MIXED PROBLEM WITH INTEGRAL CONDITIONS FOR A CERTAIN CLASS OF HYPERBOLIC EQUATIONS

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We study a mixed problem with purely integral conditions for a class of two-dimensional second-order hyperbolic equations. We prove the existence, uniqueness, and the continuous dependence upon the data of a generalized solution. We use a functional analysis method based on a priori estimate and on the density of the range of the operator generated by the considered problem.

1. Introduction

The present paper is devoted to the proof of existence and uniqueness of a generalized solution for a mixed problem with only integral conditions related to a certain class of second-order hyperbolic equations in a two-dimensional structure. That is, we consider the problem of searching a function $u = u(x,t)$, solution of the problem

$$\mathcal{L}u = u_{tt} - a(t)\Delta u = f(x,t), \quad x = (x_1, x_2) \in \Omega,\ t \in (0,T), \tag{1.1}$$

where $\Omega = (0,a) \times (0,b_i)$ and b_i, T, $i = 1,2$, are known constants and $a(t)$ is a given function satisfying the conditions

$$c_0 \le a(t) \le c_1, \qquad a'(t) \le c_2, \tag{1.2}$$

where c_i, $i = 0,1,2$, are positive constants.

To (1.1), we associate the initial conditions

$$\ell_1 u = u(x,0) = \varphi(x), \quad \ell_2 u = u_t(x,0) = \beta(x), \quad x \in \Omega, \tag{1.3}$$

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and the integral conditions

$$\int_0^{b_i} x_i^k u(x,t)\,dx_i = 0, \quad i = 1,2;\ k = 0,1, \tag{1.4}$$

where f, φ, and β are given functions such that $f \in C(\bar{Q})$ and $\varphi, \beta \in C^1(\bar{\Omega})$. The given data satisfy the consistency conditions

$$\int_0^{b_i} x_i^k \varphi\, dx_i = \int_0^{b_i} x_i^k \beta\, dx_i = 0, \quad i = 1,2;\ k = 0,1. \tag{1.5}$$

The results concerning problems with integral conditions related to one-dimensional parabolic equations are due to Batten [1], Cannon [7, 8], Cannon and van der Hoek [10, 11], Cannon et al. [9], Kamynin [13], Ionkin [12], Yurchuk [17], Benouar and Yurchuk [2], Muravey-Philinovskii [14], Shi [16], Bouziani [3, 4], and Bouziani and Benouar [6]. For problems related to one-dimensional hyperbolic equations we have the result of Bouziani [5], in which a Neumann and an integral condition are combined.

The present paper can be considered as an extension of Bouziani [5] in the way that the conditions are purely integral and the considered equation is a two-dimensional one. We first write the posed problem in its operational form $Lu = \mathcal{F}$, where the operator L is considered from the Banach space E into the Hilbert space F, which are conveniently chosen, then we establish an energy inequality for the operator L, and extend the obtained estimate to the closure $\bar{L}$, of the operator L. Finally, we prove the density of the range $R(L)$ of the operator L in the space F.

2. Energy inequality and its consequences

Problem (1.1), (1.3), and (1.4) can be considered as the resolution of the operator equation

$$Lu = \mathcal{F}, \tag{2.1}$$

where $L = (\mathcal{L}, \ell_1, \ell_2)$, $\mathcal{F} = (f, \varphi, \beta)$ and L is an operator defined on E into F, where E is the Banach space of functions $\mathfrak{I}_{x_1x_2}u \in L^2(Q)$, having the finite norm

$$\|u\|_E^2 = \sup_{0\le\tau\le 0} \int_\Omega \left(\left(\mathfrak{I}_{x_1}u(\cdot,\cdot\tau)\right)^2 + \left(\mathfrak{I}_{x_2}u(\cdot,\cdot\tau)\right)^2 + \left(\mathfrak{I}_{x_1x_2}u_t(\cdot,\cdot\tau)\right)^2\right) dx_1\, dx_2 \tag{2.2}$$

with $\mathfrak{I}_{x_1}u = \int_0^{x_1} u(\xi, x_2, t)\,d\xi$, $\mathfrak{I}_{x_1x_2}u = \int_0^{x_1}\int_0^{x_2} u(\xi,\eta,t)\,d\xi\,d\eta$, and F is the

