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CERTAIN NEAR-RINGS ARE RINGS, II

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ABSTRACT. We investigate distributively-generated near-rings R which satisfy one of the following conditions: (i) for each x,y ε R, there exist positive integers m, n for which $xy = y^m x^n$; (ii) for each x,y ε R, there exists a positive integer n such that $xy = (yx)^n$. Under appropriate additional hypotheses, we prove that R must be a commutative ring.

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1. INTRODUCTION AND TERMINOLOGY.

Consider the following two properties, either of which is known to imply commutativity for rings R [3, 4]:

(C_1) For each x, y ε R, there exist positive integers n = n(x,y) and m = m(x,y) for which $xy = y^m x^n$.

(C_2) For each x, y ε R, there exists a positive integer n = n(x,y) such that $xy = (yx)^n$.

The main purpose of this note is to show that certain distributively-generated (d-g) near-rings R with these properties must be rings. The work may be regarded as a continuation of that in [2], to which paper the reader is referred for basic definitions.

Throughout the paper, the symbols Z and Z_+ will denote the integers and the positive integers respectively. The additive group of the ring R will be denoted by R^+, its derived subgroup by R', and its center by ξ(R). The two-sided annihilator of the subset S of R will be denoted by A(S), and the ideal generated by multiplicative commutators by C(R). Such terms as center, central, commute, and commutator, unless specifically stated to refer to addition, may be assumed to refer to multiplication. The near-ring R will be called strongly-distributively-generated (s-d-g) if it contains a set of distributive elements whose squares generate R^+.

The principal results (Theorems 2 and 4) are the following:

(a) Any s-d-g near-ring with 1 which satisfies (C_1) is a commutative ring.

(b) Any d-g near-ring R which satisfies (C_2) and has $R^2 = R$ is a commutative ring.

For the proofs, we shall require the classical theorem of Fröhlich [5, p. 93] which asserts that a d-g near-ring is distributive if and only if R^2 is additively commutative. Moreover, we shall assume the easy result that in any d-g near-ring R, the derived subgroup R' of R^+ is an ideal.

2. A COMMUTATIVITY RESULT FOR ARBITRARY d-g (C_1)-NEAR-RINGS.

THEOREM 1. Let R be any d-g near-ring satisfying (C_1). Then both C(R) and R' are nil ideals. In particular, if R has no non-trivial nil ideals, then R is a commutative ring.

The proof follows from two lemmas.

LEMMA 1. If R is any d-g near-ring satisfying (C_1), then R has each of the following properties:

(a) If a, b ε R and ab = 0, then ba = 0 = axb for all x ε R.

(b) All one-sided annihilators are two-sided and are ideals of R.

(c) Idempotent elements of R are central.

(d) The set N of nilpotent elements of R forms an ideal.

PROOF. Property (a) follows at once from (C_1), and (b) follows easily from (a). To establish (c), let e be an idempotent and x ε R. Then for some p, q $\geq$ 1, $xe = e^px^q = ex^q$ and hence $exe = ex^q = xe$; similarly, $ex = x^re$ for some r $\geq$ 1, so that $exe = x^re = ex$. Thus ex = xe.

To establish (d), it will suffice to show that N forms an additive subgroup [2, Lemma 1]; and this may be done by proving that for each a, b ε N and each positive integer j, $(a-b)^j$ is a finite sum $\Sigma \pm p_i$, where each p_i is a finite product of elements of R, of which at least j belong to T = {a,b}. To see that this is enough, note that if $a^n = b^m = 0$ and if j = n + m - 1, then each of the summands p_i will be zero by (a).

We proceed by induction on j, the case j = 1 being trivial. Suppose the result holds for j and write

$$(a-b)^{j+1} = (a-b)(a-b)^j = (a-b)(\Sigma \pm p_i) = \Sigma \pm (a-b)p_i,$$

where each p_i is a product having at least j factors from T; and for each $p = p_i$ write $(a-b)p = p^k(a-b)^m = (p^ka-p^kb)(a-b)^{m-1}$. If m = 1, we are finished; otherwise, express $(a-b)^{m-1}$ in the form $\Sigma \pm d_s$ where the d_s are distributive. Since $(a-b)p = \Sigma \pm (p^kad_s - p^kbd_s)$ and since each of the products p^kad_s and p^kbd_s has at least j + 1 factors from T, the inductive step, and hence the proof of (d), is complete.

In view of (d), the proof of Theorem 1 will be complete once we establish the following lemma.

LEMMA 2. Let R be a d-g near-ring satisfying (C_1) and having no non-zero nilpotent elements. Then R is a commutative ring.

