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IRREGULAR AMALGAMS

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ABSTRACT. The amalgam of L^p and ℓ^q consists of those functions for which the sequence of L^p-norms over the intervals $[n,n+1)$ is in ℓ^q. These spaces (L^p,ℓ^q) have been studied in several recent papers. Here we replace the intervals $[n,n+1)$ by a cover $\alpha = \{I_n; n\epsilon Z\}$ of the real line consisting of disjoint half-open intervals I_n each of the form $[a,b)$, and investigate which properties of (L^p,ℓ^q) carry over to these irregular amalgams $(L^p,\ell^q)_\alpha$. In particular, we study how $(L^p,\ell^q)_\alpha$ varies as p, q, and α vary and determine conditions under which translation is continuous.

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1. INTRODUCTION.

The <u>amalgam</u> of L^p and ℓ^q is the space (L^p,ℓ^q) of functions f which are locally in L^p and satisfy

$$\|f\|_{p,q} = \{\sum_{n=-\infty}^{\infty} [\int_n^{n+1} |f(x)|^p dx]^{q/p}\}^{1/q} < \infty, \tag{1.1}$$

that is, the L^p-norms over the intervals $[n,n+1]$ form an ℓ^q-sequence. (If either p or q is infinite, the expression in (1.1) is modified in the usual way.) Special cases of amalgams were first introduced by Wiener [1], [2] and Stepanoff [3] in the 1920's but their first systematic study is due to Holland [4].

Notice that for $p = q$ we have $(L^p,\ell^p) = L^p$ and indeed some of the properties of Lebesgue spaces can be carried over to amalgams. For instance, if $1 \leq p,q < \infty$, the norm given by (1.1) makes (L^p,ℓ^q) into a Banach space with dual $(L^{p'},\ell^{q'})$, where $1/p + 1/p' = 1$, and the analogues of Holder's inequality and Young's inequality are also valid.

It is not difficult to establish the following inclusion relations between amalgams.

$$\text{If } q_1 \leq q_2, \text{ then } (L^p,\ell^{q_1}) \subset (L^p,\ell^{q_2}). \tag{1.2}$$

If $p_1 \leq p_2$, then $(L^{p_2},\ell^q) \subset (L^{p_1},\ell^q)$. (1.3)

For further information about amalgams we refer the reader to [5].

In this paper we propose to replace the cover consisting of the intervals $[n,n+1)$ in (1.1) by a cover $\alpha = \{I_n; n\varepsilon Z\}$ of the real line consisting of disjoint half-open intervals I_n, each of the form $[a,b)$, whose union is the real line. Throughout the paper we use the term cover to refer to a cover of this type. If

$$\|f\|_{p,q,\alpha} = \{\sum_n [\int_{I_n} |f(x)|^p dx]^{q/p}\}^{1/q}$$

we define the irregular amalgam

$$(L^p,\ell^q)_\alpha = \{f;\ \|f\|_{p,q,\alpha} < \infty\}$$

Such spaces have arisen in the work of Jakimovski and Russell [6] on approximation theory. Given a complex sequence $(y_n)_{n\varepsilon Z}$ and a linear space S of real-valued functions, they considered the interpolation problem: For a given increasing sequence (α_n), find a function f in S such that $f(\alpha_n) = y_n$. For certain normed sequence spaces E they took S to consist of the functions f with

$$\|(\|f\|_{L^p[\alpha_n,\alpha_{n+1})})_{n\varepsilon Z}\|_E < \infty$$

and showed the existence of optimal solutions to the interpolation problem. Notice that if we take $E = \ell^q$, then S becomes the irregular amalgam $(L^p,\ell^q)_\alpha$ where $\alpha = \{[\alpha_n,\alpha_{n+1})\}$.

An important special case of an irregular amalgam is the dyadic amalgam $(L^p,\ell^q)_\delta$ where $\delta = \{[\delta_n,\delta_{n+1})\}$ is the cover given by $\delta_o = 0$, $\delta_n = 2^n$ if $n > 0$, $\delta_n = -2^{-n}$ if $n < 0$. J.W. Wells proved that if $f \in L^p(-\infty,\infty)$, $1 < p \leq 2$, then its Fourier transform $\hat{f}$ lies in the dyadic amalgam $(L^{p'},\ell^2)_\delta$. (Kellogg [7] and Williams [8] proved similar results for the circle group and connected groups, respectively.)