Hilbert space equipped with the scalar product

$$\begin{aligned}
&\big((\mathcal{L}u,\ell_1 u,\ell_2 u),(f,\varphi,\beta)\big)_F \\
&\quad=\int_Q \mathfrak{I}_{x_1x_2}(\mathcal{L}u)\cdot\mathfrak{I}_{x_1x_2} f\,dx\,dt+\int_\Omega \mathfrak{I}_{x_1}\ell_1 u\cdot\mathfrak{I}_{x_1}\varphi\,dx \\
&\qquad+\int_\Omega \mathfrak{I}_{x_2}\ell_1 u\cdot\mathfrak{I}_{x_2}\varphi\,dx+\int_\Omega \mathfrak{I}_{x_1x_2}\ell_2 u\cdot\mathfrak{I}_{x_1x_2}\beta\,dx,
\end{aligned}\tag{2.3}$$

and the associated norm

$$\begin{aligned}
\|Lu\|_F^2&=\int_Q\big(\mathfrak{I}_{x_1x_2}(\mathcal{L}u)\big)^2dx\,dt \\
&\quad+\int_\Omega\Big(\big(\mathfrak{I}_{x_1}\ell_1 u\big)^2+\big(\mathfrak{I}_{x_2}\ell_1 u\big)^2+\big(\mathfrak{I}_{x_1x_2}\ell_2 u\big)^2\Big)dx.
\end{aligned}\tag{2.4}$$

The domain of definition $D(L)$ of the operator L is the set of functions $\mathfrak{I}_{x_1x_2}u\in L^2(Q)$ such that $\mathfrak{I}_{x_1x_2}u_t,\mathfrak{I}_{x_1x_2}u_{x_1x_1},\mathfrak{I}_{x_1x_2}u_{x_2x_2}\in L^2(Q)$, and the conditions (1.4) are fulfilled.

Theorem 2.1. *If* $a(t)$ *satisfies conditions (1.2), then for all functions* $u\in D(L)$ *we have the a priori estimate*

$$\|u\|_E\le c\|Lu\|_F,\tag{2.5}$$

where c *is a positive constant independent of the solution* u.

Proof. We consider the scalar product in $L^2(Q^\tau)$ of (1.1) and the integro-differential operator

$$Mu=\mathfrak{I}^2_{x_1x_2}u_t=\int_0^{x_1}\int_0^{x_2}\int_0^{\xi_1}\int_0^{\xi_2}u_t(\eta_1,\eta_2,t)\,d\eta_2\,d\eta_1\,d\xi_2\,d\xi_1,\tag{2.6}$$

where $Q^\tau=\Omega\times(0,\tau)$ and $\tau\in(0,T)$, we obtain

$$\begin{aligned}
&\int_{Q^\tau}u_{tt}\cdot\mathfrak{I}^2_{x_1x_2}u_t\,dx\,dt \\
&\qquad-\int_{Q^\tau}a(t)u_{x_1x_1}\cdot\mathfrak{I}^2_{x_1x_2}u_t\,dx\,dt-\int_{Q^\tau}a(t)u_{x_2x_2}\cdot\mathfrak{I}^2_{x_1x_2}u_t\,dx\,dt \\
&\quad=\int_{Q^\tau}f\cdot\mathfrak{I}^2_{x_1x_2}u_t\,dx\,dt.
\end{aligned}\tag{2.7}$$

