# PERIODIC BOUNDARY VALUE PROBLEMS FOR $n$TH-ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH $p$-LAPLACIAN

YUJI LIU AND WEIGAO GE

*Received 26 April 2004 and in revised form 5 September 2004*

We prove existence results for solutions of periodic boundary value problems concerning the $n$th-order differential equation with $p$-Laplacian $[\phi(x^{(n-1)}(t))]' = f(t,x(t),x'(t),\dots,x^{(n-1)}(t))$ and the boundary value conditions $x^{(i)}(0)=x^{(i)}(T)$, $i=0,\dots,n-1$. Our method is based upon the coincidence degree theory of Mawhin. It is interesting that $f$ may be a polynomial and the degree of some variables among $x_0,x_1,\dots,x_{n-1}$ in the function $f(t,x_0,x_1,\dots,x_{n-1})$ is allowed to be greater than 1.

## 1. Introduction

In this paper, we investigate the existence of solutions of the periodic boundary value problem for $n$th-order differential equation with $p$-Laplacian

$$\left[\phi\left(x^{(n-1)}\right)\right]' = f\left(t,x(t),x'(t),\dots,x^{(n-1)}(t)\right), \quad t\in(0,T), \tag{1.1}$$

subject to the following periodic boundary conditions:

$$x^{(i)}(0) = x^{(i)}(T), \quad i=0,1,\dots,n-1, \tag{1.2}$$

where $f:[0,1]\times\mathbb{R}^n\to\mathbb{R}$ is a continuous function, $n\geq 2$ an integer, $p>1$ a constant, and $\phi(x)=|x|^{p-2}x$ for $x\neq 0$ and $\phi(0)=0$, which is called $p$-Laplacian, whose inverse is denoted by $\phi^{-1}(x)=|x|^{q-2}x$, where $q$ satisfies $1/p+1/q=1$.

Our purpose here is to provide sufficient conditions for the existence of solutions of the periodic boundary value problem (1.1) and (1.2). This will be done by applying the well-known coincidence degree theory.

The motivation for this paper is as follows. There were many papers concerned with the solvability of the periodic boundary value problems for second-order differential equations or higher-order differential equations

$$\begin{gathered} x''(t) + f\left(t,x(t),x'(t)\right) = 0, \quad t\in(0,T), \\ x(0)=x(T), \qquad x'(0)=x'(T), \end{gathered} \tag{1.3}$$


Journal of Applied Mathematics 2005:1 (2005) 1–21
DOI: 10.1155/JAM.2005.1

or

$$
\begin{gathered}
x^{(n)}(t)=f\left(t,x(t),\dots,x^{(n-1)}(t)\right),\quad t\in(0,T),\\
x^{(i)}(0)=x^{(i)}(T),\quad i=0,\dots,n-1.
\end{gathered}
\tag{1.4}
$$

We refer the readers to [7, 13, 14, 16] and the references therein. If $n$ is even, problem (1.1)-(1.2) can be reduced to a system of $n/2$ second-order periodic problems with the last equation containing a $p$-Laplacian. Manásevich and Mawhin studied a similar problem in [12]. They established existence results for periodic solutions.

In [15], the existence of $T$-periodic solutions of the equation

$$
x^{(n)}(t)=f\left(t,x(t),x'(t),\dots,x^{(n-1)}(t)\right)
\tag{1.5}
$$

was studied, where $f$ is continuous and $f(*,x_0,\dots,x_{n-1})$ is $T$-periodic with $T>0$. The authors proved that (1.5) has at least one periodic solution if some conditions imposed on $f$ are satisfied. The main results are as follows.

(i) Let $n=2m$ and the inequalities

$$
\begin{aligned}
&p_*(t)\,|x_1|-\delta\left(t,\sum_{i=1}^{n}|x_i|\right)\\
&\quad\le(-1)^m f(t,x_1,\dots,x_n)\operatorname{sgn}x_1\le p^*(t)\,|x_1|+\delta\left(t,\sum_{i=1}^{n}|x_i|\right)
\end{aligned}
\tag{1.6}
$$

be valid on $\mathbb{R}\times\mathbb{R}^n$, where $P_*(t),p^*(t)\ge 0(\not\equiv 0)$ and $\delta(t,x)$ are $T$-periodic in $t$. If

$$
\begin{gathered}
\int_0^T p^*(t)dt\le\frac{2}{T}\left(\frac{2\pi}{T}\right)^{n-2},\\
p^*(t)\le\left(\frac{2\pi}{T}\right)^{n},\\
\lim_{x\to+\infty}\frac{1}{x}\int_0^T\delta(t,x)dt=0,
\end{gathered}
\tag{1.7}
$$

then (1.5) has at least one $T$-periodic solution.

(ii) Let the inequalities

$$
\begin{aligned}
&p_*(t)\,|x_1|-\delta\left(t,\sum_{i=1}^{n}|x_i|\right)\\
&\quad\le\sigma f(t,x_1,\dots,x_n)\operatorname{sgn}x_1\le p^*(t)\,|x_1|+\delta\left(t,\sum_{i=1}^{n}|x_i|\right)
\end{aligned}
\tag{1.8}
$$

be valid on $\mathbb{R}\times\mathbb{R}^n$, where $P_*(t),p^*(t)\ge 0(\not\equiv 0)$ and $\delta(t,x)$ are $T$-periodic in $t$. In addition, let either $n=2m-1$ or $n=2m$ and $\sigma=(-1)^{m-1}$. Then, (1.5) has at least one $T$-periodic solution.

In a recent paper [6], the author proved the existence of solutions of the following problem:

$$x^{(n)} + p_{n-1}(t)x^{(n-1)} + \cdots + p_1(t)x' + p_0(t)x = e(t),\\ x^{(i)}(0) = x^{(i)}(T), \quad i = 0,\dots,n-1, \tag{1.9}$$

where $p_i : [0,T] \to \mathbb{R}$ is continuous and $e \in C^0[0,T]$. He proved that if $\int_0^t p_0(t)dt \neq 0$ and

$$\eta_0(p_1)l_1\int_0^T |p_1(t)|dt + (1+\eta_0(p_1))\sum_{l=1}^{n-1}\int_0^T |p_l(t)|dt < 1, \tag{1.10}$$

then (1.9) has at least one solutions, where

$$l_k = \frac{T}{\sqrt{2}}\left(\frac{T}{2\pi}\right)^{n-k-1}, \quad k = 1,\dots,n-1, \qquad l_n = 1,\\ \eta_0(p) = \frac{\int_0^T |p(t)|dt}{\left|\int_0^T p(t)dt\right|} \quad \text{for } \int_0^t p(t)dt \neq 0. \tag{1.11}$$

The solvability of multipoint BVPs of $p$-Laplacian differential equations were studied by several authors, we refer the readers to [2, 4, 5, 8, 9, 10, 11]. In addition, in [1], Cabada and Pouso studied the existence of solutions of the following problem:

$$[\phi(u')]' = f(t,u,u'), \quad t \in [a,b],\\ 0 = g(u(a),u'(a),u'(b)),\\ u(b) = h(u(a)). \tag{1.12}$$

Using the methods of lower and upper solutions and Nagumo conditions, they obtained existence results for solutions of the above problem.

To the best of our knowledge, the existence of solutions of periodic boundary value problems for higher-order differential equations with $p$-Laplacian has not been well studied till now.

In this paper, we will establish some sufficient conditions for the existence of periodic solutions of problem (1.1) and (1.2) in Section 2. Our methods and results are different from the already known ones [6, 7, 14, 15, 16].

## 2. Main results

In this section, we establish sufficient conditions for the existence of at least one solution of BVP (1.1)-(1.2). For convenience, we first introduce some notations and an abstract existence theorem by Gaines and Mawhin [3]. Recently, this theorem has been reported to be more successful in solving multipoint BVPs for differential equations, see [2, 4, 5, 8, 9, 10, 11].

