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P-REPRESENTABLE OPERATORS IN BANACH SPACES

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ABSTRACT. Let E and F be Banach spaces. An operator $T \in L(E,F)$ is called p-representable if there exists a finite measure μ on the unit ball, $B(E^*)$, of E^* and a function $g \in L^q(\mu,F)$, $\frac{1}{p}+\frac{1}{q}=1$, such that

$$Tx = \int_{B(E^*)} \langle x,x^* \rangle g(x^*)d\mu(x^*)$$

for all $x \in E$. The object of this paper is to investigate the class of all p-representable operators. In particular, it is shown that p-representable operators form a Banach ideal which is stable under injective tensor product. A characterization via factorization through L^p-spaces is given.

KEY WORDS AND PHRASES. Representable Operator, Banach Space, Stable Ideal Operators.
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1. INTRODUCTION.

Let $L(E,F)$ be the space of all bounded linear operators from E into F, and $B(E^*)$ the unit ball of E^*, the dual of E. The completion of the injective tensor product of E and F is denoted by $E \check{\otimes} F$. Integral operators in $L(E,F)$ were first defined by Grothendieck, [2], as those operators which can be identified with elements in $(E \check{\otimes} F)^*$. These operators turn to have a nice integral representation. We refer to Jarchow, [4], for statements and proofs of such representations. Later on, Persson and Pietsch, [5], defined p-integral operators in $L(E,F)$ as those operators $T: E \to F$ such that $Tx = \int_{B(E^*)} \langle x,x^* \rangle dG(x^*)$, for all $x \in E^*$ where G is a vector measure on $B(E^*)$ with values in F and $\left\| \int_{B(E^*)} \phi(x^*)dG(x^*) \right\| \leq \left(\int_{B(E^*)} |\phi(x^*)|^p d\mu \right)^{1/p}$ for some finite measure μ on $B(E^*)$ and all continuous functions ϕ on $B(E^*)$. The representing vector measure for T need not be of bounded variation. Further, if G is of bounded variation and F doesn't have the Radon-Nikodym property, then T need not be a kernel integral operator.

The object of this paper is to study operators which are in some sense kernel ingegral operators. Such operators is a sub-class of Pietsch p-integral operators.

Throughout this paper, if E is a Banach space, then E^* is the dual of E and $B(E)$ the closed unit ball of E. If K is a set then 1_K is the characteristic function of K. If (Ω,μ) is a measure space, then $L^p(\Omega,\mu,E)$ is the space of all p-Bochner integrable functions defined on Ω with values in E, for $1 \le p < \infty$. If $p = \infty$, $L^\infty(\Omega,\mu,E)$ is the space of μ essentially bounded functions on Ω with values in E. The real $1 \le q \le \infty$ always denote the conjugate of $p: \frac{1}{p} + \frac{1}{q} = 1$. Most of our terminology and notations are from Pietsch [6] and Diestel and Uhl [1]. We refer to these texts for any notion cited but not defined in this paper.

2. $R_p(E,F)$.

DEFINITION 2.1. An operator $T \in L(E,F)$ is called p-representable operator if there exists a finite measure μ defined on the Borel sets of $B(E^*)$ and a function $g : B(E^*) \to F$ such that $\int_{B(E^*)} \|g(x^*)\|^q d\mu < \infty$, and $Tx = \int_{B(E^*)} \langle x,x^*\rangle g(x^*)d\mu(x^*)$ for all $x \in E$.

It follows from the definition that every p-representable operator is Pietsch-p-integral operator, but not the converse. Let $R_p(E,F)$ be the set of all p-representable operators from E into F.

LEMMA 2.2. $R_p(E,F)$ is a vector space.

