A MULTIPLICITY RESULT FOR A QUASILINEAR GRADIENT ELLIPTIC SYSTEM

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The aim of this work is to establish the existence of infinitely many solutions to gradient elliptic system problem, placing only conditions on a potential function H, associated to the problem, which is assumed to have an oscillatory behaviour at infinity. The method used in this paper is a shooting technique combined with an elementary variational argument. We are concerned with the existence of upper and lower solutions in the sense of Hernández.

1. Introduction

We prove the existence of infinitely many solutions for the following problem:

$$\begin{gathered} -\Delta_p u = f(x,u,v), \quad -\Delta_q v = g(x,u,v) \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.1}$$

We assume that Ω is a smooth bounded domain of $\mathbb{R}^N$, $N \geq 1$, $p,q > 1$, and $f,g : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ be given functions which we specify later.

The prototype model (1.1) turns up in many mathematical settings as non-Newtonian fluids, population evolution, reaction-diffusion problems, porous media, and so forth. Much attention has been given to the existence of solutions of systems (1.1), by using different approaches. When (1.1) does not have a variational structure, we can notice the existence results obtained in [3, 4]. More recently, in [1], we derived the solvability of problem (1.1), under some lower limit conditions associated to F and G, where

$$F(x,u,v) = \int_0^u f(x,t,v)\,dt, \qquad G(x,u,v) = \int_0^v g(x,u,s)\,ds. \tag{1.2}$$

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When the system has a variational structure, that is, $f = \partial H/\partial u$ and $g = \partial H/\partial v$, the existence of solutions for (1.1) can be established via variational approaches, under appropriate conditions (cf. [5, 6, 7, 11]). An interesting result in this direction was obtained in [2]. By using variational methods, the authors show how the changes in the sign of $(\partial H/\partial u)(x,\cdot,\cdot)$ and $(\partial H/\partial v)(x,\cdot,\cdot)$ lead to multiple positive solutions of the system.

The goal of this paper is to show that the same approach in [1] can be applied to deal with the question of existence of infinitely many solutions for the following gradient system:

$$-\Delta_p u = \frac{\partial H}{\partial u}(u,v) + h_1, \quad -\Delta_q v = \frac{\partial H}{\partial v}(u,v) + h_2 \quad \text{in } \Omega,$$
$$u = v = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

Placing only some lower limit conditions on the potential function H associated to (1.3), which is assumed to have an oscillatory behaviour at infinity.

2. Main result

We make the following assumptions:

$$\forall u \in \mathbb{R}, \quad \frac{\partial H}{\partial u}(u,\cdot) \text{ is an increasing function on } \mathbb{R}, \tag{2.1}$$

$$\forall v \in \mathbb{R}, \quad \frac{\partial H}{\partial v}(\cdot,v) \text{ is an increasing function on } \mathbb{R}, \tag{2.2}$$

$$\forall (u,v) \in \mathbb{R}^2, \quad \text{such that } u \cdot v \geq 0, \tag{2.3}$$

we have

$$H(u,v) \geq 0, \tag{2.4}$$

$$\liminf_{m \to +\infty} \frac{H\big(\varepsilon m^{1/p}, \varepsilon m^{1/q}\big)}{m} < \mu_{p,q}, \tag{2.5}$$

$$\limsup_{m \to +\infty} \frac{H\big(\varepsilon m^{1/p}, \varepsilon m^{1/q}\big)}{m} = +\infty, \tag{2.6}$$

where $\varepsilon = 1, -1$ and $\mu_{p,q} = \min(\mu_p, \mu_q)$ such that μ_p and μ_q are the following constants:

$$\mu_p = \frac{(p-1)}{p}\left[\frac{2}{b-a}\int_0^1 \frac{ds}{\sqrt[p]{1-s^p}}\right]^p,$$
$$\mu_q = \frac{(q-1)}{q}\left[\frac{2}{b-a}\int_0^1 \frac{dt}{\sqrt[q]{1-t^q}}\right]^q, \tag{2.7}$$

with $b-a=\min(b_i-a_i)$ and $P=\Pi[a_i,b_i]$ is the smallest cube such that $P\supset\Omega$. Observe that for $N=1$, $p\mu_p$ and $q\mu_q$ are the first eigenvalue of $-\Delta_p$ and $-\Delta_q$, respectively, when $\Omega=]a,b[$.

Example 2.1. The function H such that

$$H(u,v)=\left(\sin|u|^p\right)^2|u|^\alpha+\left(\sin|v|^q\right)^2|v|^\beta \tag{2.8}$$

satisfies the hypotheses (2.1), (2.3), (2.5), and (2.6), when $\alpha>p$ or $\beta>q$.

The main result of this paper is the following statement.

Theorem 2.2. *Under the assumptions (2.1), (2.3), (2.5), and (2.6), problem (1.3) has two sequences* $(\overline{u}_n,\overline{v}_n)$ *and* $(\underline{u}_n,\underline{v}_n)$ *solutions in* $(W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega))\cap(L^{+\infty}(\Omega)\times L^{+\infty}(\Omega))$ *for any* (h_1,h_2) *in* $L^{+\infty}(\Omega)\times L^{+\infty}(\Omega)$, *and satisfy*

$$\max\left(\sup_\Omega\overline{u}_n;\sup_\Omega\overline{v}_n\right)\longrightarrow+\infty,\qquad\min\left(\inf_\Omega\underline{u}_n;\inf_\Omega\underline{v}_n\right)\longrightarrow-\infty. \tag{2.9}$$