Let $X$ and $Y$ be Banach spaces, $L : \operatorname{dom} L \subset X \to Y$ a Fredholm operator of index zero, $P : X \to X$, $Q : Y \to Y$ projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L, \qquad X = \operatorname{Ker} L \oplus \operatorname{Ker} P, \qquad Y = \operatorname{Im} L \oplus \operatorname{Im} Q. \tag{2.1}$$

It follows that

$$L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \longrightarrow \operatorname{Im} L \tag{2.2}$$

is invertible, we denote the inverse of that map by $K_p$.

If $\Omega$ is an open bounded subset of $X$, $\operatorname{dom} L \cap \overline{\Omega} \neq \emptyset$, the map $N : X \to Y$ will be called $L$-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N : \overline{\Omega} \to X$ is compact.

Theorem 2.1 (see [3]). *Let $L$ be a Fredholm operator of index zero and let $N$ be $L$-compact on $\Omega$. Assume that the following conditions are satisfied:*

(i) *$Lx \neq \lambda Nx$ for every $(x,\lambda) \in [(\operatorname{dom} L/\operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$;*
(ii) *$Nx \notin \operatorname{Im} L$ for every $x \in \operatorname{Ker} L \cap \partial\Omega$;*
(iii) *$\deg(\Lambda QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \neq 0$, where $\Lambda : Y/\operatorname{Im} L \to \operatorname{Ker} L$ is an isomorphism.*

*Then, the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.*

We use the classical Banach space $C^k[0,T]$, let $X = C^{n-2}[0,T] \times C^0[0,T]$ and $Y = C^0[0,T] \times C^0[0,T]$. $Y$ is endowed with the norm $\|y\| = \max\{\|y_1\|_\infty, \|y_2\|_\infty\}$, where $\|y_i\|_\infty = \max_{t\in[0,T]} |y_i(t)|$, $X$ is endowed with the norm

$$\|x\| = \max\left\{\|x_1\|_\infty, \|x_1'\|_\infty, \dots, \left\|x_1^{(n-2)}\right\|_\infty, \|x_2\|_\infty\right\}. \tag{2.3}$$

Then, $X$ and $Y$ are Banach spaces. Let

$$\begin{aligned} \operatorname{dom} L = \Big\{&(x_1,x_2) \in C^{n-1}[0,T] \times C^1[0,T] : \\ &x_1^{(i)}(0) = x^{(i)}(T) \text{ for } i = 0,\dots,n-2,\ x_2(0) = x_2(T)\Big\}. \end{aligned} \tag{2.4}$$

Define the linear operator $L$ and the nonlinear operator $N$ by

$$\begin{aligned} &L : X \cap \operatorname{dom} L \longrightarrow Y, \qquad L\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} x_1^{(n-1)}(t) \\ x_2'(t) \end{pmatrix} \quad \text{for } x \in X \cap \operatorname{dom} L, \\ &N : X \longrightarrow Y, \qquad N\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \phi_q(x_2(t)) \\ f\left(t, x_1(t), x_1'(t), \dots, x_1^{(n-2)}(t), \phi_q(x_2(t))\right) \end{pmatrix} \end{aligned} \tag{2.5}$$

for $x \in X$, respectively.

Lemma 2.2. *The following results hold:*

(i) $\operatorname{Ker} L = \{(x_1(t), x_2(t)) = (a,b),\ t \in [0,T],\ a,b \in \mathbb{R}\}$;
(ii) $\operatorname{Im} L = \{(y_1(t), y_2(t)) \in Y,\ \int_0^T y_1(u)\,du = 0 = \int_0^T y_2(t)\,dt\}$;
(iii) *$L$ is a Fredholm operator of index zero;*

(iv) *there are projectors* $P : X \to X$ *and* $Q : Y \to Y$ *such that* $\operatorname{Ker} L = \operatorname{Im} P$ *and* $\operatorname{Ker} Q = \operatorname{Im} L$. *Furthermore, let* $\Omega \subset X$ *be an open bounded subset with* $\overline{\Omega} \cap \operatorname{dom} L \neq \emptyset$*, then* $N$ *is L-compact on* $\overline{\Omega}$;

(v) $x(t)$ *is a solution of BVP (1.1)-(1.2) if and only if* $x$ *is a solution of the operator equation* $Lx = Nx$ *in* $\operatorname{dom} L$.

*Proof.* The proofs are similar to those of lemmas in [2, 9, 8, 11, 10] and are omitted. For $y_1 \in C^0[0,1]$, let $x_1^{(n-1)}(t) = y_1(t)$. We get

$$x_1^{(n-2)}(t) = a_{n-2} + \int_0^t y_1(s)ds, \qquad x_1^{(n-3)}(t) = a_{n-2}t + a_{n-3} + \int_0^t (t-s)y_1(s)ds. \tag{2.6}$$

It follows from $x_1^{(n-3)}(0) = x_1^{(n-3)}(T)$ that

$$a_{n-2} = -\frac{1}{T}\left(\int_0^T (T-s)y_1(s)ds\right). \tag{2.7}$$

Similar to the above argument, we get

$$a_{n-3} = -\frac{1}{T}\left(\int_0^T \frac{(T-s)^2}{2!}y_1(s)ds + \frac{a_{n-2}}{2!}T^2\right). \tag{2.8}$$

So, let

$$\begin{aligned} a_{n-2} &= -\frac{1}{T}\left(\int_0^T (T-s)y_1(s)ds\right), \\ a_{n-3} &= -\frac{1}{T}\left(\int_0^T \frac{(T-s)^2}{2!}y_1(s)ds + \frac{a_{n-2}}{2!}T^2\right), \\ &\vdots \\ a_1 &= -\frac{1}{T}\left(\int_0^T \frac{(T-s)^{n-3}}{(n-3)!}y_1(s)ds + \sum_{i=2}^{n-2}\frac{a_i}{i!}T^i\right). \end{aligned} \tag{2.9}$$

We list $P$, $Q$, and the generalized inverse $K_p : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Im} P$:

$$\begin{aligned} P(x_1(t),x_2(t)) &= (x_1(0),x_2(0)) \quad \text{for } (x_1,x_2) \in X, \\ Q(y_1(t),y_2(t)) &= \left(\frac{1}{T}\int_0^T y_1(s)ds, \frac{1}{T}\int_0^T y_2(s)ds\right) \quad \text{for } (y_1,y_2) \in Y, \\ K_p(y_1(t),y_2(t)) &= \left(\int_0^t (t-s)^{n-2}y_1(s)ds + \sum_{i=1}^{n-2} a_i t^i, \int_0^t y_2(s)ds\right) \quad \text{for } (y_1,y_2) \in Y. \end{aligned} \tag{2.10}$$

□

Theorem 2.3. *Suppose the following conditions hold.*

($A_1$) *There are continuous functions $e(t)$ and nonnegative functions $g_i(t,x)$ $(i=0,1,\ldots,n-1)$ such that $f$ satisfies*

$$|f(t,x_0,x_1,\ldots,x_{n-1})| \leq |e(t)| + \sum_{i=0}^{n-1} |g_i(t,x_i)| \tag{2.11}$$

*for all $t\in[0,T]$ and $(x_0,x_1,\ldots,x_{n-1})\in\mathbb{R}^n$ and*

$$\lim_{x\to\infty}\sup_{t\in[0,T]} \frac{|g_i(t,x)|}{\phi(|x|)} = r_i \in [0,\infty) \quad \text{for } i=0,\ldots,n-1. \tag{2.12}$$

($A_2$) *There is a constant $M>0$ such that if $x_1\in C^{n-2}[0,T]$ and $x_2\in C^0[0,T]$ with $|x_1(t)|>M$ for all $t\in[0,T]$ and $\int_0^T \phi^{-1}(x_2(s))ds=0$, then*

$$\int_0^T f\left(s,x_1(s),\ldots,x_1^{(n-2)}(s),\phi^{-1}(x_2(s))\right)ds \neq 0. \tag{2.13}$$

($A_3$) *There is a constant $M^*>0$ so that for all $a,b\in\mathbb{R}$ either*

$$a\phi^{-1}(b) + b\int_0^T f(u,a,0,\ldots,0,\phi^{-1}(b))du > 0 \tag{2.14}$$

*for all $|a|>M^*$ or $|a|\leq M^*$ and $|b|>M^*$, or*

$$a\phi^{-1}(b) + b\int_0^T f(u,a,0,\ldots,0,\phi^{-1}(b))du < 0 \tag{2.15}$$

*for all $|a|>M^*$ or $|a|\leq M^*$ and $|b|>M^*$.*

*Then, BVP (1.1)-(1.2) has at least one solution, provided*

$$r_0T\phi(T^{n-1}) + \sum_{i=1}^{n-2} r_iT\phi(T^{n-i-1}) + r_{n-1}T < 1. \tag{2.16}$$

*Proof.* To apply Theorem 2.1, we should define an open bounded subset $\Omega$ of $X$ so that (i), (ii), and (iii) of Theorem 2.1 hold. It is based upon three steps to obtain $\Omega$. The proof of this theorem is divided into four steps.