PROOF. Let $T_1,T_2 \in R_p(E,F)$ such that

$$T_i(x) = \int_{B(E^*)} \langle x,x^*\rangle g_i(x^*)d\mu_i(x^*).$$

Set $\mu = \mu_1 + \mu_2$. Then $\mu_i << \mu$. Consequently, $d\mu_i = f_i d\mu$. Further, since $\mu_i(K) < \mu(K)$ for all Borel sets K on $B(E^*)$, it follows that $0 \le f_i(x^*) \le 1$ a.e. μ, $i = 1,2$. Let $\tilde{g}(x^*) = g_1(x^*)f_1(x^*) + g_2(x^*)f_2(x^*)$. Since $1 \le p < \infty$, and $0 \le f_i(x^*) \le 1$, we have $\tilde{g} \in L^q(B(E^*),\mu,F)$. Further $(T_1+T_2)(x) = \int_{B(E^*)} \langle x,x^*\rangle\tilde{g}(x^*)d\mu$, for all $x \in E$. This ends the proof.

For $T \in R_p(E,F)$, we define

$$\|T\|_{\sigma(p)} = \inf \{ \left(\int \|g(x^*)\|^q d\mu(x^*) \right)^{1/q} \}$$

where the infimum is taken over all g and μ for which $T(x) = \int_{B(E^*)} \langle x,x^*\rangle g(x^*)d\mu(x^*)$, $x \in E$. It is not difficult to show that $\| \ \|_{\sigma(p)}$ is a norm on $R_p(E,F)$.

LEMMA 2.3. For $T \in R_p(E,F)$, $\|T\| \le \|T\|_{\sigma(p)}$.

PROOF. Let $Tx = \int_{B(E^*)} \langle x,x^*\rangle g(x^*)d\mu(x^*)$ for some μ and g as in the Definition 2.1. Choose g and μ such that $\left(\int \|g(x^*)\|^q d\mu(x^*) \right)^{1/q} \le \|T\|_{\sigma(p)} + \epsilon$, for a given small $\epsilon > 0$. Then, using Holder's inequality:

$$\|Tx\| \le \left(\int_{B(E^*)} \|g(x^*)\|^q d\mu(x^*) \right)^{1/q}$$

$$\le \|T\|_{\sigma(p)} + \epsilon .$$

Hence $\|T\| \le \|T\|_{\sigma(p)} + \epsilon$. Since ϵ is arbitrary, the result follows.

LEMMA 2.4. Every element $T \in R_p(E,F)$ is an approximable operator in $L(E,F)$.

PROOF. Let $Tx = \int_{B(E^*)} \langle x,x^*\rangle g(x^*)d\mu(x^*)$, for some finite measure on $B(E^*)$ and some $g \in L^q(B(E^*),\mu, F)$. Choose μ and g such that

$$\left(\int_{B(E^*)} \|g(x^*)\|^q \, d\mu(x^*)\right)^{1/q} \leq \|T\|_{\sigma(p)} + \varepsilon .$$

Let g_n be a sequence of simple functions in $L^q(B(E^*),\mu, F)$ such that $\int_{B(E^*)} \|g(x^*) - g_n(X^*)\|^q \, d\mu(x^*) \to 0$. Define $T_n(x^*) = \int_{B(E^*)} \langle x,x^*\rangle g_n(x^*) \, d\mu(x^*)$. Then each T_n is a finite rank operator, and $\|T - T_n\|_{\sigma(p)} \to 0$. Then by definition of approximable operators, Pietsch [6] , T is approximable. This ends the proof.

THEOREM 2.5. Let H, E, F and G be Banach spaces, and $T \in R_p(E,F)$, $A \in L(F,G)$ and $B \in L(H,E)$. Then $ATB \in R_p(H,G)$ and $\|ATB\|_{\sigma(p)} \leq \|A\| \, \|B\| \, \|T\|_{\sigma(p)}$

PROOF. Let $Tx = \int_{B(E^*)} \langle x,x^*\rangle g(x^*) d\mu(x^*)$ for all $x \in E$ and some finite measure μ on $B(E^*)$ and some $g \in L^q(B(E^*),\mu,F)$. Then

$$ATx = \int_{B(E^*)} \langle x,x^*\rangle Ag(x^*) d\mu(x^*)$$

and $\int_{B(E^*)} \|Ag(x^*)\|^q \, d\mu(x^*) \leq \|A\| \int_{B(E^*)} \|g(x^*)\|^q \, d\mu(x^*)$. Hence $AT \in R_p(E,G)$ and $\|AT\|_{\sigma(p)} \leq \|A\| \|T\|_{\sigma(p)}$.