The method used in this paper is a shooting technique combined with an elementary variational argument. We will be concerned with the existence of a sequence of negative subsolutions $\{(u_{0n},v_{0n})\}_n$ and a sequence of nonnegative supersolutions $\{(u_n^0,v_n^0)\}_n$, in the sense of Hernández's definition [7], which are both of class C^1 and satisfy

$$\begin{aligned}+\infty\longleftarrow\min_{\overline{\Omega}}u_n^0\geq\max_{\overline{\Omega}}u_{0n}\longrightarrow-\infty,\\ +\infty\longleftarrow\min_{\overline{\Omega}}v_n^0\geq\max_{\overline{\Omega}}v_{0n}\longrightarrow-\infty.\end{aligned} \tag{2.10}$$

3. Construction of a sequence of super-subsolutions

Definition 3.1. A pair $[(u_0,v_0),(u^0,v^0)]$ is a weak sub-supersolution for the Dirichlet problem (1.3), if the following conditions are satisfied:

$$\left(u_0,v_0\right)\in\left(W^{1,p}(\Omega)\times W^{1,q}(\Omega)\right)\cap\left(L^{+\infty}(\Omega)\times L^{+\infty}(\Omega)\right),$$

$$\left(u^0,v^0\right)\in\left(W^{1,p}(\Omega)\times W^{1,q}(\Omega)\right)\cap\left(L^{+\infty}(\Omega)\times L^{+\infty}(\Omega)\right),$$

$$\begin{aligned}-\Delta_p u_0-f(x,u_0,v)\leq0\leq-\Delta_p u^0-f(x,u^0,v)\quad&\text{in }\Omega,\ \forall v\in[v_0,v^0],\\ -\Delta_q v_0-f(x,u,v_0)\leq0\leq-\Delta_q v^0-f(x,u,v^0)\quad&\text{in }\Omega,\ \forall u\in[u_0,u^0],\end{aligned} \tag{3.1}$$

$$u_0\leq u^0,\quad v_0\leq v^0\quad\text{in }\Omega,$$

$$u_0\leq0\leq u^0,\quad v_0\leq0\leq v^0\quad\text{on }\partial\Omega.$$

Similar definitions can be found in Diaz and Herrero [8]. For all $M > 0$, we note that

$$\widehat{H}(u,v) = H(u,v) + M(v+u). \tag{3.2}$$

Notice that if H satisfies assumption (2.5) then the same holds for $\widehat{H}$.

Proposition 3.2. *Under hypotheses (2.3) and (2.5) there exist the sequences* d_n, d'_n, m_n, *and* m'_n *such that*

(a) $m_n^{1/p} \geq d_n \geq 0$, $\forall n \in \mathbb{N}$,

$$\limsup \int_{d_n}^{d_{n+1}} \frac{ds}{\sqrt[p]{p\widehat{H}(d_{n+1}, m_{n+1}^{1/q}) - p\widehat{H}(s, m_{n+1}^{1/q})}} > \int_0^1 \frac{ds}{\sqrt[p]{1-s^p}} \left[p\mu_p\right]^{-1/p}, \tag{3.3}$$

and such that for all $n \in \mathbb{N}$ we have

$$\lim_{n\to+\infty} \frac{d_n}{d_{n+1}} = 0. \tag{3.4}$$

(b) ${m'_n}^{1/q} \geq d'_n \geq 0$, $\forall n \in \mathbb{N}$ we have

$$\limsup \int_{d'_n}^{d'_{n+1}} \frac{dt}{\sqrt[q]{q\widehat{H}(m_{n+1}^{1/p}, d'_{n+1}) - q\widehat{H}(m_{n+1}^{1/p}, t)}} > \int_0^1 \frac{dt}{\sqrt[q]{1-t^q}} \left[q\mu_q\right]^{-1/q}, \tag{3.5}$$

and such that for all $n \in \mathbb{N}$ we have

$$\lim_{n\to+\infty} \frac{d'_n}{d'_{n+1}} = 0. \tag{3.6}$$

Proof. We only prove (a); the proof of (b) is similar.

(1) Let a fixed real $d > 0$. Under the hypothesis (2.5), there exists some number $\mu > 0$ such that

$$\lim_{m\to+\infty} \inf \frac{p\widehat{H}(m^{1/p}, m^{1/q})}{m} < \mu < p\mu_{p,q} \leq p\mu_p, \tag{3.7}$$

then there exists some sequence $\{m_k\}_k$ such that

$$\lim_{k\to+\infty} \mu m_k - p\widehat{H}(m_k^{1/p}, m_k^{1/q}) = +\infty. \tag{3.8}$$

(2) We consider the sequence of functions $[F(\cdot, m_k)]_k$, where

$$F(s, m_k) = \mu s - p\widehat{H}(s^{1/p}, m_k^{1/q}). \tag{3.9}$$

Hence from (3.8), for $k > 0$ sufficiently large, we have

$$F(m_k,m_k)=\mu m_k-p\widehat{H}(m_k^{1/p},m_k^{1/q})>0. \tag{3.10}$$

Then for all $k\in\mathbb{N}$ there exists $d_k>0$ satisfying $d_k^p\in[d^p,m_k]$ and such that for all $s\in[d^p,m_k]$, we have

$$F(s,m_k)\leq F(d_k^p,m_k), \tag{3.11}$$

that is,

$$\mu s-p\widehat{H}(s^{1/p},m_k^{1/q})\leq \mu d_k^p-pH(d_k,m_k^{1/q}), \tag{3.12}$$

then

$$p\widehat{H}(d_k,m_k^{1/q})-p\widehat{H}(s^{1/p},m_k^{1/q})\leq \mu(d_k^p-s). \tag{3.13}$$