*Step 1.* Let

$$\Omega_1 = \{x=(x_1,x_2)\in \operatorname{dom} L/\operatorname{Ker} L,\ Lx=\lambda Nx \text{ for some } \lambda\in(0,1)\}. \tag{2.17}$$

We prove that $\Omega_1$ is bounded.

For $x \in \Omega_1$, it is easy to show that there is $\xi_i \in [0,T]$ such that $x_1^{(i)}(\xi_i) = 0$ for $i = 1,2,\dots,n-1$ and thus $x_2(\xi_{n-1}) = 0$. Hence, for $i = 1,\dots,n-2$, we get, for $t \geq \xi_i$, that

$$\left|x_1^{(i)}(t)\right| = \left|x_1^{(i)}(\xi_i) + \int_{\xi_i}^{t} x_1^{(i+1)}(s)ds\right| \leq \int_0^T \left|x_1^{(i+1)}(s)\right| ds. \tag{2.18}$$

For $t < \xi_i$, similar to the above discussion, we get

$$\left|x_1^{(i)}(t)\right| \leq \int_0^T \left|x_1^{(i+1)}(s)\right| ds. \tag{2.19}$$

So,

$$\left|x_1^{(i)}(t)\right| \leq T^{n-2-i}\int_0^T \left|x_1^{(n-1)}(s)\right| ds \leq T^{n-i-1}\left|\left|x_1^{(n-1)}\right|\right|_\infty \leq T^{n-i-1}\phi^{-1}(||x_2||_\infty). \tag{2.20}$$

Furthermore,

$$\left|x_1^{(i)}(t)\right| \leq T^{n-i-2}\int_0^T \left|x_1^{(n-1)}(s)\right| ds \quad \text{for } i = 1,\dots,n-2. \tag{2.21}$$

On the other hand, $L(x_1,x_2) = \lambda N(x_1,x_2) \in \operatorname{Im} L$ implies that

$$\int_0^T \phi^{-1}(x_2(s))ds = 0, \qquad \int_0^T f\left(s,x_1(s),\dots,x_1^{(n-2)}(s),\phi^{-1}(x_2(s))\right)ds = 0. \tag{2.22}$$

It follows from $(A_2)$ that there is $t_0 \in [0,T]$ so that $|x_1(t_0)| \leq M$. Hence, we can get

$$|x_1(t)| \leq M + \int_0^T |x_1'(s)|ds \leq M + T^{n-1}\left|\left|x_1^{(n-1)}\right|\right|_\infty \leq M + T^{n-1}\phi^{-1}(||x_2||_\infty). \tag{2.23}$$

It suffices to prove that there is a constant $B > 0$ such that

$$||(x_1,x_2)|| = \max\left\{||x_1||_\infty, ||x_1'||_\infty,\dots,\left|\left|x_1^{(n-2)}\right|\right|_\infty, ||x_2||_\infty\right\} \leq B. \tag{2.24}$$

For $x \in \Omega_1$, we have

$$\begin{gathered} x_1^{(n-1)}(t) = \lambda\phi^{-1}(x_2(t)), \\ x_2'(t) = \lambda f\left(t,x_1(t),x_1'(t),\dots,x_1^{(n-2)}(t),\phi^{-1}(x_2(t))\right). \end{gathered} \tag{2.25}$$

Integrating the second equation in (2.25) from $\xi_{n-1}$ to $t$, we get, using $(A_1)$, that

$$\begin{aligned}
|x_2(t)| &= \left| x_2(\xi_{n-1}) + \int_{\xi_{n-1}}^{t} \lambda f\left(s, x_1(s), x_1'(s), \dots, x_1^{(n-2)}(s), \phi^{-1}(x_2(s))\right) ds \right| \\
&\le \int_0^T \left| f\left(s, x_1(s), x_1'(s), \dots, x_1^{(n-2)}(s), \phi^{-1}(x_2(s))\right) \right| ds \\
&\le \int_0^T |e(s)|\, ds + \sum_{i=0}^{n-2} \int_0^T \left| g_i\left(s, x_1^{(i)}(s)\right) \right| ds + \int_0^T |g_{n-1}(s, \phi^{-1}(x_2(s)))|\, ds.
\end{aligned} \tag{2.26}$$

Integrating the first equation in (2.25) from $\xi_{n-2}$ to $t$, we get, similar to the above argument, that

$$\left| x_1^{(n-2)}(t) \right| \le \left| x^{(n-2)}(\xi_{n-2}) \right| + \int_0^T |\phi^{-1}(x_2(s))|\, ds \le T\phi^{-1}(||x_2||_\infty). \tag{2.27}$$

Then,

$$\left|\left| x_1^{(n-2)} \right|\right|_\infty \le T\phi^{-1}(||x_2||_\infty). \tag{2.28}$$

Let $\epsilon > 0$ satisfy, using (2.16),

$$1 - (r_0 + \epsilon)\phi\left(1 + \frac{M\epsilon}{T^{n-1}}\right) T\phi(T^{n-1}) - \sum_{i=1}^{n-2} (r_i + \epsilon) T\phi(T^{n-i-1}) - (r_{n-1} + \epsilon) T > 0. \tag{2.29}$$

For such $\epsilon > 0$, we find from the third part of $(A_1)$ that there is a constant $\delta > M$ such that for every $i = 0, 1, \dots, n-2$,

$$\begin{aligned}
|g_i(t,x)| &< (r_i + \epsilon)\phi(|x|) \quad \text{uniformly for } t \in [0,T],\ |x| > \delta, \\
|g_{n-1}(t, \phi^{-1}(x))| &\le (r_{n-1} + \epsilon)|x| \quad \text{uniformly for } t \in [0,T],\ \phi(|x|) > \delta.
\end{aligned} \tag{2.30}$$

Let

$$\begin{aligned}
\Delta_{1,i} &= \left\{ t : t \in [0,T],\ \left| x^{(i)}(t) \right| \le \delta \right\}, \quad i = 0, 1, \dots, n-2, \\
\Delta_{2,i} &= \left\{ t : t \in [0,T],\ \left| x^{(i)}(t) \right| > \delta \right\}, \quad i = 0, 1, \dots, n-2, \\
g_{\delta,i} &= \max_{t \in [0,T], |x| \le \delta} |g_i(t,x)|, \quad i = 0, 1, \dots, n-2, \\
\Delta_{1,n-1} &= \{ t : t \in [0,T],\ \phi(|x_2(t)|) \le \delta \}, \\
\Delta_{2,n-1} &= \{ t : t \in [0,T],\ \phi(|x_2(t)|) > \delta \}, \\
g_{\delta,n-1} &= \max_{t \in [0,T], |x| \le \delta} |g_i(t, \phi(|x_2|))|.
\end{aligned} \tag{2.31}$$