PROOF. By Lemma 3 of [1], R is a subdirect product of homomorphic images having no zero divisors; thus we assume that R has no zero divisors. Note that if R is multiplicatively commutative, hence distributive, then R^2 is additively commutative; therefore, for all a, b ε R

$$0 = a^2 + ab - a^2 - ab = a(a+b-a-b) = a + b - a - b,$$

so that R^+ is abelian. Observe also that if e is any non-zero idempotent, the fact that e(x - ex) = (x - xe)e = 0 for all x ε R shows that e must be a multiplicative identity element.

Assume, then, that R^+ is not abelian; hence R is not commutative. Let a and b be elements of R which do not commute; and let m, n, s, t be positive integers, at least one of which is greater than 1, for which $ab = b^m a^n = a^{ns} b^{mt}$. If ns = 1, then $a(b-b^{mt}) = 0$, and hence b^{mt-1} is a non-zero idempotent. If ns > 1, then $ab = a^{ns-1}ab^{mt} = (ab^{mt})^v (a^{ns-1})^w$ for appropriate positive integers v, w; and there exists an element c, which is either a or an element of the form ya, such that ab = abc. If follows that c is idempotent, and incidentally that c must have been of the form ya.

So far we have shown that any non-central element a of R either has a left inverse or has the property that for any b not commuting with a, $b^k = 1$ for some positive integer k. Suppose the latter holds, let c_1 be an element not commuting with a, and use (C_1) to obtain c for which $ac_1 = ca$. It is easily verified that ca does not commute with a, so $(ca)^k = 1$ for some integer k; and in this case also, a has a left inverse.

Now suppose that z is any non-zero central element, and that a is non-central. Then az = za is also non-central, so az has a left inverse, and therefore z has a left inverse. Thus, R is a division near-ring; and since division near-rings have commutative addition, we have contradicted our initial assumption concerning R. Hence, R^+ is abelian and R is a ring. Multiplicative commutativity follows from the result of [3].

3. COMMUTATIVITY OF s-d-g (C_1)-NEAR-RINGS WITH 1.

The major theorem of this section is the following:

THEOREM 2. If R is a strongly-distributively-generated near-ring with 1 which has property (C_1), then R must be a commutative ring.

Of course, it suffices to prove the theorem under the additional assumption that R is subdirectly irreducible (s-i). The lemmas which follow all treat the subdirectly irreducible case.

LEMMA 3. Let R be a s-i d-g near-ring with 1, in which all idempotents are central. Then 1 is the only non-zero idempotent.

PROOF. If e is any non-zero idempotent, the centrality of e enables us to show that 1-e is idempotent as well. Clearly $Re \subseteq A(1-e)$; moreover, if $x \in A(1-e)$, the representation $x = x(1-e) + xe$ shows that $x = xe \in Re$. Thus, $Re = A(1-e)$, and similarly $R(1-e) = A(e)$. But it is easy to show that A(x) is an ideal for any central x, so Re and R(1-e) are ideals, which obviously have trivial intersection. The subdirect irreducibility of R therefore forces R(1-e) to be trivial, so that e = 1.

LEMMA 4. Let R be a s-i d-g (C_1)-near-ring with 1. Then

(a) if $x \in R$, either x commutes with -1 or there exists $k \in Z_+$ such that $x^k = 1$;

(b) for each $x \in R$, $x^2(-1) = (-1)x^2$.

PROOF. (a) Suppose $x(-1) \neq (-1)x$, and chose $k, j, m, n \in Z_+$ such that

$$x(-1) = (-1)^k x^j = x^{jn}(-1)^{km}. \tag{3.1}$$

Assume first that jn > 1. If km is odd, we have $x = x^{jn}$, which implies that x^{jn-1} is a non-zero idempotent, necessarily equal to 1. If km is even, (3.1) yields $-x = x^{jn}$, hence $x = -x^{jn} = x^{jn-1}x(-1) = x^{jn-1}x^{jn}$ and $x^{2(jn-1)} = 1$.

On the other hand, if jn = 1, then (3.1) yields $-x = x(-1) = x$. Choose $q, s \in Z_+$ with $(-1)x = x^s(-1)^q$. Since $x(-1) = x$, this implies $(-1)x = x^s$ with s > 1; and we conclude that $x = (-1)x^s = ((-1)x)x^{s-1} = x^{2s-1}$, so that $x^{2(s-1)} = 1$.