Thus, from (2.3) and (3.11), we get

$$F(m_k,m_k)\leq F(d_k^p,m_k)\leq d_k. \tag{3.14}$$

Hence, from (3.8) and (3.14), we obtain

$$\lim_{k\to+\infty} d_k=+\infty. \tag{3.15}$$

Let $s=\omega^p$, where $\omega\in[d,d_k]\subset[d,m_k^{1/p}]$, we obtain

$$p\widehat{H}(d_k,m_k^{1/q})-p\widehat{H}(\omega,m_k^{1/q})\leq \mu_\epsilon(d_k^p-\omega^p), \tag{3.16}$$

that is,

$$\frac{1}{\sqrt[p]{d_k^p-\omega^p}}[\mu]^{-1/p}\leq \frac{1}{\sqrt[p]{p\widehat{H}(d_k,m_k^{1/q})-p\widehat{H}(\omega,m_k^{1/q})}}. \tag{3.17}$$

Then integrating on $[d,d_k]$, we obtain that for all $k>0$, (d,d_k,m_k) satisfies

$$\int_{d/d_k}^{1}\frac{d\omega}{\sqrt[p]{1-\omega^p}}[\mu]^{-1/p}\leq \int_d^{d_k}\frac{d\omega}{\sqrt[p]{p\widehat{H}(d_k,m_k^{1/q})-p\widehat{H}(\omega,m_k^{1/q})}}. \tag{3.18}$$

Consequently, for $d=d_0$, there exist k_0 sufficiently large, d_{k_0}, and m_{k_0} such that (d_0,d_{k_0},m_{k_0}) satisfies (3.18) and $d_0/d_{k_0}\leq 1/k_0$. Now, let $d=d_{k_0}$, then there exist k_1 sufficiently large, d_{k_1}, and m_{k_1} such that $(d_{k_0},d_{k_1},m_{k_1})$ satisfies (3.18), and $d_{k_0}/d_{k_1}\leq 1/k_1$. By iteration there exist some subsequences of $\{d_k\}_k$ and $\{m_k\}_k$, respectively, denoted $d_n:=d_{k_n}$ and $m_n:=m_{k_n}$ such that for all $n\in\mathbb{N}$, (d_n,d_{n+1},m_{n+1}) satisfies (3.18) and $d_n/d_{n+1}\leq 1/k_n$. Hence,

$$\lim_{n\to+\infty}\frac{d_n}{d_{n+1}}=0. \tag{3.19}$$

Thus, from (3.18), we have

$$\int_0^1 \frac{d\omega}{\sqrt[p]{1-\omega^p}}[\mu]^{-1/p} \le \limsup \int_{d_n}^{d_{n+1}} \frac{d\omega}{\sqrt[p]{p\hat{H}(d_{n+1}, m_{n+1}^{1/q}) - p\hat{H}(\omega, m_{n+1}^{1/q})}}. \tag{3.20}$$

This is the conclusion of Proposition 3.2. □

Remark 3.3. We observe that

$$\begin{aligned} \sqrt[p]{p-1}\int_0^1 \frac{ds}{\sqrt[p]{1-s^p}}\left[p\mu_p\right]^{-1/p} &= \sqrt[q]{q-1}\int_0^1 \frac{dt}{\sqrt[q]{1-t^q}}\left[q\mu_q\right]^{-1/q} \\ &= \frac{b-a}{2}. \end{aligned} \tag{3.21}$$

Consequently,

$$\frac{b-a}{2} < \limsup \int_{d_n}^{d_{n+1}} \frac{d\omega}{\sqrt[p]{p\hat{H}(d_{n+1}, m_{n+1}^{1/q}) - p\hat{H}(\omega, m_{n+1}^{1/q})}}. \tag{3.22}$$

3.1. Construction of a sequence of supersolutions $\{(u^0{}_n, v^0{}_n)\}_{n>1}$

Proposition 3.4. *Suppose that* $(d_n)_n$ *and* $(m_n)_n$ *satisfy Proposition 3.2*, *and that for all* $n \in \mathbb{N}$ *we have*

$$\inf_{s\in[d_{n-1}, m_n^{1/p}]} \frac{\partial H}{\partial u}(s, m_n^{1/q}) + M \ge 0. \tag{3.23}$$

Then, *there exists some* $n_0 \in \mathbb{N}$ *such that for all* $n \ge n_0$ *the following problem*:

$$\begin{aligned} -\left(|u'|^{p-2}u'\right)' &= \frac{\partial H}{\partial u}(u, m_n^{1/q}) + M \quad \text{in } (a,b), \\ u(a) = d_n, &\quad u'(a) = 0 \quad \text{on } [a,b], \end{aligned} \tag{3.24}$$

has a solution $\hat{u}_n$ *satisfying* $\hat{u}_n \in C^1([a,b])$, $(|\hat{u}_n'|^{p-2}\hat{u}_n')' \in C([a,b])$, *with* $m_n^{1/p} \ge \hat{u}_n \ge d_{n-1}$ *for all* $n \in \mathbb{N}$ *and*

$$0 < \hat{u}_0 < \cdots < \hat{u}_n < \hat{u}_{n+1} < \cdots + \infty. \tag{3.25}$$

Proof. Assume that $(d_n)_n$ and $(m_n)_n$, the sequences defined in Proposition 3.2, satisfy (3.23).
Step 1. We define the functions

$$\begin{aligned} \varphi_p(s) &:= \operatorname{sign}(s)|s|^{p-1}, \\ \Psi_p^*(s) := \int_0^s \varphi_p^{-1}(t)\,dt &= \int_0^s \operatorname{sign}(t)|t|^{1/(p-1)}\,dt = \frac{p-1}{p}|s|^{p/(p-1)}. \end{aligned} \tag{3.26}$$

Now, we consider the initial value problem

$$-\left(\varphi_n(u')\right)' = \left(\frac{\partial H}{\partial u}\left(u, m_n^{1/q}\right) + M\right), \qquad u(a) = d_n, \qquad u'(a) = 0, \tag{3.27}$$

where $m_n^{1/p} > d_{n-1}$.