To show $TB \in R_p(H,F)$, let g_n be a sequence of simple functions converging to g in $L^q(B(E^*),\mu,F)$, and T_n be the associated operators in $R_p(E,F)$. So

$$T_n x = \int_{B(E^*)} \langle x,x^*\rangle g_n(x^*) d\mu(x^*) .$$

With no loss of generality we assume $\|B\| = 1$. Define the vector measures G on $B(H^*)$ into F via:

$$G_n(K) = \int_{B(E^*)} 1_K(y^*) dG(y^*)$$
$$= \int_{B(E^*)} 1_K(B^*x^*) S_n(x^*) d\mu(x^*).$$

Clearly, G_n is a countably additive vector measure of bounded variation. Further, if we define the measure ν on $B(H^*)$ via

$$\nu(K) = \int_{B(E^*)} 1_K(B^*x^*) d\mu(x^*),$$

then, using Holder's inequality:

$$\|G_n(K)\| \leq \left(\int_{B(E^*)} \|g_n(x^*)\|^q \, d\mu(x^*)\right)^{1/q} \cdot [\nu(K)]^{1/p} .$$

Hence $G_n \ll \nu$. Since the range of G_n is finite dimensional, it has the Radon-Nikodym property, and consequently there exists $S_n \in L^1(B(H^*), \nu, F)$ such that $dG_n = S_n d\nu$. Further, it is easy to check that $S_n \in L^q(B(H^*),\nu,F)$.

An application of the Hahn-Banach theorem, we get:

$$T_n By = \int_{B(E^*)} \langle By,x^*\rangle g_n(x^*) d\mu(x^*)$$
$$= \int_{B(H^*)} \langle y,y^*\rangle S_n(y^*) d\nu(y^*).$$

Since the function $\langle By,x^*\rangle$ is bounded on $B(E^*)$, the sequence $(\langle By,\cdot\rangle g_n)$ is Cauchy in $L^q(B(E^*),\mu,F)$. Consequently the sequence $(\langle y,\cdot\rangle S_n)$ is Cauchy in $L^q(B(H^*),\nu,F)$. Let $\langle y,\cdot\rangle S$ be the limit of $(\langle y,\cdot\rangle S_n)$ in $L^q(B(H^*),\nu,F)$. It is not difficult to see that $T_n B$ converges in the operator norm to the operator $Jy = \int_{B(H^*)} \langle y,y^*\rangle S(y^*) d\nu(y^*)$.

However $T_nB \to TB$ in the operator norm. Hence $TBy = \int_{B(H^*)} \langle y,y^*\rangle S(y^*)d\nu(y^*)$, and $TB \in R_p(H,F)$. Further $\|TB\|_{\sigma} \le \|T\|_{\sigma} \|B\|$. This ends the proof.

Theorem 2.5 states that $(R_p, \| \|_{\sigma(p)})$ is a normed operator ideal, [6] .

DEFINITION 2.6. Let (Ω,μ) be a measure space and F a Banach space. An operator $T \in L(L^p(\Omega,\mu),F)$ is called B-vector integral operator if there exists $g \in L^q(\Omega,\mu,F)$ such that

$$Tf = \int_{\Omega} f(t)g(t)d\mu(t)$$

for all $f \in L^p(\Omega,\mu)$.

If the function g is only Pettis q-integrable and the integral defining Tf is the Pettis integral, then T is known to be called vector integral operator [1] .

Now using Theorem 2.5 we can prove:

THEOREM 2.7. Let E,F be Banach spaces and $T \in L(E,F)$. The following are equivalent:

(i) $T \in R_p(E,F)$

(ii) There exists operators $T_1 \in L(E,L^p(\Omega,\mu))$ and $T_2 \in L(L^p(\Omega,\mu),F)$ for some measure space (Ω,μ) such that T_2 is B-vector integral operator and $T = T_2T_1$.