(b) It follows from part (a) that zero divisors commute with -1, hence we may assume x is not a zero divisor. Now if $x(-1) = x^s$ for some $s \in Z_+$, x commutes with x(-1) and consequently x(-1) = (-1)x; therefore, we may assume that $x(-1) = (-1)x^s$ for some $s \in Z_+$. Thus, $(-1)x(-1) = x^s$; and commuting x with (-1)x(-1) gives $x(-1)x(-1) = (-1)x(-1)x$ or $(-x)^2 = (-1)(-x)(x)$. In this equality we may replace x by -x, since x commutes with -1 if and only if -x does; therefore $x^2 = (-1)x(-x)$ and $(-1)x^2 = x(-x) = -x^2 = x^2(-1)$.

LEMMA 5. Let R be a s-i d-g (C_1)-near-ring with 1, and let D be the set of zero divisors. Then

(a) D is an ideal and D^+ is abelian;

(b) $R' = C(R) \subseteq A(D)$;

(c) if $d \in D$ and x does not commute with d, then there exists $s \in Z_+$ such that $xd = dx^s$.

PROOF. (a) Let S be the heart of R —— that is, the intersection of all non-zero ideals; to show D is an ideal, we show that D = A(S). Clearly $A(S) \subseteq D$; conversely, if $d \in D$, A(d) is a non-trivial ideal, hence $S \subseteq A(d)$ and $d \in A(S)$. Therefore D = A(S). Note that by Lemma 4(a), all elements of D commute with -1; thus if $d_1, d_2 \in D$, we have $-d_1 - d_2 = (-1)(d_1 + d_2) = (d_1 + d_2)(-1) = -d_2 - d_1$, so D^+ is abelian.

(b) If $x,y \in R$ and $d \in D$, then dx and dy are in D; hence $dx + dy - dx - dy = 0 = d(x + y - x - y)$, and $R' \subseteq A(D)$. By Fröhlich's theorem and the commutativity of rings satisfying (C_1), a d-g (C_1)-near-ring with 1 is additively commutative if and only if it is multiplicatively commutative; and applying this observation to R/C(R) and R/R' gives C(R) = R'.

(c) By part (b) we have d(dy -yd) = 0 —— that is, $d^2y = dyd$ —— for all $y \in R$; moreover, since $yd = dy_1$ for some $y_1 \in R$, we get $yd^2 = dy_1d = d^2y_1 = dyd$ as well. Thus, for all $d \in D$, all $y \in R$ and all integers $i \geqslant 2$, we have $d^iy = yd^i$. Suppose that $d \in D$ and $x \in R$ are such that $xd = d^mx^s$ for some m > 1, and choose n, q such that $dx = x^nd^q$. Then $xd = d^mx^s = d^{m-1}(dx)x^{s-1} = d^{m-1}x^nd^qx^{s-1} = d^{m+q-1}x^{n+s-1}$, and $dx = x^nd^q = x^{n-1}(xd)d^{q-1} = x^{n-1}d^mx^sd^{q-1} = x^{n+s-1}d^{m+q-1}$. Since m + q - 1 > 1, we thus have xd = dx, so our proof is finished.

LEMMA 6. Let R be s-i and d-g with 1, and suppose R satisfies (C_1). Then 2R' = 0.

PROOF. By Theorem 1, $R' \subseteq D$. By Lemma 4(a), all elements of R' commute with -1; in particular, for all $x, y \in R$, x + y - x - y + y - x - y + x = x + y - x - x - y + x commutes with -1. Taking for x and y a pair r, s of distributive elements and simplifying, we get the result that 2(r + s - r - s) = 0, from which it follows that 2R' = 0.

LEMMA 7. For R a s-i d-g (C_1)-near-ring with 1, $A(2) \subseteq \xi(R)$. In particular, $R' \subseteq \xi(R)$ and therefore, $2R \subseteq \xi(R)$.

PROOF. If $2 \notin D$, R^+ is abelian by Lemma 6; hence assume $2 \in D$. Let $x \in A(2)$; and by Lemma 5(c), choose $k \in Z_+$ such that $(1 + x)2 = 2(1 + x)^k = 2$. Thus, 1 + x + 1 + x = 1 + 1, which yields x + 1 = 1 + x. If $b^k = 1$ for some $k \in Z_+$, we now get $b + x - b - x = b(1 + b^{k-1}x - 1 - b^{k-1}x) = 0$. By Lemma 4(a), the only elements b yet to be considered commute with -1; and since x commutes additively with b if and only if it commutes additively with b + x, we may assume b + x commutes with -1 also. Since $x \in D$, x commutes with -1, and it follows at once that b + x = x + b. Thus, $A(2) \subseteq \xi(R)$.