So,

$$\begin{aligned}
|x_2(t)| &= \int_0^T |e(s)|\,ds + \sum_{i=0}^{n-2}\int_0^T \left|g_i\left(s,x_1^{(i)}(s)\right)\right| ds + \int_0^T |g_{n-1}(s,\phi^{-1}(x_2(s)))|\,ds \\
&\le \int_0^T |e(s)|\,ds + \sum_{i=0}^{n-2}\int_{\Delta_{1,i}} \left|g_i\left(s,x_1^{(i)}(s)\right)\right| ds + \sum_{i=0}^{n-2}\int_{\Delta_{2,i}} \left|g_i\left(s,x_1^{(i)}(s)\right)\right| ds \\
&\quad + \int_{\Delta_{1,n-1}} |g_{n-1}(s,\phi^{-1}(x_2(s)))|\,ds + \int_{\Delta_{1,n-1}} |g_{n-1}(s,\phi^{-1}(x_2(s)))|\,ds \\
&\le \int_0^T |e(s)|\,ds + T\sum_{i=0}^{n-1} g_{\delta,i} + \sum_{i=0}^{n-2}(r_i+\epsilon)\int_0^T \phi\left(\left|x_1^{(i)}(s)\right|\right)ds + (r_{n-1}+\epsilon)\int_0^T |x_2(s)|\,ds \\
&\le \int_0^T |e(s)|\,ds + T\sum_{i=0}^{n-1} g_{\delta,i} + (r_0+\epsilon)T\phi[M+T^{n-1}\phi^{-1}(||x_2||_\infty)] \\
&\quad + \sum_{i=1}^{n-2}(r_i+\epsilon)T\phi(T^{n-i-1}\phi^{-1}(||x_2||_\infty)) + (r_{n-1}+\epsilon)T||x_2||_\infty \\
&\le \int_0^T |e(s)|\,ds + T\sum_{i=0}^{n-1} g_{\delta,i} + (r_0+\epsilon)T\phi(T^{n-1}\phi^{-1}(||x_2||_\infty))\phi\left(\frac{M}{T^{n-1}\phi^{-1}(||x_2||_\infty)}+1\right) \\
&\quad + \sum_{i=1}^{n-2}(r_i+\epsilon)T\phi(T^{n-i-1}\phi(T)||x_2||_\infty) + (r_{n-1}+\epsilon)T||(x_2(s))||_\infty.
\end{aligned} \tag{2.32}$$

Without loss of generality, suppose $\|x_2\|_\infty > 1/\epsilon$. Hence,

$$\begin{aligned}
||x_2||_\infty \le{}& \int_0^T |e(s)|\,ds + T\sum_{i=0}^{n-1} g_{\delta,i} + (r_0+\epsilon)\phi\left(1+\frac{M\epsilon}{T^{n-1}}\right)T\phi(T^{n-1})||x_2||_\infty \\
&+ \sum_{i=1}^{n-2}(r_i+\epsilon)T\phi(T^{n-i-1})||x_2||_\infty + (r_{n-1}+\epsilon)T||(x_2(s))||_\infty.
\end{aligned} \tag{2.33}$$

We get

$$\begin{aligned}
&\left(1-(r_0+\epsilon)\phi\left(1+\frac{M\epsilon}{T^{n-1}}\right)T\phi(T^{n-1}) - \sum_{i=1}^{n-2}(r_i+\epsilon)T\phi(T^{n-i-1}) - (r_{n-1}+\epsilon)T\right)||(x_2(s))||_\infty \\
&\quad \le \int_0^T |e(s)|\,ds + T\sum_{i=0}^{n-1} g_{\delta,i}.
\end{aligned} \tag{2.34}$$

By the definition of $\epsilon$, we get that there is constant $A_{n-1} > 0$ so that $\|x_2\|_\infty \le A_{n-1}$.

Now, we see that

$$\begin{gathered}
\left\|x_1^{(i)}\right\|_\infty \le T^{n-i-1}\phi^{-1}(||x_2||_\infty) \le T^{n-i-1}\phi^{-1}(A_{n-1}) \quad \text{for } i = 1,\dots,n-2, \\
||x_1||_\infty \le M + T^{n-1}\phi^{-1}(||x_2||_\infty) \le M + T^{n-1}\phi^{-1}(A_{n-1}).
\end{gathered} \tag{2.35}$$

This implies that there is $B > 0$ so that

$$||(x_1, x_2)|| \leq B. \tag{2.36}$$

Hence, $\Omega_1$ is bounded. This completes Step 1.

*Step 2.* Let

$$\Omega_2 = \{x = (x_1, x_2) \in \operatorname{Ker} L,\ Nx \in \operatorname{Im} L\}. \tag{2.37}$$

We prove $\Omega_2$ is bounded. Suppose $x \in \Omega_2$, then $x(t) = (x_1(t), x_2(t)) = (a, b) \in \mathbb{R}^2$. We prove that $|a| \leq M$ and $|b| \leq M$. Suppose that either $|a| > M$ or $|a| \leq M$ and $|b| > M$. $Nx \in \operatorname{Im} L$ implies that

$$\int_0^T \phi^{-1}(x_2(t))\,dt = 0, \qquad \int_0^T f\Big(t, x_1(t), \ldots, x_1^{(n-2)}(t), \phi^{-1}(x_2(t))\Big)\,dt = 0. \tag{2.38}$$

Thus, we get $b = 0$ and

$$\int_0^T f(t, a, 0, \ldots, 0)\,dt = 0. \tag{2.39}$$

From $(A_2)$, we know that $|a| \leq M$, this contradicts $|a| > M$. It follows that $\Omega_2$ is bounded.

*Step 3.* If the first case in $(A_3)$ holds, let

$$\Omega_3 = \{x = (x_1, x_2) \in \operatorname{Ker} L,\ \lambda x + (1-\lambda) QNx = 0,\ \lambda \in [0,1]\}. \tag{2.40}$$

Now, we show that $\Omega_3$ is bounded. Suppose that there is sequence $y_n(t) = (a_n, b_n) \in \Omega_3$ and $|a_n| \to +\infty$ or $|b_n| \to +\infty$ as $n$ tends to infinity. Then, there exists $\lambda_n \in [0,1]$ such that

$$\lambda_n(a_n, b_n) + (1-\lambda_n)\left(\frac{1}{T}\int_0^T \phi^{-1}(b_n)\,ds, \frac{1}{T}\int_0^T f(s, a_n, 0, \ldots, \phi^{-1}(b_n))\,ds\right) = 0. \tag{2.41}$$

So,

$$\begin{aligned} \lambda_n a_n &= -(1-\lambda_n)\phi^{-1}(b_n), \\ \lambda_n b_n T &= -(1-\lambda_n)\int_0^T f(u, a_n, 0, \ldots, 0, \phi^{-1}(b_n))\,du. \end{aligned} \tag{2.42}$$

We get

$$\lambda_n a_n^2 + \lambda_n b_n^2 T = -(1-\lambda_n)\left(a_n\phi^{-1}(b_n) + b_n\int_0^T f(u, a_n, 0, \ldots, 0, \phi^{-1}(b_n))\,du\right). \tag{2.43}$$

By the definition of $y_n$, we know that either $|a_n| > M^*$ or $|a_n| \le M^*$ and $|b_n| > M^*$ for sufficiently large $n$. We claim, for this $n$, that $\lambda_n \neq 1$. Suppose that $\lambda_n = 1$. Then, we get $a_n = b_n = 0$, a contradiction. So, $\lambda_n \neq 1$. Now, using $(A_3)$, we get

$$0 \le \lambda_n a_n^2 + \lambda_n b_n^2 T = -(1-\lambda_n)\left(a_n\phi^{-1}(b_n) + b_n\int_0^T f(u,a_n,0,\dots,0,\phi^{-1}(b_n))\,du\right) < 0, \tag{2.44}$$

a contradiction. Hence, $\Omega_3$ is bounded.

If the second case in $(A_3)$ holds, let

$$\Omega_3 = \{x = (x_1,x_2) \in \operatorname{Ker} L,\ -\lambda x + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\}. \tag{2.45}$$

Similar to the above argument, we get that $\Omega_3$ is bounded by $(A_3)$.