Since problem (3.27) is equivalent to the system

$$u' = \varphi_p^{-1}(v), \qquad v' = -\left(\frac{\partial H}{\partial u}\left(u, m_n^{1/q}\right) + M\right), \tag{3.28}$$

with initial conditions

$$u(a) = d_n, \qquad v(a) = 0, \tag{3.29}$$

it follows that the existence of a solution u_n of (3.27) and its continuity on the same maximal interval are standard facts (see [1]). We set

$$t_n := \sup\left\{t \in\,]a,b], \text{ such that } u_n \text{ is defined and } u_n > d_{n-1} \text{ on } [a,t]\right\}. \tag{3.30}$$

Of course, it is $t_n > a$. Integrating (3.27) on $[a,t]$, for any $t \in\,]a, t_n[$, we obtain that

$$\varphi_p\left(u_n'(t)\right) = \varphi_p\left(u_n'(a)\right) - \int_a^t \left(\frac{\partial H}{\partial u}\left(u_n(s), m_n^{1/q}\right) + M\right) ds. \tag{3.31}$$

Hence, from (3.23), we get

$$u_n'(t) \le 0. \tag{3.32}$$

This implies that $u_n' = \varphi_p^{-1}(v_n)$ is of class C^1 on $[a, t_n[$. So that $\varphi_p(u_n') = -|u_n'|^{p-1}$ can be differentiated.

Assume now by contradiction that

$$t_n < \frac{b+a}{2}. \tag{3.33}$$

By (3.32) there exists

$$\lim_{t \to t_n^-} u_n(t) = d_{n-1}. \tag{3.34}$$

Hence, we can denote

$$u_n\left(t_n\right) := d_{n-1}, \tag{3.35}$$

and hence u_n can be continued as a solution to t_n.

Accordingly, multiplying (3.27) by u_n', we obtain

$$\frac{p-1}{p}\left(-|u_n'(t)|^p\right)' = \frac{d}{dt}\left(\hat{H}(u_n(t), m_n^{1/q})\right), \tag{3.36}$$

where

$$\hat{H}(u,v) = H(u,v) + Mu. \tag{3.37}$$

Integrating (3.36) on $[a,t] \subset [a,t_n]$, we obtain

$$-\sqrt[p]{p-1}\,u_n'(t) = \sqrt[p]{p\hat{H}(d_n, m_n^{1/q}) - p\hat{H}(u_n(t), m_n^{1/q})}. \tag{3.38}$$

Integrating again (3.38) on $[a,t_n]$, we deduce that

$$\sqrt[p]{p-1}\int_a^{t_n} \frac{-u_n'(t)}{\sqrt[p]{p\hat{H}(d_n, m_n^{1/q}) - p\hat{H}(u_n(t), m_n^{1/q})}}\,dt \le t_n - a. \tag{3.39}$$

Then we obtain

$$\sqrt[p]{p-1}\int_{d_{n-1}}^{d_n} \frac{ds}{\sqrt[p]{p\hat{H}(d_n, m_n^{1/q}) - p\hat{H}(s, m_n^{1/q})}} \le t_n - a. \tag{3.40}$$

It follows from Proposition 3.2 and Remark 3.3 that for all $n \ge n_0$, we have

$$\frac{b-a}{2} < \sqrt[p]{p-1}\int_{d_{n-1}}^{d_n} \frac{ds}{\sqrt[p]{p\hat{H}(d_n, m_n^{1/q}) - p\hat{H}(s, m_n^{1/q})}} \le t_n - a. \tag{3.41}$$

This implies that $t_n > (b+a)/2$. Hence we obtain a contradiction.

This shows that, there exits a sequence $\{u_n\}_n$ satisfying for all $n \ge n_0$,

$$\begin{gathered}
u_n \in C^1\left(\left[a, \frac{a+b}{2}\right]\right), \qquad \left(|u_n'|^{p-2}u_n'\right)' \in C\left(\left[a, \frac{a+b}{2}\right]\right), \\
-\left(|u_n'|^{p-2}u_n'\right)'(t) = \frac{\partial H}{\partial u}(u_n(t), m_n^{1/q}) + M \quad \text{in } \left[a, \frac{a+b}{2}\right], \\
m_n^{1/p} \ge u_n \ge d_{n-1} \quad \text{in } \left[a, \frac{a+b}{2}\right], \qquad u_n'(a) = 0.
\end{gathered} \tag{3.42}$$

Step 2. We note by $\{\hat{u}_n\}_n$ the following functions such that

$$\hat{u}_n(t) = \begin{cases} u_n\left(\dfrac{3a+b}{2} - t\right) & \text{if } t \in \left[a, \dfrac{a+b}{2}\right], \\ u_n\left(t - \dfrac{b-a}{2}\right) & \text{if } t \in \left[\dfrac{a+b}{2}, b\right]. \end{cases} \tag{3.43}$$

