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RESEARCH NOTES

ON COLLINEATION GROUPS OF TRANSLATION PLANES OF ORDER q^4

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ABSTRACT. Let P be an affine translation plane of order q^4 admitting a nonsolvable group G in its translation complement. If G fixes more than q+1 slopes, the structure of G is determined. In particular, if G is simple then q is even and $G = L_2(2^s)$ for some integer s at least 2.

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1. INTRODUCTION.

Let π be a translation plane of square order p^{2r}. If π admits a collineation group isomorphic to $SL(2,p^t)$ and the Sylow p-subgroups are planar, then usually (in the known cases) the group fixes p^r+1 components (or slopes). Generally, however, simply knowing that a group fixes a number of slopes says essentially nothing concerning the structure of the group. However, for planes of order q^4, we can make some progress. That is, in this note our objective is to prove the following.

THEOREM A. Let π be an affine translation plane of order q^4 admitting a nonsolvable group G in its translation complement. Suppose G fixes more than q+1 slopes.

(1) If q is odd then $8 \mid |G|$ and the 2-rank of $G \leq 2$. Furthermore, G always contains the kern involution.

(2) If q is even then G contains a normal subgroup N such that $N \cong L_2(2^s)$, for some s, and G/N is of odd order. Now the Sylow 2-subgroups fix Baer subplanes elementwise. Furthermore, the Baer subplanes share the same points at infinity and so N fixes exactly q^2+1 slopes.

COROLLARY 1. If G is simple then q is even and $G = L_2(2^s)$ for some integer $s \geq 2$.

PROOF. When q is odd the kern involution is central in G and so G is not simple.

REMARKS. (1) When q is odd it is possible to find G's with 2-rank one and 2-rank two such that they satisfy the hypothesis of Theorem A. For example, $G_1 = SL(2,q)$, acting on Hall planes of order q^4, has 2-rank one; if we now choose $G_2 = \langle G_1, \alpha \rangle$, where α is any Baer involution, we get a 'G' with 2-rank two.

(2) Both the theorem and its corollary cease to be unconditionally true if we allow G to fix "at least $q+1$ slopes", instead of "more than $q+1$ slopes": the only known counterexamples seem to be the Lorimer-Rahilly planes [1] and its transpose, the Johnson-Walker plane [2].

The following well known consequences of Foulser [3] will be used on several occasions in the proof of Theorem A.

RESULT 0. Suppose B is a collineation group of an affine translation plane of order p^{2r} that fixes a Baer subplane elementwise. Then

(i) B is solvable;

(ii) the Sylow p-subgroups of B are elementary abelian; and

(iii) the Hall p' subgroups of B are cyclic.

2. PROOF OF THEOREM A.

We begin by dealing with the case when q is odd. The first step is folklore and corresponds to Ostrom's ideas in [4].

LEMMA 1. If q is odd then any Klein 4-group in G must contain the unique kern involution of π; we shall always denote this involution by $\hat{\imath}$.

PROOF. Let $K = \{1, \alpha, \beta, \alpha\beta\}$ be a Klein group in G and suppose, if possible that $\hat{\imath} \notin K$. For any involution x in K write π_x for its fixed Baer subplane. Now we claim $\pi_x \cap \pi_y$ cannot be a subplane of π if x,y are distinct involutions in K. If $\pi_x = \pi_y$, we have a Klein group fixing elementwise a Baer subplane of π, contrary to Result 0. So $\pi_x \cap \pi_y$ is a fourth root subplane of π and now we contradict the assumption that G fixes more than $q+1$ slopes. Thus xy acts like -1 on the fixed components of G, in the spread associated with π; i.e., xy is the required involution.

LEMMA 2. If q is odd then $|G|$ is divisible by 8.

PROOF. If 2 exactly divides $|G|$ then, by Burnside's theorem, G has a normal 2-complement [5, 6.2.11] and we contradict the assumption that G is nonsolvable. For the same reason the Sylow 2-subgroups of G cannot be cyclic of order 4. Hence $4 \parallel |G|$ only if the Sylow 2-subgroups are Klein groups. So by Lemma 1, G contains the kern involution $\hat{\imath}$ and $G/\langle \hat{\imath} \rangle$ is solvable. Thus we contradict the nonsolvability of G when $8 \nmid |G|$. The result follows.

LEMMA 3. Suppose q is odd. Then G cannot contain an elementary abelian 2-group of order 8.