In the following, we will show that all conditions of Theorem 2.1 are satisfied. Let $\Omega$ be an open bounded subset of $X$ such that $\Omega \supset \cup_{i=1}^3 \overline{\Omega_i}$. By Lemma 2.2, $L$ is a Fredholm operator of index zero and $N$ is $L$-compact on $\overline{\Omega}$. By the definition of $\Omega$, we have the following:

(a) $Lx \neq \lambda Nx$ for $x \in (\operatorname{dom} L/\operatorname{Ker} L) \cap \partial\Omega$ and $\lambda \in (0,1)$;
(b) $Nx \notin \operatorname{Im} L$ for $x \in \operatorname{Ker} L \cap \partial\Omega$.

*Step 4.* We prove $\deg(QN|_{\operatorname{Ker} L},\Omega \cap \operatorname{Ker} L,0) \neq 0$.

In fact, let $H(x,\lambda) = \pm\lambda x + (1-\lambda)QNx$. According the definition of $\Omega$, we know $H(x,\lambda) \neq 0$ for $x \in \partial\Omega \cap \operatorname{Ker} L$, thus by homotopy property of degree,

$$\begin{aligned} &\deg\left(QN|_{\operatorname{Ker} L},\Omega \cap \operatorname{Ker} L,0\right) \\ &\quad= \deg\left(H(\cdot,0),\Omega \cap \operatorname{Ker} L,0\right) \\ &\quad= \deg\left(H(\cdot,1),\Omega \cap \operatorname{Ker} L,0\right) \\ &\quad= \deg(\pm I,\Omega \cap \operatorname{Ker} L,0) \neq 0. \end{aligned} \tag{2.46}$$

Thus, by Theorem 2.1, $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$, which is a solution of BVP (1.1)-(1.2). The proof is complete. □

Theorem 2.4. *Suppose the following condition holds.*

$(A_1')$ *There are continuous functions $h(t,x_0,x_1,\dots,x_{n-1})$, $e(t)$, nonnegative functions $g_i(t,x)$ $(i = 0,1,\dots,n-1)$ and positive number $\beta$ and $m$ such that $f$ satisfies*

$$f(t,x_0,x_1,\dots,x_{n-1}) = e(t) + h(t,x_0,x_1,\dots,x_{n-1}) + \sum_{i=0}^{n-1} g_i(t,x_i), \tag{2.47}$$

*and also that $h$ satisfies*

$$x_{n-1}h(t,x_0,x_1,\dots,x_{n-1}) \le -\beta\left|x_{n-1}\right|^{m+1} \tag{2.48}$$

*for all* $t \in [0,T]$ *and* $(x_0,x_1,\dots,x_{n-1}) \in \mathbb{R}^n$ *and that* $g_i$ *satisfies*

$$\limsup_{|x|\to\infty,\, t\in[0,T]} \frac{|g_i(t,x)|}{|x|^m} = r_i, \quad \text{for } i = 0,1,\dots,n-2,$$
$$\limsup_{|x|\to\infty,\, t\in[0,T]} \frac{|g_{n-1}(t,x)|}{|\phi(x)|^m} = r_{n-1} \tag{2.49}$$

*with* $r_i \geq 0$ *for* $i = 0,1,\dots,n-1$. *Furthermore,* $(A_2)$ *and* $(A_3)$ *of Theorem 2.3 hold. Then, BVP (1.1)-(1.2) has at least one solution provided*

$$r_0 T^{m(n-1)} + \sum_{i=1}^{n-2} r_i T^{m(n-i-2)} + r_{n-1} < \beta. \tag{2.50}$$

*Proof.* To apply Theorem 2.1, we should define an open bounded subset $\Omega$ of $X$ so that (i), (ii), and (iii) of Theorem 2.1 hold. It is based upon three steps to obtain $\Omega$. The proof of this theorem is divide into four steps.

*Step 1.* Let

$$\Omega_1 = \{x \in \operatorname{dom} L/\operatorname{Ker} L,\ Lx = \lambda N x \text{ for some } \lambda \in (0,1)\}. \tag{2.51}$$

We prove $\Omega_1$ is bounded. Similar to the proof of Theorem 2.3, we get (2.25). It suffices to prove there is a constant $B > 0$ such that

$$\|(x_1,x_2)\| = \max\left\{\|x_1\|_\infty, \|x_1'\|_\infty, \dots, \left\|x_1^{(n-2)}\right\|_\infty, \|x_2\|_\infty\right\} \leq B. \tag{2.52}$$

We divide this step into two substeps.

*Substep 1.1.* We prove that there is constant $\overline{M} > 0$ such that $\int_0^T \phi^{-1}(|x_2(s)|)^{m+1} ds \leq \overline{M}$.

Multiplying the two sides of the second equation in (2.25) by $\phi^{-1}(x_2(t))$ and integrating it from 0 to $T$, using $(A_1')$, we get

$$\begin{aligned}
0 &= \int_0^T \phi^{-1}(x_2(t)) x_2'(t)\,dt \\
&= \lambda \int_0^T f\left(s, x_1(s), x_1'(s), \dots, x_1^{(n-2)}(s), \phi^{-1}(x_2(s))\right) \phi^{-1}(x_2(s))\,ds \\
&= \lambda \Bigg( \int_0^T h\left(s, x_1(s), x_1'(s), \dots, x_1^{(n-2)}(s), \phi^{-1}(x_2(s))\right) \phi^{-1}(x_2(s))\,ds \\
&\quad + \sum_{i=0}^{n-2} \int_0^T g_i\left(s, x_1^{(i)}(s)\right) \phi^{-1}(x_2(s))\,ds + \int_0^T e(s)\phi^{-1}(x_2(s))\,ds \\
&\quad + \int_0^T g_{n-1}(s, \phi^{-1}(x_2(s)))\phi^{-1}(x_2(t))\,ds \Bigg).
\end{aligned} \tag{2.53}$$

Thus, from the second part of $(A_1')$,

$$\begin{aligned}
&\lambda\beta\int_0^T[\phi^{-1}(|x_2(s)|)]^{m+1}ds\\
&\quad\le -\lambda\int_0^T h\Big(s,x_1(s),x_1'(s),\dots,x_1^{(n-2)}(s),\phi^{-1}(x_2(s))\Big)\phi^{-1}(x_2(s))\,ds\\
&\quad=\lambda\sum_{i=0}^{n-2}\int_0^T g_i\Big(s,x_1^{(i)}(s)\Big)\phi^{-1}(x_2(s))\,ds+\lambda\int_0^T g_{n-1}(s,\phi^{-1}(x_2(s)))\phi^{-1}(x_2(s))\,ds\\
&\qquad+\lambda\int_0^T e(s)\phi^{-1}(x_2(s))\,ds.
\end{aligned}\tag{2.54}$$

Hence,

$$\begin{aligned}
&\beta\int_0^T[\phi^{-1}(|x_2(s)|)]^{m+1}ds\\
&\quad\le\sum_{i=0}^{n-2}\int_0^T\Big|g_i\Big(s,x_1^{(i)}(s)\Big)\Big|\,|\phi^{-1}(x_2(s))|\,ds\\
&\qquad+\int_0^T|g_{n-1}(s,\phi^{-1}(x_2(s)))|\,|\phi^{-1}(x_2(s))|\,ds+\int_0^1|e(s)|\phi^{-1}(x_2(s))\,ds.
\end{aligned}\tag{2.55}$$

Let $\epsilon>0$ satisfy

$$\beta>(r_0+\epsilon)\Big(\epsilon+T^{n-2}T^{m/(m+1)}\Big)^m T^{m/(m+1)}+\sum_{i=1}^{n-2}(r_i+\epsilon)T^{m(n-i-2)}+(r_{n-1}+\epsilon).\tag{2.56}$$