Certain facts about irregular amalgams are straightforward generalizations of known results about regular amalgams. Thus we state the following theorems without proof.

THEOREM 1. If $1 \leq p,q < \infty$, the irregular amalgam $(L^p,\ell^q)_\alpha$ is a Banach space whose dual space is $(L^p,\ell^q)^*_\alpha = (L^{p'},\ell^{q'})_\alpha$.

THEOREM 2. If $f \in (L^p,\ell^q)$ and $g \in (L^{p'},\ell^{q'})$, where $p,q \geq 1$, then $fg \in L^1$ and

$$\|fg\|_1 \leq \|f\|_{p,q,\alpha} \|g\|_{p',q',\alpha}$$

Other aspects of regular amalgams do not always generalize to irregular amalgams. For instance, as stated in [5], although the inclusion (1.2) holds for irregular amalgams, (1.3) does not hold in general. In fact we show in Section 2 that the analogue of (1.3) holds if and only if the lengths of the intervals I_n are bounded above.

In Section 3 we discuss inclusion relations between irregular amalgams when p and q are fixed and the sequence α varies. We write $\alpha \prec \beta$ if the intervals $I_n \in \alpha$ intersect boundedly many of the intervals of β and show that if $q \leq p$ and $\alpha \prec \beta$, then $(L^p,\ell^q)_\alpha \subset (L^p,\ell^q)_\beta$ with strictness if $\beta \not\prec \alpha$. If $q \geq p$, the inclusion is reversed. It then follows that $(L^p,\ell^q)_\alpha = (L^p,\ell^q)_\beta$ if and only if $\alpha \prec \beta$ and $\beta \prec \alpha$.

We write the latter conditions as $\alpha \sim \beta$ and this defines an equivalence relation on the set of all amalgams.

In Section 4 we investigate conditions under which translation is a continuous operator on $(L^p,\ell^q)_\alpha$. In particular we show that all the translation operators are continuous, with uniform bound on their norms, if and only if $\alpha \sim \rho$, where $\rho_n = \{[n,n+1)\}$ is the cover that gives the regular amalgam.

Finally, in Section 5, we discuss generalizations to functions on measure spaces and groups.

2. THE VARIATION OF $(L^p,\ell^q)_\alpha$ WITH p AND q

In this section we consider the irregular amalgam $(L^p,\ell^q)_\alpha$ defined by a fixed cover $\alpha = \{I_n\}$ and investigate how it varies when p and q vary.

The variation with q is an immediate consequence of Jensen's inequality [9, p. 28]:

$$[\sum |a_n|^{q_2}]^{1/q_2} \leq [\sum |a_n|^{q_1}]^{1/q_1}, \; 0 < q_1 \leq q_2 \tag{2.1}$$

Taking $a_n = [\int_{I_n} |f(x)|^p dx]^{1/p}$, we have the following inclusion.

THEOREM 3. If $1 \leq q_1 \leq q_2$, then

$$(L^p,\ell^{q_1}) \subset (L^p,\ell^{q_2})$$

and

$$\|f\|_{p,q_2,\alpha} \leq \|f\|_{p,q_1,\alpha}$$

When $(L^p,\ell^q)_\alpha$ varies with p, the truth of the corresponding inclusion will depend on the lengths $|I_n|$ of the intervals $I_n \in \alpha$.

THEOREM 4. Let $1 \leq p_1 \leq p_2$. Then $(L^{p_2},\ell^q)_\alpha \subset (L^{p_1},\ell^q)_\alpha$ if and only if the set $\{|I_n|; n\in Z\}$ is bounded above.

PROOF: Let f_n be the function which agrees with f on I_n and is 0 elsewhere. Suppose that the set $\{|I_n|; n\in Z\}$ is bounded above. It follows from Holder's inequality that

$$\|f_n\|_{p_1} \leq \|f_n\|_{p_2} |I_n|^{1/p_1 - 1/p_2}$$

and so

$$\|f\|^q_{p_1,q,\alpha} = \sum_n \|f_n\|^q_{p_1} \leq \sum_n \|f_n\|^q_{p_2} |I_n|^{q(1/p_1-1/p_2)} \leq K \|f\|^q_{p_2,q,\alpha}$$

where K is an upper bound for the set $\{|I_n|^{q(1/p_1-1/p_2)}; n\in Z\}$. It follows that $(L^{p_2},\ell^q)_\alpha \subset (L^{p_1},\ell^q)_\alpha$.

