \ . \tag{2.7}$$

The graph K_{8m+1} has an F_2-decomposition. In the case $K = 4$ we saw that $K_{4,8m} \cup K_4$ has an F_2-packing leaving two non-packed edges.

Let $V(K_4) = \{a,b,c,d\}$ and $V(K_{8m+1}) = Z_{8m+1}$. Denote the non-packed edges by (a,b) and $(8m-1,d)$. We show that $G = K_{4,1} \cup \{(a,b),(8m-1,d)\}$ has an F_2-packing leaving two non-packed edges. The F_2 of this packing is $(8m-1,d,8m,a,b)$. The non-packed edges are: $(c,8m)$ and $(8m,b)$.

k = 6.
Write

$$k_{8m+6} = K_{8m} \cup K_{6,8m} \cup K_6.$$

The graph K_{8m} has and F_2-decomposition. Observe that $K_{6,8m} = 3K_{2,8m}$. In the case $k = 2$ we saw that $F_2|K_{2,8m}$. Table 1 shows that K_6 has F_2 - packing leaving three non-packed edges as required, and these three can be included in one more F_2.

k = 7.
Let

$$K_{8m+7} = K_{8m+1} \cup K_{6,8m} \cup K_7.$$

The graph K_{8m+1} has an F_2-decomposition, and $F_2|K_{6,8m}$, as was shown above. By Table 1 we know that the graph K_7 has an F_2-packing leaving one non-packed edge. The Theorem has now been proved for F_2 since all cases have been considered.

F_3: The proof will consider the same cases as the proof for F_2.

n	packing	remains for covering
5	(0,1,2;3,4);(1,4,0;2,3)	(1,3),(3,4)
6	(0,1,2;3,4);(3,4,5;0,2);(0,3,1;4,5)	(0,2),(0,4),(3,5)
7	(3,2,0;1,6);(5,4,1;2,3);(1,6,3;4,5)	
	(2,4,0;3,5);(1,5,6;2,4)	(2,5)
8m,8m+1	F_3 - decomposition [4]	

Table 2

We now have to prove the theorem for the cases:

$$n = 8m+k \ , \ k = 2,\dots,7, \ m\geq 1 \ .$$

k = 2.
Let K_{8m+2} be as in (2.3). The graph K_{8m} has an F_3-decomposition. Since $K_{2,8m} = 2mK_{2,4}$ and $K_{2,4}$ can be decomposed easily into two F_3 graphs, it follows that $K_{2,8m}$ has an F_3-decomposition. Only K_2 in (2.3) is left non-packed. Hence, the Theorem is proved in this case.

k = 3.
Let K_{8m+3} be as in (2.4). K_{8m+1} has an F_3-decomposition. $K_{2,8m+1} = K_{2,8m} \cup K_{2,1}$. The graph $K_{2,8m}$ has an F_3-decomposition as was shown above. Replace the edge (8m-4, 8m+2) which appears in some F_3 in the decomposition of $K_{2,8m}$, with the edge (8m,8m+2). Then the edges (8m-4, 8m+2), (8m+2, 8m+1), (8m+1, 8m) remain non-packed, but could be included in one additional F_3.

k = 4.
Let K_{8m+4} be as in (2.5). The graph K_{8m} has an F_3-decomposition. Let, $K_{4,8m} \cup K_4$ be as in (2.6). Since $K_{2,4}$ has an F_3-decomposition, so does $2(2m-1)K_{2,4}$. We show that $2K_{2,4} \cup K_4$ can be packed by five F_3 graphs, leaving two non-packed edges.

Let $V(2K_{2,4}) = \{1,2,\ldots 8,a,b,c,d\}$ and $V(K_4) = \{a,b,c,d\}$.
Then the five graphs F_3 are:

$$(c,6,d;7,8),\ (d,5,c;7,8),\ (4,a,c;d,b),\ (1,a,b;4,d),\ (a,3,b;1,2).$$

The edges $(a,2)$ and (a,d) are left non-packed.

<u>$k = 5$.</u>

Let K_{8m+5} be as in (2.7). The graph K_{8m+1} has an F_3-decomposition. In the case $k = 4$ we saw that $K_{4,8m} \cup K_4$ has an F_3-packing leaving two non-packed edges. Let $V(K_4) = \{a,b,c,d\}$ and $V(K_{8m+1}) = Z_{8m+1}$. Denote the non-packed edges by (a,d) and $(a,8m-1)$. The F_3 graph in $K_{4,1} \cup \{(a,d,),\ (a,8m-1)\}$ is $(b,8m,a;d,8m-1)$. The edges $(d,8m)$ and $(c,8m)$ remain non-packed.

The proofs for $k = 6,7$ are accomplished in the same ways as for F_2.