For such a $\epsilon>0$, we find from $(A_1')$ that there is a constant $\delta>M$ such that for every $i=0,1,\dots,n-2$,

$$\begin{aligned}
|g_i(t,x)|&<(r_i+\epsilon)|x|^m\quad\text{uniformly for } t\in[0,T],\ |x|>\delta,\\
|g_{n-1}(t,x)|&<(r_{n-1}+\epsilon)|\phi(x)|^m\quad\text{uniformly for } t\in[0,T],\ |x|>\delta.
\end{aligned}\tag{2.57}$$

Let, for $i=0,1,\dots,n-2$,

$$\begin{aligned}
\Delta_{1,i}&=\{t:t\in[0,T],\ |x^{(i)}(t)|\le\delta\},\\
\Delta_{2,i}&=\{t:t\in[0,T],\ |x^{(i)}(t)|>\delta\},\\
g_{\delta,i}&=\max_{t\in[0,T],|x|\le\delta}|g_i(t,x)|;\\
\Delta_{1,n-1}&=\{t:t\in[0,T],\ \phi(|x_2(t)|)\le\delta\},\\
\Delta_{2,i}&=\{t:t\in[0,T],\ \phi(|x_2(t)|)>\delta\},\\
g_{\delta,i}&=\max_{t\in[0,T],|x|\le\delta}|g_i(t,\phi(|x_2|))|.
\end{aligned}\tag{2.58}$$

Then,

$$\begin{aligned}
\beta\int_0^T &[\phi^{-1}(|x_2(s)|)]^{m+1}ds \\
&\le \sum_{i=0}^{n-2}\int_{\Delta_{1,i}}\left|g_i\left(s,x_1^{(i)}(s)\right)\right||\phi^{-1}(x_2(s))|ds \\
&\quad+\sum_{i=0}^{n-2}\int_{\Delta_{2,i}}\left|g_i\left(s,x_1^{(i)}(s)\right)\right||\phi^{-1}(x_2(s))|ds \\
&\quad+\int_{\Delta_{1,n-1}}|g_{n-1}(s,\phi^{-1}(x_2(s)))||\phi^{-1}(x_2(s))|ds \\
&\quad+\int_{\Delta_{2,n-1}}|g_{n-1}(s,\phi^{-1}(x_2(s)))||\phi^{-1}(x_2(s))|ds \\
&\quad+\int_0^T|e(s)||\phi^{-1}(x_2(s))|ds \\
&\le \sum_{i=0}^{n-2}g_{\delta,i}\int_0^T|\phi^{-1}(x_2(s))|ds+\sum_{i=0}^{n-2}(r_i+\epsilon)\int_0^T\left|x_1^{(i)}(s)\right|^m|\phi^{-1}(x_2(s))|ds \\
&\quad+g_{\delta,n-1}\int_0^T|\phi^{-1}(x_2(s))|ds+(r_{n-1}+\epsilon)\int_0^T|\phi^{-1}(x_2(s))|^{m+1}ds \\
&\quad+\int_0^T|e(s)||\phi^{-1}(x_2(s))|ds.
\end{aligned} \tag{2.59}$$

It is easy to see that there is $\xi_i\in[0,T]$ so that $x_1^{(i)}(\xi_i)=0$ for $i=1,\dots,n-1$. Hence, for $i=1,\dots,n-2$, we get

$$\left|x_1^{(i)}(t)\right| = \left|x_1^{(i)}(\xi_i)+\int_{\xi_i}^t x_1^{(i+1)}(s)ds\right| \le \int_0^T\left|x_1^{(i+1)}(s)\right|ds. \tag{2.60}$$

So, we have

$$\begin{aligned}
\left|x_1^{(i)}(t)\right| &\le T^{n-i-2}\int_0^T\left|x_1^{(n-1)}(s)\right|ds \\
&\le T^{n-i-2}\int_0^T\phi^{-1}(|x_2(s)|)ds \quad \text{for } i=1,\dots,n-2.
\end{aligned} \tag{2.61}$$

Similar to that of the proof of Theorem 2.3, from $(A_2)$, we see that

$$|x_1(t)| \le M+\int_0^T|x_1'(s)|ds \le M+T^{n-3}\int_0^T\left|x_1^{(n-1)}(s)\right|ds \le M+T^{n-2}\int_0^T|\phi^{-1}(x_2(s))|ds. \tag{2.62}$$

Using

$$\int_0^T \phi^{-1}(|x_2(s)|)\,ds \le T^{m/(m+1)}\left(\int_0^T |\phi^{-1}(x_2(s))|^{m+1} ds\right)^{1/(m+1)}, \tag{2.63}$$

we get

$$\begin{aligned}
\beta\int_0^T &[\phi^{-1}(|x_2(s)|)]^{m+1} ds \\
&\le \sum_{i=0}^{n-2} g_{\delta,i}T^{m/(m+1)}\left(\int_0^T |\phi^{-1}(x_2(s))|^{m+1} ds\right)^{1/(m+1)} \\
&\quad + \sum_{i=1}^{n-2}(r_i+\epsilon)T^{m(n-i-2)}\left(\int_0^T |\phi^{-1}(x_2(s))|\,ds\right)^{m+1} \\
&\quad + (r_0+\epsilon)\left(M+T^{n-2}\int_0^T |\phi^{-1}(x_2(s))|\,ds\right)^{m}\int_0^T |\phi^{-1}(x_2(s))|\,ds \\
&\quad + g_{\delta,n-1}T^{m/(m+1)}\left(\int_0^T |\phi^{-1}(x_2(s))|^{m+1} ds\right)^{1/(m+1)} \\
&\quad + (r_{n-1}+\epsilon)\int_0^T |\phi^{-1}(x_2(s))|^{m+1} ds \\
&\quad + \|e\|_\infty T^{m/(m+1)}\left(\int_0^T |\phi^{-1}(x_2(s))|^{m+1} ds\right)^{1/(m+1)} \\
&\le \sum_{i=0}^{n-2} g_{\delta,i}T^{m/(m+1)}\left(\int_0^T |\phi^{-1}(x_2(s))|^{m+1} ds\right)^{1/(m+1)} \\
&\quad + \sum_{i=1}^{n-2}(r_i+\epsilon)T^{m(n-i-2)}\int_0^T |\phi^{-1}(x_2(s))|^{m+1} ds \\
&\quad + (r_0+\epsilon)\left[M+T^{n-2}T^{m/(m+1)}\left(\int_0^T |\phi^{-1}(x_2(s))|^{m+1} ds\right)^{1/(m+1)}\right]^{m} \\
&\quad \times T^{m/(m+1)}\left(\int_0^T |\phi^{-1}(x_2(s))|^{m+1} ds\right)^{1/(m+1)} \\
&\quad + g_{\delta,n-1}T^{m/(m+1)}\left(\int_0^T |\phi^{-1}(x_2(s))|^{m+1} ds\right)^{1/(m+1)} \\
&\quad + (r_{n-1}+\epsilon)\int_0^T |\phi^{-1}(x_2(s))|^{m+1} ds \\
&\quad + \|e\|_\infty T^{m/(m+1)}\left(\int_0^T |\phi^{-1}(x_2(s))|^{m+1} ds\right)^{1/(m+1)}
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=0}^{n-2} g_{\delta,i} T^{m/(m+1)} \left( \int_0^T |\phi^{-1}(x_2(s))|^{m+1} ds \right)^{1/(m+1)} \\
&\quad + \sum_{i=1}^{n-2} (r_i + \epsilon) T^{m(n-i-2)} \int_0^T |\phi^{-1}(x_2(s))|^{m+1} ds \\
&\quad + (r_0 + \epsilon) T^{m/(m+1)} \left( \frac{M}{\left( \int_0^T |\phi^{-1}(x_2(s))|^{m+1} ds \right)^{1/(m+1)}} + T^{n-2} T^{m/(m+1)} \right)^m \\
&\qquad \times \int_0^T |\phi^{-1}(x_2(s))|^{m+1} ds + g_{\delta,n-1} T^{m/(m+1)} \left( \int_0^T |\phi^{-1}(x_2(s))|^{m+1} ds \right)^{1/(m+1)} \\
&\quad + (r_{n-1} + \epsilon) \int_0^T |\phi^{-1}(x_2(s))|^{m+1} ds \\
&\quad + \|e\|_\infty T^{m/(m+1)} \left( \int_0^T |\phi^{-1}(x_2(s))|^{m+1} ds \right)^{1/(m+1)} .
\end{aligned}
\tag{2.64}
$$