Now assume that $\{|I_n|; n\in Z\}$ is not bounded above. Without loss of generality we may assume that $|I_n| \leq |I_{n+1}|$. (If not, we take a subsequence, reorder if necessary, and assume the function constructed below is zero on intervals not in this subsequence.) Observe that if f is constant on I_n, say $f(x) = c_n$ on I_n, then $\|f_n\|_p = c_n|I_n|^{1/p}$. In order to construct $f \in (L^{p_2},\ell^q)$ with $f \notin (L^{p_1},\ell^q)$ we shall choose the c_n's in such a way that

$$\|f\|^q_{p_2,q,\alpha} = \sum_n \|f_n\|^q_{p_2} = \sum_n [c_n|I_n|^{1/p_2}]^q < \infty \tag{2.2}$$

while

$$\|f\|^q_{p_1,q,\alpha} = \sum_n \|f_n\|^q_{p_1} = \sum_n [c_n|I_n|^{1/p_1}]^q = \infty \tag{2.3}$$

To do this we use the following fact due to Stieltjes [10, p. 41]: If $a_n \searrow 0$, then there is a sequence (d_n) of positive numbers such that $\sum d_n$ diverges while $\sum a_n d_n$ converges. If we choose

$$a_n = |I_n|^{q(1/p_2 - 1/p_1)}$$

then $a_n \searrow 0$ since $|I_n| \nearrow \infty$. By the result of Stieltjes there exist numbers d_n with $\sum a_n d_n < \infty$ and $\sum d_n = \infty$. Let

$$c_n = \frac{d_n^{1/q}}{|I_n|^{1/p_1}}$$

Then (2.2) and (2.3) give

$$\|f\|^q_{p_2,q,\alpha} = \sum_n c_n^q \, |I_n|^{q/p_2} = \sum a_n d_n < \infty$$

$$\|f\|^q_{p_1,q,\alpha} = \sum_n c_n^q \, |I_n|^{q/p_1} = \sum d_n = \infty$$

This shows that $(L^{p_2},\ell^q)_\alpha$ is not contained in $(L^{p_1},\ell^q)_\alpha$.

3. THE VARIATION OF $(L^p,\ell^q)_\alpha$ WITH α.

In this section we fix p and q and consider how the irregular amalgam $(L^p,\ell^q)_\alpha$ varies when the cover α varies. For that purpose we need the inequalities provided by the following lemmas.

LEMMA 1. If $a_n > 0$ and $p > 1$, then

$$N^{-1/p'} \sum_{n=1}^N a_n < [\sum_{n=1}^N a_n^p]^{1/p} < \sum_{n=1}^N a_n$$

PROOF: The left inequality follows from Holder's inequality and the right one is just Jensen's inequality (2.1).

LEMMA 2. Suppose $f_n = f|E_n$, where $E_1, \cdots, E_N$ are disjoint measurable subsets of R and $f = \sum f_n$.

(a) If $q > p > 1$, then $\sum_{n=1}^N \|f_n\|_p^q < \|f\|_p^q$.

(b) If $1 < q < p$, then $\|f\|_p^q < \sum_{n=1}^N \|f_n\|_p^q$.

(c) If $p,q > 1$, then $N^{-q/p'}(\sum_{n=1}^N \|f_n\|_p^q) < \|f\|_p^q < N^{q/q'}(\sum_{n=1}^N \|f_n\|_p^q)$

PROOF: (a) Since $q/p > 1$, Jensen's inequality gives

$$\|f\|_p^q = [\sum \|f_n\|_p^p]^{q/p} > \sum \|f_n\|_p^q$$

(b) Here $q/p < 1$ and the result again follows from Jensen's inequality.

(c) From Lemma 1 we have

$$\|f\|_p^q = [\sum \|f_n\|_p^p]^{q/p} \geq N^{-q/p'}(\sum \|f_n\|_p)^q \geq N^{-q/p'}(\sum \|f_n\|_p^q)$$

The inequality on the right follows from Minkowski's inequality and Lemma 1:

$$\|f\|_p^q \leq (\sum \|f_n\|_p)^q \leq N^{q/q'}[\sum \|f_n\|_p^q]$$

DEFINITION. Let $\alpha = \{I_n\}$ and $\beta = \{J_n\}$ be covers of the real line of the type described in Section 1. We say that α has index N in β and write $\alpha \prec_N \beta$ if each I_n intersects at most N of the J_m's. The notation $\alpha \prec \beta$ will mean that $\alpha \prec_N \beta$ for some N.