PROOF. (i) $\to$ (ii). Let $T \in R_p(E,F)$ and

$$Tx = \int_{B(E^*)} \langle x,x^* \rangle g(x^*)d\mu(x^*)$$

for some finite measure μ on $B(E^*)$ and $g \in L^q(B(E^*),\mu,F)$. Define

$$T_1 : E \longrightarrow L^p(B(E^*))$$

$$(T_1x)(x^*) = \langle x,x^* \rangle ,$$

and

$$T_2 : L^p(B(E^*), \mu) \longrightarrow F$$

$$T_2(f) = \int_{B(E^*)} f(x^*)g(x^*)d\mu .$$

Then T_2 is a B-vector integral operator and $T = T_2T_1$.

(ii) $\to$ (i). Let $T = T_2T_1$, $T_1(E,L^p(\Omega,\mu))$ and T_2 is a B-vector integral operator in $L(L^p(\Omega,\mu), F)$. Then $T_2 \in R_p(L^p(\Omega,\mu), F)$. Using Theorem 2.6, $T_2T_1 \in R_p(E,F)$. This ends the proof.

Let $I_p(E,F)$ be the space of Pietsch p-integral operators from E into F, and $\|T\|_{i(p)}$ be the p-integral norm for $T \in I_p(E,F)$. Clearly $R_p(E,F) \subseteq I_p(E,F)$ and $\|T\|_{i(p)} \le \|T\|_{\sigma(p)}$ for all $T \in R_p(E,F)$. This, together with the fact that $I_p(E,F)$ is complete, [5] , one can prove:

THEOREM 2.8. $(R_p(E,F), \| \|_{\sigma(p)})$ is a Banach space.

If F has the Radon Nikodym property, then $R_1(E,F) = I_1(E,F)$, and by using Corollary 5 in [1], we see that $R_1(C(\Omega),F) = I_1(E,F) = N_1(C(\Omega),F)$, where $N_1(E,F)$ is the class of nuclear operators from E into F.

Further if $\Pi_p(E,F)$ is the class of p-summing operators from E into F, then it follows from the Grothendieck-Pietsch represenation theorem [6], that $R_p(E,F) \subseteq \Pi_p(E,F)$

3. IDEAL PROPERTIES OF R_p.

We let R_p denote the operator ideal of all p-representable operators. The following notions are taken from Pietsch [5] and Holub [3].

(i) An operator ideal J is called regular if for all Banach spaces E and F, $T \in J(E,F)$ if and only if $K_F T \in J(E,F^{**})$, where K_F is the natural embedding of F into F^{**}.

(ii) J is called closed if the closure of $J(E,F)$ in $L(E,F)$ is $J(E,F)$ for all Banach spaces E and F.

(iii) J is called injective if whenever $J_F T \in J(E,\ell^\infty(B(F^*)))$, then $T \in (E,F)$ for all Banach spaces E and F. Here J_F is the natural embedding of F into $\ell^\infty(B(F^*))$.

(iv) J is called stable with respect to the injective tensor product if $T_i \in J(E_i,F_i)$, then $T_1 \otimes T_2 \in J(E_1 \check{\otimes} E_2,\ F_1 \check{\otimes} F_2)$, for all Banach spaces E_1,E_2,F_1,F_2.

THEOREM 3.1. R_p is regular.

PROOF. Let E and F be any Banach spaces and let $K_F T \in R_p(E,F^*)$, for $T \in L(E,F)$. Then $K_F Tx = \int_{B(E^*)} \langle x,x^* \rangle g(x^*)d\mu(x^*)$ for some μ and g as in Definition 2.1.