Once again all cases have been considered and the proof is complete for F_3.

F_4: it is easy to see that the theorem does not hold for $n = 5$ and $n = 6$. For K_7 the graphs F_4 of the packing are: $(0;1,2,3,4)$, $(1;2,3,4,5)$, $(2;3,4,5,6)$, $(5;4,3,0,6)$, $(6;0,1,3,4)$. The edge $(3,4)$ is left non-packed. Hence, the theorem is proved for $n = 7$. For $n=8m$, $8m+1$ we have an F_4-decomposition [5,6]. Hence, we again have to prove the theorem for the cases:

$$n = 8m + k,\quad k = 2,\ldots,7,\quad m \geq 1.$$

<u>$k = 2$.</u>

Let $K_{8m+2} = K_{8m+1} \cup K_{1,8m+1}$. The graph K_{8m+1} has an F_4-decomposition. $K_{1,8m+1}$ is a star that can easily be packed by $2m$ stars F_4, leaving one non-packed edged.

<u>$k = 3$.</u>

Let $K_{8m+3} = K_{8m} \cup K_{3,8m} \cup K_3$. The graph K_{8m} has an F_4-decomposition. Let $K_{3,8m} = 3K_{1,8m}$. Since the graph $K_{1,8m}$ can be decomposed into $2m$ stars F_4, it follows that $K_{3,8m}$ also has an F_4-decomposition. Let $V(K_3)=\{a,b,c\}$, and create a decomposition of $K_{3,8m}$ which includes the three stars $(a;x,y,z,u)$,$(b;x,y,z,u)$, $(c;x,y,z,u)$. Replace the edge (a,u) by (a,b), the edge (b,u) by (b,c), and the edge (c,u) by (c,a). We did not spoil any star of the decomposition of $K_{3,8m}$ and the star $(u;a,b,c)$ of three branches is left non-packed.

<u>$k = 4$.</u>

Let $K_{8m+4} = K_{8m} \cup K_{4,8m} \cup K_4$. The graph K_{8m} has an F_4-decomposition. Let $K_{4,8m} = 4K_{1,8m}$. The graph $K_{1,8m}$ can be decomposed into $2m$ stars F_4 so $K_{4,8m}$ has an F_4-decomposition. Let $V(K_4)=\{a,b,c,d\}$, and consider the subgraph $K_{4,4}$ of $K_{4,8m}$ whose vertices are given by $V(K_{4,4}) = \{a,b,c,d\} \cup \{8m-1,\ 8m-2,\ 8m-3,\ 8m-4\}$. The F_4 decomposition of $K_{4,8m}$ can be arranged in such a way that our $K_{4,4}$ is made up of the four F_4 graphs $(a;8m-1,8m-2,8m-3,8m-4)$, $(b;8m-1,8m-2,8m-3,8m-4)$, $(c;8m-1,8m-2,8m-3,8m-4)$ and $(d;8m-1,8m-2,8m-3,8m-4)$. Replace the edges $(a,8m-1)$, $(b,8m-1)$, $(c,8m-1)$, $(d,8m-1)$ with the edges (a,c), (a,b), (b,c), (c,d), respectively we now have a new F_4 graph, namely $(8m-1;a,b,c,d)$. The edges (a,d) and (b,d) are the only one which remain non-packed.

<u>$k = 5$</u>.

Let $K_{8m+5} = K_{8m} \cup K_{5,8m} \cup K_5$. As before K_{8m} and $K_{5,8m}$ have F_4-decompositions. Now $K_5 = K_4 \cup K_{1,4}$ so we can complete the proof in the same way as in the case $k = 4$.

<u>$k = 6$</u>.

Let $K_{8m+6} = K_{8m} \cup K_{6,8m} \cup K_6$. The graphs K_{8m} and $K_{6,8m}$ have F_4-decompositions. Let $V(K_6) = \{v_1,v_2,v_3,v_4,v_5,v_6\}$. Graph K_6 can be packed with the two F_4 $\{v_1;v_2,v_3,v_4,v_5\}$ and $\{v_2;v_3,v_4,v_5,v_6\}$. The induced graph on $\{v_3,v_4,v_5,v_6\}$ is K_4. Hence, we can complete the proof here as in the case $k = 4$, leaving the edges (v_5,v_6), (v_4,v_6) non-packed. Those edges together with the non-packed edge (v_1,v_6) accomplish the proof of the theorem in this case.

<u>$k = 7$</u>.

Let $K_{8m+7} = K_{8m} \cup K_{7,8m} \cup K_7$. The graphs K_{8m} and $K_{7,8m}$ have F_4-decompositions and we apply the F_4-packing shown for K_7 at the beginning of this case.

This completes the proof of the theorem for F_4. □

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