THEOREM 5. Suppose α has index N in β.

(i) If $q > p \geq 1$, then $(L^p,\ell^q)_\beta \subset (L^p,\ell^q)_\alpha$ and

$$\|f\|_{p,q,\alpha} \leq N^{1/q'} \|f\|_{p,q,\beta}$$

(ii) If $1 \leq q < p$, then $(L^p,\ell^q)_\alpha \subset (L^p,\ell^q)_\beta$ and

$$\|f\|_{p,q,\beta} \leq N^{1/q} \|f\|_{p,q,\alpha}$$

PROOF: (i) Using parts (c) and (a) of Lemma 2, we have

$$\|f\|_{p,q,\alpha}^q = \sum_n \|f|I_n\|_p^q \leq \sum_n N^{q/q'} \sum_m \|f|I_n \cap J_m\|_p^q$$

$$\leq N^{q/q'} \sum_m \|f|J_m\|_p^q = N^{q/q'} \|f\|_{p,q,\beta}^q$$

Thus $$\|f\|_{p,q,\alpha} \leq N^{1/q'} \|f\|_{p,q,\beta}$$

(ii) Using Theorems 1 and 2 we write

$$\|f\|_{p,q,\beta} = \sup \{|\int fg|;\ \|g\|_{p',q',\beta} \leq 1\}$$

$$\leq \sup \{\|f\|_{p,q,\alpha} \|g\|_{p',q',\alpha};\ \|g\|_{p',q',\beta} \leq 1\}$$

But $q' > p'$ and so, by part (i),

$$\|f\|_{p,q,\beta} \leq \sup \{\|f\|_{p,q,\alpha} \|g\|_{p',q',\alpha};\ \|g\|_{p',q',\alpha} \leq N^{1/q}\} = N^{1/q} \|f\|_{p,q,\alpha}$$

We note that part (ii) could also be proved directly (using Lemmas 1 and 2), but the constant $N^{1/q}$ would be replaced by $N^{1/p'}$.

THEOREM 6. If $\beta \not\prec \alpha$, then the inclusions in Theorem 5 are strict.

PROOF: (i) If $\beta \not\prec \alpha$, there are two cases. First we consider the case where there is an interval, say J_1, which intersects infinitely many I_n's. Let $I_n' = I_n \cap J_1$. Define f to be 0 outside J_1 and

$$f|I_n' = c_n = [n|I_n'|]^{-1/p}$$

Then

$$\|f\|_{p,q,\alpha}^q = \sum_n \|f|I_n'\|_p^q = \sum_n c_n^q |I_n'|^{q/p} = \sum \frac{1}{n^{q/p}} < \infty$$

since $q > p$, but

$$\|f\|_{p,q,\beta}^p = \int_{J_1} |f|^p = \sum_n c_n^p |I_n'| = \sum \frac{1}{n} = \infty$$

If no J_m intersects infinitely many I_n's, then we may assume there exist $J_1, J_2, \cdots$ such that J_m intersects at least m of the I_n's, say $I_{m_1}, \cdots, I_{m_n}$, and the intervals

I_{k_ℓ} are all disjoint. Let $I'_{m_j} = I_{m_j} \cap J_m$ and define

$$f|I'_{m_j} = c_j = n^{-(1/p+1/q)}|I'_{m_j}|^{-1/p}$$

and $f \equiv 0$ off $J_1 \cup J_2 \cup \cdots$. Then

$$\|f\|^q_{p,q,\alpha} = \sum_k \|f|I_k\|^q_p = \sum_n \sum_{k=1}^n \|f|I'_{m_k}\|^q_p$$

$$= \sum_n \sum_{k=1}^n c_n^q\, |I'_{m_k}|^{q/p} = \sum_n \sum_{k=1}^n n^{-q/p-1} = \sum_n n^{-q/p} < \infty$$

and

$$\|f\|^q_{p,q,\beta} = \sum_m \|f|J_m\|^q_p = \sum_m [\sum_{j=1}^m \|f|I'_{m_j}\|^p_p]^{q/p}$$

$$= \sum_m [\sum_{j=1}^m c_j^p\, |I'_{m_j}|]^{q/p} = \sum_m [\sum_{j=1}^m m^{-1-p/q}]^{q/p} = \sum_m \frac{1}{m} = \infty$$