We separately consider the integrals of the equality (2.7). Integrating by parts and taking into account conditions (1.3) and (1.4), we get

$$\begin{aligned}&\int_{Q^\tau} u_{tt}\cdot \mathfrak{I}^2_{x_1x_2}u_t\,dx\,dt\\&\quad=\frac{1}{2}\int_\Omega \left(\mathfrak{I}_{x_1x_2}u_t(\xi_1,\xi_2,\tau)\right)^2dx-\frac{1}{2}\int_\Omega\left(\mathfrak{I}_{x_1x_2}\beta\right)^2dx,\end{aligned}\tag{2.8}$$

$$\begin{aligned}&-\int_{Q^\tau} a(t)u_{x_1x_1}\cdot \mathfrak{I}^2_{x_1x_2}u_t\,dx\,dt\\&\quad=\frac{1}{2}\int_\Omega a(\tau)\left(\mathfrak{I}_{x_2}u(x_1,\xi_2,\tau)\right)^2dx\\&\qquad-\frac{1}{2}\int_\Omega a(0)\left(\mathfrak{I}_{x_2}\varphi\right)^2dx-\frac{1}{2}\int_{Q^\tau}a'(t)\left(\mathfrak{I}_{x_2}u\right)^2dx\,dt,\end{aligned}\tag{2.9}$$

$$\begin{aligned}&-\int_{Q^\tau} a(t)u_{x_2x_2}\cdot \mathfrak{I}^2_{x_1x_2}u_t\,dx\,dt\\&\quad=\frac{1}{2}\int_\Omega a(\tau)\left(\mathfrak{I}_{x_1}u(\xi_1,x_2,\tau)\right)^2dx\\&\qquad-\frac{1}{2}\int_\Omega a(0)\left(\mathfrak{I}_{x_1}\varphi\right)^2dx-\frac{1}{2}\int_{Q^\tau}a'(t)\left(\mathfrak{I}_{x_1}u\right)^2dx\,dt,\end{aligned}\tag{2.10}$$

$$\int_{Q^\tau} f\cdot\mathfrak{I}^2_{x_1x_2}u_t\,dx\,dt=\int_{Q^\tau}\mathfrak{I}_{x_1x_2}f\cdot\mathfrak{I}_{x_1x_2}u_t\,dx\,dt.\tag{2.11}$$

Substitution of (2.8), (2.9), (2.10), and (2.11) into (2.7) yields

$$\begin{aligned}&\int_\Omega\left(\mathfrak{I}_{x_1x_2}u_t(\xi_1,\xi_2,\tau)\right)^2dx+\int_\Omega a(\tau)\left(\mathfrak{I}_{x_2}u(x_1,\xi_2,\tau)\right)^2dx\\&\qquad+\int_\Omega a(\tau)\left(\mathfrak{I}_{x_1}u(\xi_1,x_2,\tau)\right)^2dx\\&\quad=2\int_{Q^\tau}\mathfrak{I}_{x_1x_2}f\cdot\mathfrak{I}_{x_1x_2}u_t\,dx\,dt+\int_\Omega a(0)\left(\mathfrak{I}_{x_1}\varphi\right)^2dx\\&\qquad+\int_\Omega a(0)\left(\mathfrak{I}_{x_2}\varphi\right)^2dx+\int_\Omega\left(\mathfrak{I}_{x_1x_2}\beta\right)^2dx\\&\qquad+\int_{Q^\tau}a'(t)\left(\mathfrak{I}_{x_1}u\right)^2dx\,dt+\int_{Q^\tau}a'(t)\left(\mathfrak{I}_{x_2}u\right)^2dx\,dt.\end{aligned}\tag{2.12}$$

Using the Cauchy inequality and taking into account conditions (1.2), it follows that

$$\begin{aligned}&\|\Im_{x_1}u(\xi_1,x_2,\tau)\|^2_{L^2(\Omega)}+\|\Im_{x_2}u(x_1,\xi_2,\tau)\|^2_{L^2(\Omega)}+\|\Im_{x_1x_2}u_t(\xi_1,\xi_2,\tau)\|^2_{L^2(\Omega)}\\&\quad\le c_3\Big(\|\Im_{x_1x_2}f\|^2_{L^2(Q)}+\|\Im_{x_1}\varphi\|^2_{L^2(\Omega)}+\|\Im_{x_2}\varphi\|^2_{L^2(\Omega)}+\|\Im_{x_1x_2}\beta\|^2_{L^2(\Omega)}\Big)\\&\qquad+c_4\int_0^\tau\Big(\|\Im_{x_1}u\|^2_{L^2(\Omega)}+\|\Im_{x_2}u\|^2_{L^2(\Omega)}+\|\Im_{x_1x_2}u_t\|^2_{L^2(\Omega)}\Big)dt,\end{aligned}\tag{2.13}$$