It is a trivial matter to claim that the sequence $\{\hat{u}_n\}_n$ satisfies

$$\forall n \geq n_0, \quad \hat{u}_n \in C^1\big([a,b]\big), \quad \Big(\big|\hat{u}_n'\big|^{p-2}\hat{u}_n'\Big)' \in C\big([a,b]\big),$$

$$-\Big(\big|\hat{u}_n'\big|^{p-2}\hat{u}_n'\Big)'(t) = \frac{\partial H}{\partial u}\big(\hat{u}_n(t), m_n^{1/q}\big) + M \quad \text{in } [a,b], \tag{3.44}$$

$$m_n^{1/p} \geq \hat{u}_n \geq d_{n-1} \quad \text{in } [a,b],$$

moreover, we have

$$0 < \cdots < \hat{u}_n < \hat{u}_{n+1} < \cdots, \qquad \sup_{[a,b]} \hat{u}_n = d_n \longrightarrow +\infty. \tag{3.45}$$

Hence, Proposition 3.4 is proved. □

Proposition 3.5. *Let* $M > 0$. *Under the hypothesis (2.3) and (2.5) there exists some sequence of the positive numbers* $(m_n)_n$ *such that there exists* $(\hat{u}_n, \hat{v}_n) \in (C^1([a,b]))^2$ *satisfying*

$$\Big(\Big(\big|\hat{u}_n'\big|^{p-2}\hat{u}_n'\Big)', \Big(\big|\hat{v}_n'\big|^{p-2}\hat{v}_n'\Big)'\Big) \in \big(C[a,b]\big)^2,$$

$$-\Big(\big|\hat{u}_n'\big|^{p-2}\hat{u}_n'\Big)' \geq \frac{\partial H}{\partial u}\big(\hat{u}_n, {m_n}^{1/q}\big) + M \quad \text{a.e. in } (a,b),$$

$$-\Big(\big|\hat{v}_n'\big|^{q-2}\hat{v}_n'\Big)' \geq \frac{\partial H}{\partial v}\big({m_n}^{1/p}, \hat{v}_n\big) + M \quad \text{a.e. in } (a,b), \tag{3.46}$$

$${m_n}^{1/p} \geq \hat{u}_n \geq 0, \quad {m_n}^{1/q} \geq \hat{v}_n \geq 0 \quad \text{on } [a,b],$$

$$\max_{[a,b]} \hat{u}_n \leq \min_{[a,b]} \hat{u}_{n+1} \longrightarrow +\infty, \qquad \max_{[a,b]} \hat{v}_n \leq \min_{[a,b]} \hat{v}_{n+1} \longrightarrow +\infty.$$

Proof. Let (d_n) and (m_n) be as defined in Proposition 3.2. We study three cases.
Case 1. We suppose that for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \inf_{s \in [d_{n-1}, m_n^{1/p}]} \frac{\partial H}{\partial u}\big(s, {m_n}^{1/q}\big) + M < 0, \\ \inf_{t \in [d'_{n-1}, m_n^{1/q}]} \frac{\partial H}{\partial v}\big(m_n^{1/p}, t\big) + M < 0. \end{aligned} \tag{3.47}$$

Then, from (3.47) we get $\forall n \in \mathbb{N}$, there exist $s_n \in [d_{n-1}, m_t^{1/p}]$ and $t_n \in [d'_{n-1}, m_n^{1/q}]$ satisfying

$$\frac{\partial H}{\partial u}\big(s_n, {m_n}^{1/q}\big) + M < 0, \qquad \frac{\partial H}{\partial v}\big({m_n}^{1/p}, t_n\big) + M < 0. \tag{3.48}$$

Consequently, the sequence $(\hat{u}_n, \hat{v}_n) = (s_n, t_n)$ is a sequence of supersolutions satisfying

$$\lim s_n = +\infty, \qquad \lim t_n = +\infty. \tag{3.49}$$

Case 2. Assume that for all $n \in \mathbb{N}$, we have

$$\inf_{s \in [d_{n-1}, {m_n}^{1/p}]} \frac{\partial H}{\partial u}(s, {m_n}^{1q}) + M \geq 0, \tag{3.50}$$

$$\inf_{t \in [d'_{n-1}, {m_n}^{1/q}]} \frac{\partial H}{\partial v}(m_n^{1/p}, t) + M < 0. \tag{3.51}$$

(a) From (3.50) and Proposition 3.4, there exist some $n_0 \in \mathbb{N}$ and some sequence $(\hat{u}_n)_n$ such that, for all $n \geq n_0$, we have

$$\begin{gathered} \hat{u}_n \in C^1([a,b]), \qquad \left(|\hat{u}'_n|^{p-2}\hat{u}'_n\right)' \in C([a,b]), \\ -\left(|\hat{u}'_n|^{p-2}\hat{u}'_n\right)' \geq \frac{\partial H}{\partial u}(\hat{u}_n, {m_n}^{1/q}) + M \quad \text{a.e. in } (a,b), \\ {m_n}^{1/p} \geq \hat{u}_n \geq d_{n-1} \quad \text{in } [a,b]. \end{gathered} \tag{3.52}$$

(b) From (3.51), there exists a sequence $(t_n)_{n \geq n_0}$ such that

$$ {m_n}^{1/p} \geq t_n \geq d'_{n-1}, \qquad \frac{\partial H}{\partial v}({m_n}^{1/p}, t_n) + M < 0. \tag{3.53}$$