Thus, in both cases,

$$(L^p,\ell^q)_\beta \subsetneqq (L^p,\ell^q)_\alpha$$

(ii) We consider the same two cases as in (i). In the first case we define

$$f|I'_n = c_n = n^{-1/q}\, |I'_n|^{-1/p}$$

Then

$$\|f\|^q_{p,q,\alpha} = \sum_n c_n^q\, |I'_n|^{q/p} = \sum_n \frac{1}{n} = \infty$$

while

$$\|f\|^p_{p,q,\beta} = \sum_n c_n^p\, |I'_n| = \sum_n n^{-p/q} < \infty$$

since $q < p$. In the second case we take

$$f\ |I'_{m_j} = c_j = n^{-2/q}\, |I'_{m_j}|^{-1/p}$$

so that

$$\|f\|^q_{p,q,\alpha} = \sum_n \sum_{k=1}^n c_n^q\, |I'_{m_k}|^{q/p} = \sum_n \sum_{k=1}^n n^{-2} = \sum_n n^{-1} = \infty$$

while

$$\|f\|^q_{p,q,\beta} = \sum_m [\sum_{j=1}^m c_j^p\, |I'_{m_j}|]^{q/p} = \sum_m [\sum_{j=1}^m m^{-2p/q}]^{q/p} = \sum_m m^{(q/p-2)} < \infty$$

This completes the proof.

By combining Theorems 5 and 6 we obtain the following.

THEOREM 7. (i) If $q > p \geq 1$, then $(L^p,\ell^q)_\beta \subset (L^p,\ell^q)_\alpha$ if and only if $\alpha \prec \beta$. (ii) If $1 \leq q < p$, then $(L^p,\ell^q)_\alpha \subset (L^p,\ell^q)_\beta$ if and only if $\alpha \prec \beta$. (iii) If $p,q \geq 1$, $p \neq q$, then $(L^p,\ell^q)_\alpha = (L^p,\ell^q)_\beta$ if and only if $\alpha \prec \beta$ and $\beta \prec \alpha$. Furthermore, the norms are equivalent.

DEFINITION. We write $\alpha \sim \beta$ if $\alpha \prec \beta$ and $\beta \prec \alpha$.

From Theorem 7 (iii) we note that $\sim$ is an equivalence relation on the set of all irregular amalgams.

For example, consider the class of amalgams given by $\alpha^{(r)} = \{[\alpha_n, \alpha_{n+1})\}$ where $\alpha_n = n^r$, $n > 0$. (In these and the following examples we assume that $\alpha_0 = 0$ and $\alpha_{-n} = -\alpha_n$.) Then $\alpha^{(r)} \prec \alpha^{(s)}$ whenever $0 < r < s$ and $\alpha^{(1)} = \rho$, where $(L^p,\ell^q)_\rho$ is the regular amalgam (L^p,ℓ^q) discussed in Section 1. Thus we have infinitely many mutually inequivalent amalgams both $\prec \rho$ and $\succ \rho$. Clearly the dyadic amalgam $(L^p,\ell^q)_\delta$ with $\delta_n = 2^n$ satisfies $\alpha^{(r)} \not\prec \delta$, but $\delta \sim \tau$ where $\tau = \{[\tau_n,\tau_{n+1})\}$, $\tau_n = 3^n$.

Another amalgam of some interest is defined by taking intervals I_n such that $|I_{2n-1}| = 1$, $|I_{2n}| = 1/n$, $n = 1, 2, \cdots$. This amalgam is equivalent to the regular amalgam but has intervals of arbitrarily small length. We shall see in the next section, however, that no amalgam with arbitrarily large intervals is equivalent to the regular amalgam.

4. TRANSLATION.

For the case of regular amalgams, Holland [4] showed that the translation operator is continuous on (L^p,ℓ^q) and used this fact to investigate the Fourier transform of functions in this space. Here we determine when the translation operator is continuous on an irregular amalgam. In order to state the first result we use the notation α_t to mean the cover α shifted t units to the right and $f_t(x) = f(x-t)$.

THEOREM 8. (i) If $q > p \geq 1$, the translation operator $T_t\, f = f_t$ is continuous on $(L^p,\ell^q)_\alpha$ if and only if $\alpha \prec \alpha_t$. (ii) If $1 \leq q < p$, the operator T_{-t} is continuous on $(L^p,\ell^q)_\alpha$ if and only if $\alpha \prec \alpha_t$.