Now $g(x^*) \in K_F(F)$ for all $x^* \in B(E^*)$. Since $K_F : F \longrightarrow K_F(F)$ is an isometric onto operator, the function $g(x^*) = K_F^{-1}(g(x^*))$ is well defined measurable and $\tilde{g} \in L^q(B(E^*),\mu, F)$. Further

$$Tx = \int_{B(E^*)} \langle x,x^* \rangle \tilde{g}(x^*)d\mu(x^*).$$

Hence $T \in R_p(E,F)$. This ends the proof.

In a similar way one can prove:

THEOREM 3.2. R_p is injective

THEOREM 3.3. R_p is stable.

PROOF. Let $T_i \in R_p(E_i,F_i)$, $i = 1,2$ and

$$T_1 x = \int_{B(E_1^*)} \langle x,x^* \rangle g_1(x^*)d\mu_1(x^*)$$

$$T_2 x = \int_{B(E_2^*)} \langle x,x^* \rangle g_2(x^*)d\mu_2(x^*),$$

where μ_i and g_i be the associated measures and functions as in Definition 2.1. If $E_i \check{\otimes} F_i$, $i = 1,2$, is the completion of the injective tensor product of E_i with F_i, [1], then $T_1 \otimes T_2 \in L(E_1 \otimes F_1,\ E_2 \otimes F_2)$. Further:

$$(T_1 \otimes T_2)(x \otimes y) = \int_{B(E_1^*)} \langle x,x^* \rangle g_1(x^*)d\mu_1(x^*)\ )\ \int_{B(E_2^*)} \langle y,y^* \rangle g_2(y^*)d\mu_2(y^*).$$

Let K be the w*-closure of $B(E_1^*) \otimes B(E_2^*) = \{x^* \otimes y^*: x^* \in B(E_1^*),\ y^* \in B(E_2^*)\}$ in $(E_1 \otimes E_2)^*$. Since the map $\tau : E_1^* \otimes E_2^* \longrightarrow E_1^* \otimes E_2^*$, the projective tensor product of E_1 with E_2, is continuous, [7], it follows that the map $\gamma : B(E_1^*) \times B_2(E_2^*) \longrightarrow B(E_1^*) \otimes B(E_2^*)$, $\gamma(x^*,y^*) = x^* \otimes y^*$ is continuous. This induces an isometric into operator $\psi: C(K) \longrightarrow C(B(E_1^*) \times B(E_2^*))$ defined by $\psi(f) = f \circ \gamma$. Consequently, there exists a measure μ on K such that

$$\int_K f(z^*)d\mu(z^*) = \iint_{B(E_1^*) \times B(E_2^*)} f \circ \gamma(x^*,y^*)d(\mu_1 \times \mu_2)(x^*,y^*).$$

Extend μ to $B(E_1 \otimes E_2)^*$ by putting $\mu \equiv 0$ on $B(E_1 \otimes E_2)^* \setminus K$. Further define $g : B(E_1 \otimes E_2)^* \longrightarrow F_1 \otimes F_2$ via $g(x^* \otimes y^*) = g_1(x^*) \otimes g_2(y^*)$ if $x^* \in B(E_1^*)$, $y^* \in B(E_2^*)$, and $g(z^*) = 0$ otherwise. Then it is not difficult to see that

$$(T_1 \otimes T_2)(z) = \int_{B(E_1 \otimes E_2)^*} \langle z,z^* \rangle g(z^*)d\mu(z^*)$$

for all $z \in E_1 \otimes E_2$. Since $g \in L^q(B(E_1 \otimes E_2)^*, \mu, F_1 \otimes F_2)$, it follows that $T_1 \otimes T_2 \in R_p(E_1 \otimes E_2, F_1 \otimes F_2)$. This ends the proof.

A negative result for R_p is the following:

THEOREM 3.4. R_p is not closed.

PROOF: Assume R_p is closed. Since the ideal of finite rank operator is contained in R_p, one has the ideal of approximable operators is contained in R_p. By Lemma 2.4, one gets R_p = the ideal of approximable operators. Theorem 2.8, together with the open mapping theorem we get that $\| \ \|_{\sigma(p)}$ and $\| \ \|$ are equivalent on R_p. This is a contradiction. Hence R_p is not closed.

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