where

$$c_3=\max\frac{(1,c_1)}{c_0},\qquad c_4=\max\frac{(1,c_2)}{c_0}.\tag{2.14}$$

Applying the Gronwall's lemma [4] to inequality (2.13), we get

$$\begin{aligned}&\|\Im_{x_1}u(\xi_1,x_2,\tau)\|^2_{L^2(\Omega)}+\|\Im_{x_2}u(x_1,\xi_2,\tau)\|^2_{L^2(\Omega)}+\|\Im_{x_1x_2}u_t(\xi_1,\xi_2,\tau)\|^2_{L^2(\Omega)}\\&\quad\le c_3e^{c_4T}\Big(\|\Im_{x_1x_2}f\|^2_{L^2(Q)}+\|\Im_{x_1}\varphi\|^2_{L^2(\Omega)}\|\Im_{x_1}\varphi\|^2_{L^2(\Omega)}+\|\Im_{x_1x_2}\beta\|^2_{L^2(\Omega)}\Big).\end{aligned}\tag{2.15}$$

Since the right-hand side of (2.15) does not depend on τ, then by taking the supremum with respect to τ over the interval $[0,T]$, we obtain the desired inequality (2.5), with $c=c_3/2\exp(c_4T/2)$. This completes the proof of Theorem 2.1. □

Proposition 2.2. *The operator* $L:E\to F$ *is closable.*

Proof. The proof of this proposition is analogous to Proposition 3.1 in [4]. □

Let $\bar{L}$ be the closure of the operator L, and $D(\bar{L})$ its domain of definition.

Definition 2.3. The solution of the equation

$$\bar{L}u=\mathcal{F}\tag{2.16}$$

is called strong solution of problem (1.1), (1.3), and (1.4).

We extend inequality (2.5) to the set of solutions $u\in D(\bar{L})$ by passing to the limit and thus establish uniqueness of a strong solution and closedness of the range $R(\bar{L})$ of the operator L in the space F.

3. Solvability of the problem

Theorem 3.1. *If conditions (1.2) are satisfied, then for all* $\mathcal{F}=(f,\varphi,\beta)\in F$, *there exists a unique strong solution* $u=\bar{L}^{-1}\mathcal{F}=\overline{L^{-1}}\mathcal{F}$ *of problem (1.1), (1.3), and (1.4).*

Proof. To prove that problem (1.1), (1.3), and (1.4) has a unique strong solution for all $\mathcal{F} \in F$, it suffices to prove that $R(L)$ is dense in F. For this we need the following proposition.

Proposition 3.2. *If conditions (1.2) are satisfied, and if for* $\mathfrak{I}_{x_1x_2}\omega \in L^2(Q)$,

$$\int_Q \mathfrak{I}_{x_1x_2}(\mathcal{L}u)\cdot\mathfrak{I}_{x_1x_2}\omega\,dx\,dt = 0, \tag{3.1}$$

for all the functions $u \in D_0(L) = \{u/u \in D(L),\ \ell_1 u = \ell_2 u = 0\}$, *then* $\mathfrak{I}_{x_1x_2}\omega = 0$ *almost everywhere in* Q.

Using the fact that relation (3.1) is given for all $u \in D_0(L)$, we can express it in a particular form.

Let u be defined as

$$u = \begin{cases} 0, & 0 \le t \le s, \\ \displaystyle\int_s^t (t-\tau)u_{\tau\tau}\,d\tau, & s \le t \le T, \end{cases} \tag{3.2}$$

and let u_{tt} be the solution of the equation

$$a(t)\mathfrak{I}_{x_1x_2}u_{tt} = \int_t^T \mathfrak{I}_{x_1x_2}\omega\,d\tau. \tag{3.3}$$

We now have

$$\mathfrak{I}_{x_1x_2}\omega = -\big(a(t)\mathfrak{I}_{x_1x_2}u_{tt}\big)_t. \tag{3.4}$$

To continue the proof of the proposition, we need the following lemma.