PROOF. If S is an elementary abelian subgroup of G, whose order is 8, we may write

$$S = \{1, \alpha, \beta, \alpha\beta, \gamma, \alpha\gamma, \beta\gamma, \alpha\beta\gamma\} \qquad (2.1)$$

and assume that

$$L = \{1,\alpha,\beta,\alpha\beta\}, \quad M = \{1,\alpha,\gamma,\alpha\gamma\} \tag{2.2}$$

are distinct subgroups of order 4. But by Lemma 1, both L and M contain the kern involution $\hat{\imath}$. Hence $\alpha = \hat{\imath}$. Interchanging the role of α and β, we find that β is also $\hat{\imath}$. The lemma follows, since we have contradicted the assumption that $|S| = 8$.

LEMMA 4. G contains $\hat{\imath}$, the kern involution of π.

PROOF. Let S denote a Sylow 2-subgroup of G. So $|S| \geq 8$ (Lemma 2) and noncyclic, because G is nonsolvable. Now Lemma 1 applies unless the 2-rank of S is one. Thus S is the generalized quaternion group

$$\langle x,y \mid x^{2^n} = 1,\ y^2 = x^{2^{n-1}},\ y^{-1}xy = x^{-1} \text{ for } n \geq 2\rangle \tag{2.3}$$

and so contains $Q = \langle x^{2^{n-2}}, y\rangle$, the quaternion group of order 8.

Now let α denote the unique involution in Q and, to get a contradiction, assume α is a Baer involution with fixed plane π_α. Now Q leaves π_α invariant but does not fix it elementwise because of Result 0. Moreover, no element of Q can induce a Baer involution on π_α because Q fixes $> q+1$ slopes of π_α. Thus the restriction map

$$\rho : Q \longrightarrow Q \mid \pi_\alpha \tag{2.4}$$

has as its image $\langle \beta \rangle$, where β is the kern involution of π_α. So $\ker \rho$ is clearly a noncyclic group Σ of order 4, contrary to Result 0.

The lemmas proved so far add up to Theorem A, Part (1). To deal with the case when q is even we need the following version of a theorem of Johnson [6], deduced from Hering [7].

RESULT 5. Suppose ψ is an affine translation plane of even order admitting a nonsolvable group H in its translation complement. Assume a Sylow 2-subgroup of H fixes a Baer subplane elementwise. Then H contains a normal subgroup N such that H/N is of odd order and $N \cong L_2(2^s)$ for some integer s.

PROOF. Use Johnson's argument in [6, Theorem 2.3].

LEMMA 6. If q is even then the Sylow 2-subgroups of G fix Baer subplanes of π elementwise.

PROOF. Let S be a Sylow 2-subgroup of G and note that elements of π fixed by S form a subplane π_S, because S fixes many slopes. To get a contradiction we assume π_S is not a Baer subplane of π. Now let α be any involution in the center of S and let π_α be its fixed Baer subplane. Then we have a chain of planes $\pi \supset \pi_\alpha \supset \pi_S$. Hence the order of $\pi_S \leq q$ and we contradict the assumption that G fixes more than $q+1$ slopes. The result follows.

Now by Result 5 and Lemma 6 we immediately have

LEMMA 7. Suppose q is even. Then the Sylow 2-subgroups of G fix Baer subplanes elementwise and generate of a subgroup $N \cong L_2(2^s)$ where $2^s \,\|\, |G|$.

To complete the proof of Theorem A we restrict ourselves from now on to the situation described in Lemma 7.

LEMMA 8. N fixes a unique affine point 0.

PROOF. Suppose N, which is in the translation complement of π, fixes a second affine point of π. Then N is a planar group of π of order $> q$. Now by Lemma 7 we have a Baer chain $\pi \supset \pi_S \supset \pi_N$. If $\pi_N \neq \pi_S$ we have the same contradiction as in Lemma 6; otherwise we have a nonsolvable group fixing a Baer subplane elementwise, contrary to Result 0.

LEMMA 9. The only affine point fixed by distinct Sylow 2-subgroups of N is 0, the unique point fixed by N.

PROOF. N is generated by any two of its Sylow 2-subgroups, so we contradict Lemma 8 unless the lemma is valid.

For $2^s \neq 4$, Lemma 9, when combined with Foulser and Johnson [8, Proposition 3.4], shows than N fixes q^2+1 slopes. From Johnson [9, Theorem 2.1], the same is true if $2^s = 4$ as N fixes at least 3 slopes. This proves Part (2) of Theorem A.

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