PROOF: (i) If $\alpha \prec_N \alpha_t$ and $q > p \geq 1$, then, by Theorem 5 (i) applied to f_t, we have

$$\|f_t\|_{p,q,\alpha} \leq N^{1/q'}\|f_t\|_{p,q,\alpha_t} = N^{1/q'}\|f\|_{p,q,\alpha}$$

If $\alpha \not\prec \alpha_t$, then by Theorem 6, $(L^p,\ell^q)_{\alpha_t} \not\subset (L^p,\ell^q)_\alpha$ and so there exists $f \in (L^p,\ell^q)_{\alpha_t}$ such that $f \notin (L^p,\ell^q)_\alpha$. Thus $f_{-t} \in (L^p,\ell^q)_\alpha$ and $(f_{-t})_t \notin (L^p,\ell^q)_\alpha$.

(ii) If $\alpha \prec_N \alpha_t$ and $1 \leq q < p$, then Theorem 5 (ii) gives

$$\|f_{-t}\|_{p,q,\alpha} = \|f\|_{p,q,\alpha_t} \leq N^{1/q}\|f\|_{p,q,\alpha}$$

If $\alpha \not\prec \alpha_t$, then by Theorem 6, $(L^p,\ell^q)_\alpha \not\subset (L^p,\ell^q)_{\alpha_t}$, so there exists $f \in (L^p,\ell^q)_\alpha$ such that $f \notin (L^p,\ell^q)_{\alpha_t}$, that is, $f_{-t} \notin (L^p,\ell^q)_\alpha$.

REMARK. It is possible for the translation operator T_t to be continuous for some values of t but not for other values. For example, consider the amalgam obtained by splitting each unit interval $[n,n+1)$ at the points $n+2^{-k}$, $k = 1, 2, 3, \cdots$. By Theorem 8, T_t is continuous for integer values of t but for no other value.

We also note that it is possible for all translation operators T_t to be continuous, but without a uniform bound on their norms. It is not hard to see that the dyadic amalgam has this property. Another example is obtained by taking intervals of successive lengths $\cdots, 4, 3, 2, 1, 1/2, 1/3, 1/4, \cdots$. In fact we show in Theorem 10 that $\{\|T_t\|;\ t \in R\}$ is bounded if and only if $\alpha \sim \rho$.

THEOREM 9. If there exists N such that $\alpha \prec_N \alpha_t$ for all t, then $\alpha \sim \rho$ and conversely.

PROOF. If $\alpha = \{I_n\}$, we claim that $\sup |I_n| < \infty$. For if $\sup |I_n| = \infty$, then given any positive integer k and k adjacent intervals $I_{n_1},\ldots,I_{n_k}$, we can choose an interval I_N such that $|I_N| > |I_{n_1}| + \cdots + |I_{n_k}|$. Thus there exists t such that

$(I_{n_1} \cup \cdots \cup I_{n_k})_t \subset I_N$. Since k is arbitrary, this contradicts our hypothesis. Therefore we can write $\sup |I_n| \le M$ where M is an integer. It follows that any interval I_n meets at most M + 1 intervals from ρ, i.e., $\alpha \prec_{M+1} \rho$.

Suppose now that $\rho \not\prec \alpha$. Then there are intervals in ρ which contain arbitrarily many I_j's. It follows from the pigeon-hole principle that there are intervals of R of any given size which contain arbitrarily many I_j's. In particular, a given interval in α can be translated to cover arbitrarily many I_j's, so $\alpha \not\prec_N \alpha_t$.

To prove the converse we assume that $\alpha \sim \rho$. Then $\sup |I_n| = K < \infty$. Suppose there is no N such that $\alpha \prec_N \alpha_t$ for all t. This means that we can find an interval in α containing arbitrarily many I_n+t's for some t. But, since $|I_n| \le K$, it follows (as above) that there is a unit interval which intersects arbitrarily many I_n+t's and therefore a unit interval [k,k+1) meeting arbitrarily many I_n's. This is a contradiction.

THEOREM 10. The translation operators T_t, $t \in R$, are continuous on $(L^p, \ell^q)_\alpha$, with uniform bound on their norms, if and only if $\alpha \sim \rho$. In this case we have

$$\sup \|T_t\| \le N^{1/q'} \qquad \text{if } q \ge p \ge 1$$

$$\sup \|T_t\| \le N^{1/q} \qquad \text{if } 1 \le q \le p$$

where N is the smallest integer such that $\alpha \prec_N \alpha_t$ for all t.