Lemma 3.3. *If conditions (1.2) are satisfied, then the function* u *defined by relations (3.2) and (3.3) possesses derivatives with respect to* t *up to the third order belonging to* $L^2(Q)$.

The proof of this lemma is analogous to that of [3, Lemma 4.1].

We now prove the proposition. Replacing $\mathfrak{I}_{x_1x_2}\omega$ in (3.1) by its representation (3.4), we have

$$\begin{aligned} &-\int_Q \mathfrak{I}_{x_1x_2}u_{tt}\big(a(t)\mathfrak{I}_{x_1x_2}u_{tt}\big)_t\,dx\,dt \\ &\quad+\int_Q \mathfrak{I}_{x_1x_2}u_{x_1x_1}\big(a(t)\mathfrak{I}_{x_1x_2}u_{tt}\big)_t\,dx\,dt \\ &\quad+\int_Q \mathfrak{I}_{x_1x_2}u_{x_2x_2}\big(a(t)\mathfrak{I}_{x_1x_2}u_{tt}\big)_t\,dx\,dt = 0. \end{aligned} \tag{3.5}$$

We write the terms of (3.5) in the form

$$-\int_Q \mathfrak{I}_{x_1x_2}u_{tt}\big(a(t)\mathfrak{I}_{x_1x_2}u_{tt}\big)_t\,dx\,dt = \frac{1}{2}\int_\Omega a(s)\big(\mathfrak{I}_{x_1x_2}u_{tt}(x,s)\big)^2dx - \int_{Q_s} a'(t)\big(\mathfrak{I}_{x_1x_2}u_{tt}\big)^2dx\,dt, \tag{3.6}$$

$$\begin{aligned}\int_Q \mathfrak{I}_{x_1x_2}u_{x_1x_1}\big(a(t)\mathfrak{I}_{x_1x_2}u_{tt}\big)_t\,dx\,dt &= \frac{1}{2}\int_\Omega a(T)\big(\mathfrak{I}_{x_2}u_t(x,T)\big)^2dx - \frac{1}{2}\int_{Q_s} a'(t)\big(\mathfrak{I}_{x_2}u_t\big)^2dx\,dt \\ &\quad - \int_{Q_s} a'(t)\mathfrak{I}_{x_2}u\mathfrak{I}_{x_2}u_{tt}\,dx\,dt,\end{aligned} \tag{3.7}$$

$$\begin{aligned}\int_Q \mathfrak{I}_{x_1x_2}u_{x_2x_2}\big(a(t)\mathfrak{I}_{x_1x_2}u_{tt}\big)_t\,dx\,dt &= \frac{1}{2}\int_\Omega a(T)\big(\mathfrak{I}_{x_1}u_t(x,T)\big)^2dx - \frac{1}{2}\int_{Q_s} a'(t)\big(\mathfrak{I}_{x_1}u_t\big)^2dx\,dt \\ &\quad - \int_{Q_s} a'(t)\mathfrak{I}_{x_1}u\mathfrak{I}_{x_1}u_{tt}\,dx\,dt.\end{aligned} \tag{3.8}$$