Consequently, the sequence $(\hat{u}_n, t_n)_n$ satisfies the result.
Case 3. Assume that for all $n \in \mathbb{N}$,

$$\inf_{s \in [d_{n-1}, {m_n}^{1/p}]} \frac{\partial H}{\partial u}(s, {m_n}^{1/q}) + M \geq 0, \tag{3.54}$$

$$\inf_{t \in [d'_{n-1}, {m_n}^{1/q}]} \frac{\partial H}{\partial v}({m_n}^{1/p}, t) + M \geq 0. \tag{3.55}$$

Then from Proposition 3.4, for all $n \geq n_0$ there exists $(\hat{u}_n, \hat{v}_n) \in (C^1([a,b]))^2$ such that

$$\begin{gathered} \left(\left(|\hat{u}'_n|^{p-2}\hat{u}'_n\right)', \left(|\hat{v}'_n|^{p-2}\hat{v}'_n\right)'\right) \in \left(C[a,b]\right)^2, \\ -\left(|\hat{u}'_n|^{p-2}\hat{u}'_n\right)' \geq \frac{\partial H}{\partial u}(\hat{u}_n, {m_n}^{1/q}) + M \quad \text{a.e. in } (a,b), \\ -\left(|\hat{v}'_n|^{p-2}\hat{v}'_n\right)' \geq \frac{\partial H}{\partial v}({m_n}^{1/p}, \hat{v}_n) + M \quad \text{a.e. in } (a,b), \\ {m_n}^{1/p} \geq \hat{u}_n \geq 0, \quad {m_n}^{1/q} \geq \hat{v}_n \geq 0 \quad \text{on } [a,b], \end{gathered} \tag{3.56}$$

and the sequence $\{(\hat{u}_n, \hat{v}_n)\}_n$ satisfies

$$\max_{[a,b]} \hat{u}_n \le \min_{[a,b]} \hat{u}_{n+1} \longrightarrow +\infty, \qquad \max_{[a,b]} \hat{v}_n \le \min_{[a,b]} \hat{v}_{n+1} \longrightarrow +\infty. \tag{3.57}$$

This proves the results. □

Now, for problem (1.3) we consider a smooth bounded domain Ω in $\mathbb{R}^N$, we have the following result.

Proposition 3.6. *Under hypotheses (2.1), (2.3), and (2.5), problem (1.3) has a nonnegative sequence of supersolutions* $\{(u_n^0, v_n^0)\}$ *in* $W^{1,p}(\Omega) \times W^{1,q}(\Omega)$ *such that*

$$\begin{aligned} 0 < \max_{\overline{\Omega}} u_n^0 \le \min_{\overline{\Omega}} u_{n+1}^0 \longrightarrow +\infty, \\ 0 < \max_{\overline{\Omega}} v_n^0 \le \min_{\overline{\Omega}} v_{n+1}^0 \longrightarrow +\infty. \end{aligned} \tag{3.58}$$

Proof. Let $M \ge \|h_1\|_\infty + \|h_2\|_\infty$; $P = \prod [a_i, b_i]$ is a cube containing Ω and

$$b - a = \inf_{1 \le i \le N} b_i - a_i = b_1 - a_1. \tag{3.59}$$

From Proposition 3.5, there exist $(m_n)_n$ and $(\hat{u}_n, \hat{v}_n)$ in $W^{1,p}((a,b)) \times W^{1,q}((a,b))$ such that

$$\begin{aligned} -\left(\left|\hat{u}_n'\right|^{p-2} \hat{u}_n'\right)' &\ge \frac{\partial H}{\partial u}\left(\hat{u}_n, {m_n}^{1/q}\right) + M \quad \text{a.e. in } (a,b), \\ -\left(\left|\hat{v}_n'\right|^{q-2} \hat{v}_n'\right)' &\ge \frac{\partial H}{\partial v}\left({m_n}^{1/p}, \hat{v}_n\right) + M \quad \text{a.e. in } (a,b), \\ {m_n}^{1/p} \ge \hat{u}_n \ge 0, &\quad {m_n}^{1/q} \ge \hat{v}_n \ge 0 \quad \text{on } [a,b]. \end{aligned} \tag{3.60}$$

We denote by u_n^0 and v_n^0 the functions such that for all $x \in \Omega$ with $x = (x_1, x_2, \ldots, x_N)$,

$$u_n^0(x) = \hat{u}_n(x_1), \qquad v_n^0(x) = \hat{v}_n(x_1), \tag{3.61}$$

(u_n^0, v_n^0) is clearly in $W^{1,p}(\Omega) \times W^{1,q}(\Omega)$, moreover by (2.1), we obtain easily, for all $n \in \mathbb{N}$

$$\begin{aligned} -\Delta_p u_n^0 &\ge \frac{\partial H}{\partial u}\left(u_n^0, v\right) + h_1 \quad \text{for } v \le v_n^0 \text{ on } \Omega, \\ -\Delta_q v_n^0 &\ge \frac{\partial H}{\partial v}\left(u, v_n^0\right) + h_2 \quad \text{for } u \le u_n^0 \text{ on } \Omega, \\ u_n^0 \ge 0, &\quad v_n^0 \ge 0 \quad \text{on } \Omega. \end{aligned} \tag{3.62}$$

Thus the result follows. □

3.2. Construction of a sequence of subsolutions $\{(u_{0n}, v_{0n})\}_{n>1}$

Similar to the construction of a sequence of supersolutions we can prove the following proposition.