Without loss of generality, suppose that

$$
\left( \int_0^T |\phi^{-1}(x_2(s))|^{m+1} ds \right)^{1/(m+1)} \geq \frac{M}{\epsilon}. \tag{2.65}
$$

So, we get

$$
\begin{aligned}
&\beta \int_0^T [\phi^{-1}(|x_2(s)|)]^{m+1} ds \\
&\quad \leq \sum_{i=0}^{n-2} g_{\delta,i} T^{m/(m+1)} \left( \int_0^T |\phi^{-1}(x_2(s))|^{m+1} ds \right)^{1/(m+1)} \\
&\qquad + \sum_{i=1}^{n-2} (r_i + \epsilon) T^{m(n-i-2)} \int_0^T |\phi^{-1}(x_2(s))|^{m+1} ds \\
&\qquad + (r_0 + \epsilon) \left( \epsilon + T^{n-2} T^{m/(m+1)} \right)^m T^{m/(m+1)} \int_0^T |\phi^{-1}(x_2(s))|^{m+1} ds \\
&\qquad + g_{\delta,n-1} T^{m/(m+1)} \left( \int_0^T |\phi^{-1}(x_2(s))|^{m+1} ds \right)^{1/(m+1)} \\
&\qquad + (r_{n-1} + \epsilon) \int_0^T |\phi^{-1}(x_2(s))|^{m+1} ds \\
&\qquad + \|e\|_\infty T^{m/(m+1)} \left( \int_0^T |\phi^{-1}(x_2(s))|^{m+1} ds \right)^{1/(m+1)} .
\end{aligned}
\tag{2.66}
$$

It follows that

$$\begin{aligned}
&\left(\beta-(r_0+\epsilon)\left(\epsilon+T^{n-2}T^{m/(m+1)}\right)^{m}T^{m/(m+1)}-\sum_{i=1}^{n-2}(r_i+\epsilon)T^{m(n-i-2)}-(r_{n-1}+\epsilon)\right)\\
&\qquad\times\int_0^T|\phi^{-1}(x_2(s))|^{m+1}ds\\
&\quad\le\sum_{i=0}^{n-2}g_{\delta,i}T^{m/(m+1)}\left(\int_0^T|\phi^{-1}(x_2(s))|^{m+1}ds\right)^{1/(m+1)}\\
&\qquad+g_{\delta,n-1}T^{m/(m+1)}\left(\int_0^T|\phi^{-1}(x_2(s))|^{m+1}ds\right)^{1/(m+1)}\\
&\qquad+\|e\|_\infty T^{m/(m+1)}\left(\int_0^T|\phi^{-1}(x_2(s))|^{m+1}ds\right)^{1/(m+1)}.
\end{aligned}\tag{2.67}$$

By the definition of $\epsilon$, we know that there is $\overline{M}>0$ so that

$$\int_0^T|\phi^{-1}(x_2(s))|^{m+1}ds\le\overline{M}.\tag{2.68}$$

It follows that

$$\int_0^T|\phi^{-1}(x_2(s))|^{m+1}ds\le\max\{\overline{M},\,M\}=:A.\tag{2.69}$$

*Substep 1.2.* We prove that there is $B>0$ such that $\|(x_1,x_2)\|\le B$.

From Substep 1.1, we have

$$\begin{aligned}
\left\|x_1^{(i)}\right\|_\infty&\le T^{n-i-2}\int_0^T\left|x_1^{(n-1)}(s)\right|ds\\
&\le T^{n-i-2}T^{m/(m+1)}\left(\int_0^T[\phi^{-1}(|x_2(s)|)]^{m+1}ds\right)^{1/(m+1)}\\
&\le T^{n-i-2}T^{m/(m+1)}A^{1/(m+1)}\quad\text{for } i=1,\dots,n-2,\\
\|x_1\|_\infty&\le M+T^{n-3}T^{m/(m+1)}\left(\int_0^T[\phi^{-1}(|x_2(s)|)]^{m+1}ds\right)^{1/(m+1)}\\
&\le M+T^{n-3}T^{m/(m+1)}A^{1/(m+1)}.
\end{aligned}\tag{2.70}$$

Now, we consider $\|x_2\|_\infty$. Multiplying the two sides of the second equation in (2.25) by $\phi^{-1}(x_2(t))$, integrating it from $\xi_{n-1}$ to $t$, for $\xi_{n-1}<t$, and using $(A_1')$, we get

$$\begin{aligned}
&\frac{1}{2}\left|\phi^{-1}(x_2(t))\right|^2\\
&\quad=\lambda\int_{\xi_{n-1}}^{t} f\Big(s,x_1(s),x_1'(s),\dots,x_1^{(n-2)}(s),\phi^{-1}(x_2(s))\Big)\phi^{-1}(x_2(s))\,ds\\
&\quad=\lambda\int_{\xi_{n-1}}^{t} h\Big(s,x_1(s),x_1'(s),\dots,x_1^{(n-2)}(s),\phi^{-1}(x_2(s))\Big)\phi^{-1}(x_2(s))\,ds\\
&\qquad+\lambda\sum_{i=0}^{n-2}\int_{\xi_{n-1}}^{t} g_i\Big(s,x_1^{(i)}(s)\Big)\phi^{-1}(x_2(s))\,ds+\lambda\int_{\xi_{n-1}}^{t} g_{n-1}(s,\phi^{-1}(x_2(s)))\phi^{-1}(x_2(s))\,ds\\
&\qquad+\lambda\int_{\xi_{n-1}}^{t} e(s)\phi^{-1}(x_2(s))\,ds\\
&\quad\le-\lambda\beta\int_{\xi_{n-1}}^{t}\left|\phi^{-1}(x_2(s))\right|^{m+1}ds+\lambda\sum_{i=0}^{n-2}\int_{\xi_{n-1}}^{t} g_i\Big(s,x_1^{(i)}(s)\Big)\phi^{-1}(x_2(s))\,ds\\
&\qquad+\lambda\int_{\xi_{n-1}}^{t} g_{n-1}(s,\phi^{-1}(x_2(s)))\phi^{-1}(x_2(s))\,ds+\lambda\int_{\xi_{n-1}}^{t} e(s)\phi^{-1}(x_2(s))\,ds\\
&\quad\le\lambda\sum_{i=0}^{n-2}\int_{\xi_{n-1}}^{t} g_i\Big(s,x_1^{(i)}(s)\Big)\phi^{-1}(x_2(s))\,ds+\lambda\int_{\xi_{n-1}}^{t} e(s)\phi^{-1}(x_2(s))\,ds\\
&\qquad+\lambda\int_{\xi_{n-1}}^{t} g_{n-1}(s,\phi^{-1}(x_2(s)))\phi^{-1}(x_2(s))\,ds\\
&\quad\le\sum_{i=0}^{n-2}\int_0^T\left|g_i\Big(s,x_1^{(i)}(s)\Big)\right|\left|\phi^{-1}(x_2(s))\right|ds+\int_0^T|e(s)|\left|\phi^{-1}(x_2(s))\right|ds\\
&\qquad+\int_0^T\left|g_{n-1}(s,\phi^{-1}(x_2(s)))\right|\left|\phi^{-1}(x_2(s))\right|ds\\
&\quad\le\sum_{i=0}^{n-2}\int_{\Delta_{1,i}}\left|g_i\Big(s,x_1^{(i)}(s)\Big)\right|\left|\phi^{-1}(x_2(s))\right|ds+\sum_{i=0}^{n-2}\int_{\Delta_{2,i}}\left|g_i\Big(s,x_1^{(i)}(s)\Big)\right|\left|\phi^{-1}(x_2(s))\right|ds\\
&\qquad+\int_0^T|e(s)|\left|\phi^{-1}(x_2(s))\right|ds\\
&\qquad+\int_{\Delta_{1,n-1}} g_{n-1}(s,\phi^{-1}(x_2(s)))\left|\phi^{-1}(x_2(s))\right|ds\\
&\qquad+\int_{\Delta_{2,n-1}} g_{n-1}(s,\phi^{-1}(x_2(s)))\left|\phi^{-1}(x_2(s))\right|ds\\
&\quad\le\sum_{i=0}^{n-2}g_{\delta,i}\int_0^T\left|\phi^{-1}(x_2(s))\right|ds+\sum_{i=0}^{n-2}(r_i+\epsilon)\int_0^T\left|x_1^{(i)}(s)\right|^m\left|\phi^{-1}(x_2(s))\right|ds\\
&\qquad+g_{\delta,n-1}\int_0^T\left|\phi^{-1}(x_2(s))\right|ds+(r_{n-1}+\epsilon)\int_0^T\left|\phi^{-1}(x_2(s))\right|^{m+1}ds\\
&\qquad+\|e\|_\infty\int_0^T\left|\phi^{-1}(x_2(s))\right|ds.
\end{aligned}\tag{2.71}$$