Combining conditions (3.5), (3.6), (3.7), and (3.8) and using conditions (1.2), we obtain the inequality

$$\begin{aligned}&\big\|\mathfrak{I}_{x_1x_2}u_{tt}(x,s)\big\|^2_{L^2(\Omega)} + \big\|\mathfrak{I}_{x_1}u_t(x,T)\big\|^2_{L^2(\Omega)} + \big\|\mathfrak{I}_{x_2}u_t(x,T)\big\|^2_{L^2(\Omega)} \\ &\quad \le c_5\Big\{\big\|\mathfrak{I}_{x_1x_2}u_{tt}\big\|^2_{L^2(Q_s)} + \big\|\mathfrak{I}_{x_1}u_t\big\|^2_{L^2(Q_s)} \\ &\qquad + \big\|\mathfrak{I}_{x_2}u_t\big\|^2_{L^2(Q_s)} + \big\|\mathfrak{I}_{x_1}u\big\|^2_{L^2(Q_s)} + \big\|\mathfrak{I}_{x_2}u\big\|^2_{L^2(Q_s)}\Big\},\end{aligned} \tag{3.9}$$

where

$$c_5 = \max\frac{c_0}{2}\left(\frac{c_2}{2}+\frac{c_2^2}{2},1\right). \tag{3.10}$$

Using now the Friedrichs inequality [15], to express the norms of $\mathfrak{I}_{x_1}u$ and $\mathfrak{I}_{x_2}u$, in terms of the norms of $\mathfrak{I}_{x_1}u_t$ and $\mathfrak{I}_{x_2}u_t$, respectively, then it follows from (3.9) that

$$\begin{aligned}&\big\|\mathfrak{I}_{x_1x_2}u_{tt}(x,s)\big\|^2_{L^2(\Omega)} + \big\|\mathfrak{I}_{x_1}u_t(x,T)\big\|^2_{L^2(\Omega)} + \big\|\mathfrak{I}_{x_2}u_t(x,T)\big\|^2_{L^2(\Omega)} \\ &\quad \le c_6\Big\{\big\|\mathfrak{I}_{x_1x_2}u_{tt}\big\|^2_{L^2(Q_s)} + \big\|\mathfrak{I}_{x_1}u_t\big\|^2_{L^2(Q_s)} + \big\|\mathfrak{I}_{x_2}u_t\big\|^2_{L^2(Q_s)}\Big\}.\end{aligned} \tag{3.11}$$

To continue, we introduce the new function θ defined by

$$\theta(x,t) = \int_t^T u_{\tau\tau}\,d\tau, \tag{3.12}$$

then

$$u_t(x,t) = \theta(x,s) - \theta(x,t), \qquad u_t(x,T) = \theta(x,s). \tag{3.13}$$

Hence

$$\begin{aligned}(1-2c_6(T-s))\Big(\|\mathfrak{I}_{x_1}\theta(x,s)\|^2_{L^2(\Omega)} + \|\mathfrak{I}_{x_2}\theta(x,s)\|^2_{L^2(\Omega)}\Big)\\ + \|\mathfrak{I}_{x_1x_2}u_{tt}(x,s)\|^2_{L^2(\Omega)}\\ \le 2c_6\Big\{\|\mathfrak{I}_{x_1x_2}u_{tt}\|^2_{L^2(Q_s)} + \|\mathfrak{I}_{x_1}\theta\|^2_{L^2(Q_s)} + \|\mathfrak{I}_{x_2}\theta\|^2_{L^2(Q_s)}\Big\}.\end{aligned} \tag{3.14}$$

Consequently, if $s_0 > 0$ satisfies

$$(1-2c_6(T-s)) = \frac{1}{2}, \tag{3.15}$$

then (3.14) implies

$$\begin{aligned}\|\mathfrak{I}_{x_1x_2}u_{tt}(x,s)\|^2_{L^2(\Omega)} + \|\mathfrak{I}_{x_1}\theta(x,s)\|^2_{L^2(\Omega)} + \|\mathfrak{I}_{x_1}\theta(x,s)\|^2_{L^2(\Omega)}\\ \le 2c_6\Big\{\|\mathfrak{I}_{x_1x_2}u_{tt}\|^2_{L^2(Q_s)} + \|\mathfrak{I}_{x_1}\theta\|^2_{L^2(Q_s)} + \|\mathfrak{I}_{x_2}\theta\|^2_{L^2(Q_s)}\Big\},\end{aligned} \tag{3.16}$$

for all $s \in [T-s_0, T]$.