Proposition 3.7. *Under hypotheses (2.1), (2.3), and (2.6), problem (1.3) has a sequence of subsolutions* $(u_{0n}, v_{0n})_n$ *in* $W^{1,p}(\Omega)\times W^{1,q}(\Omega)$, *such that*

$$\begin{aligned} &0 \geq \min_{\overline{\Omega}} u_{0n} \geq \max_{\overline{\Omega}} u_{0n+1} \longrightarrow -\infty, \\ &0 \geq \min_{\overline{\Omega}} v_{0n} \geq \max_{\overline{\Omega}} v_{0n+1} \longrightarrow -\infty. \end{aligned} \tag{3.63}$$

4. Proof of Theorem 2.2

We closely follow an argument introduced in [11]. We define the functional

$$\Phi : {W_0}^{1,p} \times {W_0}^{1,q} \longrightarrow \mathbb{R} \tag{4.1}$$

by setting

$$\Phi(u,v) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{q}\int_\Omega |\nabla u|^q\,dx - \int_\Omega H(u,v)dx. \tag{4.2}$$

Claim 4.1. *Let a lower solution* (u_0,v_0) *and an upper solution* (u^0,v^0) *of problem (1.3) satisfy* $u_0 \leq u^0$ *and* $v_0 \leq v^0$ *in* Ω. *Then, problem (1.3) has a solution* (u,v) *belonging to* $C^{1,\sigma}$, *for some* $\sigma > 0$, *such that*

$$\begin{aligned} &u_0 \leq u \leq u^0, \qquad v_0 \leq v \leq v^0, \\ &\Phi(u,v) = \min_{(w_1,w_2)\in K} \Phi\big(w_1,w_2\big), \end{aligned} \tag{4.3}$$

with

$$K = \left[u_0,u^0\right] \times \left[v_0,v^0\right] \subset {W_0}^{1,p} \times {W_0}^{1,q}. \tag{4.4}$$

Proof. We argue as in [10]. By minimization of the functional associated with truncated system (1.3). The validity of a weak comparison principle (see [11]) gives the regularity of solutions. Consider the following problem:

$$\begin{aligned} -\Delta_p u = \frac{\partial \overline{H}}{\partial u}(x,u,v), \quad -\Delta_q v = \frac{\partial \overline{H}}{\partial v}(x,u,v) \quad \text{in } \Omega, \\ u = 0, \quad v = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} \frac{\partial \overline{H}}{\partial u}(x,u,v) = \frac{\partial H}{\partial u}(U,V) + h_1(x), \\ \frac{\partial \overline{H}}{\partial v}(x,u,v) = \frac{\partial H}{\partial v}(U,V) + h_2(x), \end{aligned} \tag{4.6}$$

with

$$U(x) = u(x) + (u_0 - u)_+ - (u - u^0)_+,$$
$$V(x) = v(x) + (v_0 - v)_+ - (v - v^0)_+. \tag{4.7}$$

Minimization of the functional $\overline{\phi}$ associated to (4.5)

Denote by $\overline{\phi}$ the functional associated to (4.5)

$$\overline{\phi}(u,v) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{q}\int_\Omega |\nabla u|^q\,dx - \int_\Omega \overline{H}(x,u,v)\,dx. \tag{4.8}$$

It is easy to show that there exist some constants $M_1 > 0$ and $M_2 > 0$ such that

$$\left|\overline{H}(x,u,v)\right| \le M_1 + M_2\left[|u| + |v|\right]. \tag{4.9}$$

Hence, the functional $\overline{\phi}$ is weakly lower semicontinuous. It follows from a standard theorem in the calculus of variations (see Vainberg [9]) that $\overline{\phi}$ attains its minimum at $(\overline{u},\overline{v})$ solution of problem (4.5), that is,

$$\min_{(w_1,w_2)\in W_0{}^{1,p}\times W_0{}^{1,q}} \overline{\phi}(w_1,w_2) = \overline{\phi}(\overline{u},\overline{v}). \tag{4.10}$$

Weak comparison principle

We show, for example, that $\overline{u} \le u^0$. From (4.7), we denote by $\overline{U}$ and $\overline{V}$ the functions associated to $\overline{u}$ and $\overline{v}$. Then we have

$$0 \ge -\Delta_p\overline{u} - \frac{\partial\overline{H}}{\partial u}(x,\overline{u},\overline{v}) \ge \Delta_p\overline{u} - \frac{\partial H}{\partial u}(\overline{U},\overline{V}) - h_1(x)$$
$$\ge \left[-\Delta_p\overline{u} + \Delta_p u^0\right] + \left[\frac{\partial H}{\partial u}(u^0,\overline{V}) - \frac{\partial H}{\partial u}(\overline{U},\overline{V})\right], \tag{4.11}$$

multiplying (4.11) by $(\overline{u} - u^0)_+$ and integrating over Ω, we obtain

$$0 \ge \int_\Omega \left[|\nabla\overline{u}|^{p-2}\nabla\overline{u} - \left|\nabla u^0\right|^{p-2}\nabla u^0\right]\nabla\left(\overline{u} - u^0\right)_+ dx$$
$$+ \int_\Omega \left[\frac{\partial H}{\partial u}(u^0,\overline{V}) - \frac{\partial H}{\partial u}(\overline{U},\overline{V})\right]\left(\overline{u} - u^0\right)_+ dx. \tag{4.12}$$

Denote by Ω_+ the set

$$\Omega_+ = \left\{x \in \Omega;\ \overline{u} - u^0 > 0\right\}. \tag{4.13}$$

We have $\overline{U} = u^0$ in Ω_+. Then

$$\begin{aligned}\int_\Omega \left[\frac{\partial H}{\partial u}(u^0,\overline{V}) - \frac{\partial H}{\partial u}(\overline{U},\overline{V})\right](\overline{u}-u^0)_+\,dx \\ = \int_\Omega \left[\frac{\partial H}{\partial u}(u^0,\overline{V}) - \frac{\partial H}{\partial u}(u^0,\overline{V})\right](\overline{u}-u^0)_+\,dx = 0.\end{aligned} \tag{4.14}$$

By the monotonicity of $-\Delta_p$ in $L^p(\Omega)$ we get that $0 \geq \|(\overline{u}-u^0)_+\|_{L^p(\Omega)}$.