That $R' \subseteq \xi(R)$ is now clear from Lemma 6. Noting that $2R = R2$, we complete the proof of the lemma by showing that $x + x+y = y + x + x$ for all $x, y \in R$. But since $y - x = -x + y+c$ for some $c \in R'$ and since $R' \subseteq \xi(R)$, we have $x + x + y - x - x - y = x + y + c - x - y = x + y - x - y + c = y + c - y + c = c + c = 0$, the last equality following from Lemma 6.

PROOF OF THEOREM 2. We need only show that R^+ is abelian, since the theorem then follows from Fröhlich's theorem and the theorem of [3]. Let r, s be arbitrary distributive elements of R. By Lemma 4(b), $(r + s)^2$ commutes with -1, which means that

$$-r^2 - sr - rs - s^2 = -s^2 - rs - sr - r^2. \tag{3.2}$$

If we write $rs = c + sr$, where $c = rs - sr \in \xi(R)$, and recall that $2sr \in \xi(R)$, we can write (3.2) as

$$-r^2 - s^2 - 2sr - c = -s^2 - r^2 - 2sr - c.$$

It follows at once that $r^2 + s^2 = s^2 + r^2$, and the fact that R is strongly-distributively-generated implies that R^+ is abelian.

I conjecture that Theorem 2 remains true if R is merely assumed to be d-g rather than s-d-g, but a proof eludes me. However, all the machinery is in place to establish two interesting cases of the conjecture.

THEOREM 3. Let R be a d-g near-ring with 1, and suppose R satisfies one of the following specialized versions of (C_1):

(C_3) For each $x, y \in R$, there exists an integer $n = n(x, y) \geq 1$ for which $xy = yx^n$.

(C_4) For each $x, y \in R$, either $xy = yx$ or there exist $m, n \in Z_+$ with $m \geq 2$, such that $xy = y^m x^n$.

Then R is a commutative ring.

PROOF. Again we may assume that R is subdirectly irreducible and (by Lemma 6) that $2 \in D$. Arguments similar to that of Lemma 5(c) show that zero divisors are central, and commuting 2 with r + s for arbitrary distributive r, s now shows R^+ is abelian.

4. COMMUTATIVITY OF d-g (C_2)-NEAR-RINGS.

THEOREM 4. If R is any d-g near-ring satisfying (C_2), then R is commutative. Moreover, if $R^2 = R$, then R is a ring.

PROOF. Note first that idempotents are central, for if e is idempotent and $x \in R$, and if we choose $n, m \in Z_+$ such that $ex = (xe)^n$ and $xe = (ex)^m$, right-multiplying the first of these equalities by e and left-multiplying the second by e yields $ex = exe = xe$.

Now if R is any d-g (C_2)-near-ring and $a, b \in R$ are such that $ab \neq ba$, there exist $m, n > 1$ such that

$$ab = (ba)^n \text{ and } ba = (ab)^m; \tag{4.1}$$

it follows that

$$ab = (ab)^{nm} \text{ and } ba = (ba)^{nm}, \tag{4.2}$$

and hence that $(ab)^{nm-1}$ and $(ba)^{nm-1}$ are both idempotent. In fact, if $(ab)^t = e$ is idempotent, (4.1) shows that $(ba)^t = (ab)^{mt} = e$, hence $(ab)^{nm-1}$ and $(ba)^{nm-1}$ are equal to the same idempotent, say e_1.

We treat first the case of R with 1, and as usual consider the subdirectly irreducible case. If we suppose R contains a pair of non-commuting elements a, b and choose n, m as above, then Lemma 3 guarantees that $(ab)^{nm-1} = (ba)^{nm-1} = 1$, so that a and b are both invertible. We choose $q \in Z_+$ such that $(b^{-1}a)b = (b(b^{-1}a))^q$, which reduces at once to $ab = ba^q$. Thus, R is a commutative ring by Theorem 3.

We now drop the hypothesis that R has 1. Again suppose $ab \neq ba$ and let n, m and e_1 be as above. The near-ring e_1R is a d-g (C_2)-near-ring having e_1 as multiplicative identity, hence is commutative. Therefore $e_1ae_1b - e_1be_1a = 0 = e_1(ab - ba)$; and since $e_1 = (ab)^{nm-1} = (ba)^{nm-1}$, an appeal to (4.2) yields the contradiction ab = ba. Hence R must be commutative; and if $R^2 = R$, Fröhlich's theorem shows that R is a ring.

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