PROOF. This follows from the two preceding theorems together with the proof of Theorem 4.1.

REMARK. Continuity of translation is an essential ingredient in the proof that if $f \in (L^p, \ell^q)$, $1 \le p,q \le 2$, then the Fourier transform $\hat{f} \in (L^{q'}, \ell^{p'})$ [4, Theorem 8]. Thus, in view of Theorem 10, it seems unlikely that any such result will hold for amalgams other than those equivalent to the regular amalgam.

Translation also arises in the consideration of convolution and Young's inequality. If $\alpha = \{I_n\}$, $\beta = \{J_m\}$, and γ are covers and $\alpha + \beta = \{I_n + J_m;\ n,m \in Z\}$, we write $\gamma \sim_N (\alpha + \beta)$ if $\gamma \prec_N (\alpha + \beta)$ and $(\alpha + \beta) \prec_N \gamma$. With this notation we can state the following version of Young's inequality for irregular amalgams. The proof is similar to that of Theorem 4.2 in [11].

THEOREM 11. Suppose $\gamma \sim_N (\alpha+\beta)$. If $f \in (L^{p_1}, \ell^{p_2})_\alpha$ and $g \in (L^{q_1}, \ell^{q_2})_\beta$, where $1/p_i + 1/q_i \ge 1$, then $f*g \in (L^{r_1}, \ell^{r_2})_\gamma$, where $1/r_i = 1/p_i + 1/q_i - 1$, and

$$\|f*g\|_{r_1,r_2,\gamma} \le N \|f\|_{p_1,p_2,\alpha} \|g\|_{q_1,q_2,\beta}$$

Theorem 11 holds for amalgams on groups (see the next section for definitions) but on R the condition $\gamma \sim_N (\alpha + \beta)$ holds if and only if α, β, and γ are all equivalent to the regular amalgam, as we now show.

THEOREM 12. If $\gamma \sim (\alpha + \beta)$, then $\alpha, \beta, \gamma \sim \rho$.

PROOF. If $\gamma \sim (\alpha + \beta)$, then clearly $\gamma \sim \alpha_t$ for all t and so $\alpha \sim \alpha_t$ for all t. Then $\alpha \sim \rho$ by Theorem 9. Similarly $\beta \sim \rho$ and it follows that $\gamma \sim \rho$ also.

5. AMALGAMS ON MORE GENERAL SPACES.

In this section we discuss the extent to which our results can be generalized from functions on R to more general functions. The amalgam spaces themselves make sense on any measure space.

Let (X,μ) be a measure space and $E = \{E_\lambda;\ \lambda \in J\}$ any covering of X by disjoint measurable sets of finite measure:

$$X = \bigcup_{\lambda\in J} E_\lambda, \qquad \mu(E_\lambda) < \infty$$

In terms of this decomposition E of X we define the amalgam $(L^p,\ell^q)_E$ to consist of functions f such that

$$\|f\|_{p,q,E} = \Big[\sum_{\lambda\in J} \|f\|^q_{L^p(E_\lambda)}\Big]^{1/q} < \infty$$

Then all of the results of the first three sections extend to this setting. In particular, the extension of Theorem 1 follows from a general result [12, p. 359] concerning the dual of the space $\ell^q(B_\lambda)$ of nets $x = (x_\lambda)$, where $x_\lambda \in B_\lambda$, each B_λ is a Banach space, and $\|x\| = [\sum \|x_n\|^q]^{1/q} < \infty$. Taking $B_\lambda = L^p(E_\lambda)$, and using this result, we have

$$(L^p,\ell^q)^*_E = \ell^q(L^p(E_\lambda))^* = \ell^{q'}(L^{p'}(E_\lambda)) = (L^{p'},\ell^{q'})_E$$

Theorems 3, 4, 5, 6, and 7 have verbatim proofs in the general context of measure spaces.

For translation, of course, we need algebraic structure and so we assume that G is a locally compact abelian group (with Haar measure) and easily see that Theorem 8 is valid for amalgams on G. The question then arises as to an analogue of Theorem 10 for groups. Regular amalgams on groups have been defined and studied in [13], [14], and [11]. Nonetheless we do not see how to extend Theorem 10 to groups without imposing severe restrictions on the shape of the sets E_λ, even for groups as simple as $G = R^2$.

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