Thus $\overline{u} \leq u^0$ on Ω and similarly $\overline{v} \leq v^0$ on Ω. Then, we conclude that $u_0 \leq \overline{u} \leq u^0$ and $v_0 \leq \overline{v} \leq v^0$. Consequently, we obtain

$$\overline{\phi}(\overline{u},\overline{v}) = \phi(\overline{u},\overline{v}) = \min_{(w_1,w_2)\in K} \phi(w_1,w_2). \tag{4.15}$$

This ends the proof of Claim 4.1. □

Proof of Theorem 2.2. We are in position to build a sequence $\{(\overline{u}_n,\overline{v}_n)\}_n$ of solutions of (1.3) such that

$$\max\left(\sup_\Omega \overline{u}_n; \sup_\Omega \overline{v}_n\right) \longrightarrow +\infty. \tag{4.16}$$

Take an upper solution (u_1^0,v_1^0) and a lower solution (u_0,v_0) of (1.3). We get a solution $(\overline{u}_1,\overline{v}_1)$ in $C^{1,\sigma}(\overline{\Omega})$, for some $\sigma > 0$, of (1.3), with

$$\begin{aligned}(\overline{u}_1,\overline{v}_1) &\in [u_0,u_1^0]\times[v_0,v_1^0] = K_1,\\ \phi(\overline{u}_1,\overline{v}_1) &= \min_{(w_1,w_2)\in K_1} \phi(w_1,w_2).\end{aligned} \tag{4.17}$$

Step 1. Let $(\varphi,\psi) \in {W_0}^{1,p} \times {W_0}^{1,q}$ be positive in Ω, such that $\varphi = 1$ and $\psi = 1$ on $\Omega_0 \subset_{\neq} \Omega$, $\varphi = \psi = 0$, $\partial\varphi/\partial\nu < 0$, and $\partial\psi/\partial\nu < 0$, where ν is the outer normal to $\partial\Omega$. Moreover, from (2.5) there exists some positive sequence (s_n) such that

$$\lim_{n\to+\infty} \frac{H({s_n}^{1/p},{s_n}^{1/q})}{s_n} = +\infty. \tag{4.18}$$

Consequently, from (2.3), (4.18), and the definitions of φ and ψ we have

$$\lim_{n\to+\infty} \Phi({s_n}^{1/p}\varphi,{s_n}^{1/q}\psi) = -\infty \tag{4.19}$$

with

$$\Phi({s_n}^{1/p}\varphi,{s_n}^{1/q}\psi) = \frac{s_n}{p}\|\varphi\|_{1,p}^p + \frac{s_n}{q}\|\psi\|_{1,q}^q - \int_\Omega H({s_n}^{1/p}\varphi,{s_n}^{1/q}\psi). \tag{4.20}$$

Step 2. Select a number, say s_1, such that

$$\begin{aligned} &\overline{u}_1 \le {s_1}^{1/p}\varphi, \qquad \overline{v}_1 \le {s_1}^{1/q}\psi, \\ &\Phi\big({s_1}^{1/p}\varphi, {s_1}^{1/q}\psi\big) < \Phi\big(\overline{u}_1, \overline{v}_1\big). \end{aligned} \tag{4.21}$$

Now, take an upper solution (u_2^0, v_2^0) such that $u_2^0 \ge {s_1}^{1/p}\varphi$ and $v_2^0 \ge {s_1}^{1/q}\psi$ in Ω. We find a solution $(\overline{u}_2, \overline{v}_2)$ in $[\overline{u}_1, u_2^0] \times [\overline{v}_1, v_2^0] = K_2$ and

$$\Phi\big(\overline{u}_2, \overline{v}_2\big) = \min_{(w_1, w_2) \in K_2} \phi\big(w_1, w_2\big). \tag{4.22}$$

Thus, since

$$\Phi\big(\overline{u}_2, \overline{v}_2\big) \le \Phi\big({s_1}^{1/p}\varphi, {s_1}^{1/q}\psi\big) < \Phi\big(u_1, v_1\big), \tag{4.23}$$

we conclude that $(\overline{u}_2, \overline{v}_2) \ne (\overline{u}_1, \overline{v}_1)$,

$$\max\Big(\max_{\overline{\Omega}} \overline{u}_2, \max_{\overline{\Omega}} \overline{v}_2\Big) \ge \min\Big(\min_{\overline{\Omega}} u_1^0, \min_{\overline{\Omega}} v_1^0\Big). \tag{4.24}$$

Iterating this argument, we construct the required sequence of solutions of problem (1.3) such that

$$\max\Big(\max_{\overline{\Omega}} \overline{u}_n, \max_{\overline{\Omega}} \overline{v}_n\Big) \longrightarrow +\infty. \tag{4.25}$$

In completely similar way we construct a sequence $\{(\underline{u}_n, \underline{v}_n)\}_n$ of solutions of problem (1.3) satisfying

$$\min\Big(\inf_{\Omega} \underline{u}_n; \inf_{\Omega} \underline{v}_n\Big) \longrightarrow -\infty. \tag{4.26}$$

Hence, Theorem 2.2 is proved. □

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