If we denote the sum of terms involving norms on the right-hand side of (3.16) by $y(s)$, we obtain

$$-\frac{dy(s)}{ds} \le 4c_6 y(s). \tag{3.17}$$

Integrating (3.17) over (s,T) and taking into account that $y(T) = 0$, we get

$$y(s)e^{4c_6 s} \le 0. \tag{3.18}$$

It follows then from (3.18) that $\mathfrak{I}_{x_1x_2}\omega = 0$ almost everywhere in Q_{T-s_0}. Proceeding in this way step by step, we prove that $\mathfrak{I}_{x_1x_2}\omega = 0$ in Q.

To conclude, we prove Theorem 3.1. We should prove the validity of the equality $\overline{R(L)} = F$.

Since F is a Hilbert space, $\overline{R(L)} = F$ holds, if

$$\begin{aligned}(Lu, W)_F &= \int_Q \mathfrak{I}_{x_1x_2}(\mathcal{L}u)\cdot\mathfrak{I}_{x_1x_2}\omega\,dx\,dt + \int_\Omega \mathfrak{I}_{x_1}\ell_1 u\cdot\mathfrak{I}_{x_1}\omega_0\,dx\\ &\quad + \int_\Omega \mathfrak{I}_{x_2}\ell_1 u\cdot\mathfrak{I}_{x_2}\omega_0\,dx + \int_\Omega \mathfrak{I}_{x_1x_2}\ell_2 u\cdot\mathfrak{I}_{x_1x_2}\omega_1\,dx\\ &= 0,\end{aligned} \tag{3.19}$$

it follows that $\omega = 0$, $\omega_0 = 0$, and $\omega_1 = 0$, almost everywhere in Q, where $W = (\omega, \omega_0, \omega_1) \in R(L)^\perp$.

Putting $u \in D_0(L)$ into (3.19), we obtain

$$\int_Q \mathfrak{I}_{x_1x_2}(\mathcal{L}u)\cdot\mathfrak{I}_{x_1x_2}\omega\,dx\,dt = 0. \tag{3.20}$$

Hence, Proposition 3.2 implies that $\omega = 0$. Thus (3.19) takes the form

$$\begin{aligned}\int_\Omega \mathfrak{I}_{x_1}\ell_1 u\cdot\mathfrak{I}_{x_1}\omega_0\,dx + \int_\Omega \mathfrak{I}_{x_2}\ell_1 u\cdot\mathfrak{I}_{x_2}\omega_0\,dx&\\ + \int_\Omega \mathfrak{I}_{x_1x_2}\ell_2 u\cdot\mathfrak{I}_{x_1x_2}\omega_1\,dx = 0, \quad \forall u\in D_0(L).&\end{aligned} \tag{3.21}$$

Since the sets $\ell_1 u$ and $\ell_2 u$ are independent and the ranges of the trace operators ℓ_1 and ℓ_2 are everywhere dense in the Hilbert spaces having the norms $(\int_\Omega((\mathfrak{I}_{x_1}\omega_0)^2+(\mathfrak{I}_{x_2}\omega_0)^2)dx)^{1/2}$ and $(\int_\Omega(\mathfrak{I}_{x_1x_2}\omega_1)^2dx)^{1/2}$, respectively, then $\omega_0 = 0$, $\omega_1 = 0$, almost everywhere in Ω. This completes the proof of Theorem 3.1. □

Remark 3.4. The above used method can be easily applied to solve the following differential problem of higher order

$$\begin{gathered}\mathcal{L}u = u_{tt} + (-1)^m a(t)\Delta^{2m}u = f(x,t),\\ \ell_1 u = u(x,0) = \varphi(x), \quad \ell_2 u = u_t(x,0) = \beta(x), \quad x\in\Omega,\\ \int_0^{b_i} x_i^k u(x_1,x_2,t)\,dx_1\,dx_2 = 0, \quad k = 0,\dots,2m-1;\ i = 1,2,\\ x = (x_1,x_2)\in\Omega = (0,b_1)\times(0,b_2)\subset\mathbb{R}^2, \qquad t\in(0,T).\end{gathered} \tag{3.22}$$

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