Similar to Substep 1.1, we can get

$$\begin{aligned}
&\frac{1}{2}\left|\phi^{-1}(x_2(t))\right|^2\\
&\quad\le \sum_{i=0}^{n-2} g_{\delta,i}T^{m/(m+1)}\left(\int_0^T\left|\phi^{-1}(x_2(s))\right|^{m+1}ds\right)^{1/(m+1)}\\
&\qquad+\sum_{i=1}^{n-2}(r_i+\epsilon)T^{m(n-i-2)}\left(\int_0^T\left|\phi^{-1}(x_2(s))\right|ds\right)^m\int_0^T\left|\phi^{-1}(x_2(s))\right|ds\\
&\qquad+g_{\delta,n-1}T^{m/(m+1)}\left(\int_0^T\left|\phi^{-1}(x_2(s))\right|^{m+1}ds\right)^{1/(m+1)}\\
&\qquad+(r_{n-1}+\epsilon)\int_0^T\left|\phi^{-1}(x_2(s))\right|^{m+1}ds\\
&\qquad+(r_0+\epsilon)\left(M+T^{n-2}\int_0^T\left|\phi^{-1}(x_2(s))\right|ds\right)^m\int_0^T\left|\phi^{-1}(x_2(s))\right|ds\\
&\qquad+\|e\|_\infty T^{m/(m+1)}\left(\int_0^T\left|\phi^{-1}(x_2(s))\right|^{m+1}ds\right)^{1/(m+1)}\\
&\quad\le \sum_{i=0}^{n-2} g_{\delta,i}T^{m/(m+1)}A^{1/(m+1)}+\sum_{i=1}^{n-2}(r_i+\epsilon)T^{m(n-i-2)}T^mA^{m+1}+g_{\delta,n-1}T^{m/(m+1)}A^{1/(m+1)}\\
&\qquad+(r_{n-1}+\epsilon)A+(r_0+\epsilon)\left(M+T^{n-2}T^{m/(m+1)}A^{1/(m+1)}\right)^mA^{1/(m+1)}\\
&\qquad+\|e\|_\infty T^{m/(m+1)}A^{1/(m+1)}.
\end{aligned}\tag{2.72}$$

So, there is $\overline{M}' > 0$ such that $|x_2(t)| \le \overline{M}'$ for $t > \xi_n$.

Especially, we get $|x_2(0)| = |x_2(T)| \le \overline{M}'$. Thus, one gets by (2.25), after multiplying the two sides of the second equation in (2.25) by $\phi^{-1}(x_2(t))$ and integrating it from 0 to $t$, for $t \le \xi_{n-1}$,

$$\begin{aligned}
&\frac{1}{2}\left|\phi^{-1}(x_2(t))\right|^2\\
&\quad=\frac{1}{2}\left|\phi^{-1}(x_2(0))\right|^2+\lambda\int_0^t f\left(s,x_1(s),x_1'(s),\dots,x_1^{(n-2)}(s),\phi^{-1}(x_2(s))\right)\phi^{-1}(x_2(s))ds\\
&\quad\le\frac{1}{2}\phi^{-1}(\overline{M}')^2+\lambda\int_0^t h\left(s,x_1(s),x_1'(s),\dots,x_1^{(n-2)}(s),\phi^{-1}(x_2(s))\right)\phi^{-1}(x_2(s))ds\\
&\qquad+\lambda\sum_{i=0}^{n-2}\int_0^t g_i\left(s,x_1^{(i)}(s)\right)\phi^{-1}(x_2(s))ds+\lambda\int_0^t g_{n-1}(s,\phi^{-1}(x_2(s)))\phi^{-1}(x_2(s))ds\\
&\qquad+\lambda\int_0^t e(s)\phi^{-1}(x_2(s))ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\phi^{-1}(\overline{M}')^2 - \lambda\beta\int_0^t |\phi^{-1}(x_2(s))|^{m+1}ds + \lambda\sum_{i=0}^{n-2}\int_0^t g_i\left(s,x_1^{(i)}(s)\right)\phi^{-1}(x_2(s))ds \\
&\quad + \lambda\int_0^t g_{n-1}(s,\phi^{-1}(x_2(s)))\phi^{-1}(x_2(s))ds + \lambda\int_0^t e(s)\phi^{-1}(x_2(s))ds \\
&\leq \frac{1}{2}\phi^{-1}(\overline{M}')^2 + \sum_{i=0}^{n-2}\int_0^T \left|g_i\left(s,x^{(i)}(s)\right)\right| |\phi^{-1}(x_2(s))|ds \\
&\quad + \int_0^T |g_{n-1}(s,\phi^{-1}(x_2(s)))||\phi^{-1}(x_2(s))|ds \\
&\quad + \int_0^1 |e(s)\phi^{-1}(x_2(s))|ds.
\end{aligned}
\tag{2.73}
$$

Similar to the above discussion, there is $\overline{M}'' > 0$ such that $|x_2(t)| \leq \overline{M}''$ for $t \leq \xi_n$. It follows that

$$
\begin{aligned}
||(x_1,x_2)|| \leq \max\Big\{&T^{n-i-2}T^{m/(m+1)}A^{1/(m+1)},\ i = 1,\dots,n-2, \\
&M + T^{n-3}T^{m/(m+1)}A^{1/(m+1)}, \overline{M}, \overline{M}', \overline{M}''\Big\} =: B.
\end{aligned}
\tag{2.74}
$$

Hence, $\Omega_1$ is bounded. This completes Step 1.

*Step 2.* Let

$$
\Omega_2 = \{x \in \operatorname{Ker} L,\ Nx \in \operatorname{Im} L\}. \tag{2.75}
$$

Similar to the proof of Step 2 of Theorem 2.3, we can prove $\Omega_2$ is bounded.

*Step 3.* Let

$$
\Omega_3 = \{x \in \operatorname{Ker} L,\ \pm\lambda x + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\}. \tag{2.76}
$$

Similar to that of the proof of Step 3 of Theorem 2.3, we can show that $\Omega_3$ is bounded.

The remaining step, Step 4, is similar to that of the proof of Step 4 of Theorem 2.3 and is omitted.

Thus, by Theorem 2.1, $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$, which is a solution of BVP (1.1)-(1.2). The proof is complete. □

*Remark 2.5.* In Theorem 2.4, if $f$ is a polynomial, the degrees of the variables $x_0, x_1, \dots, x_{n-1}$ in function $f$ are $m$, $m$ may be greater than 1.

*Remark 2.6.* It is easy to obtain the existence results for solutions of problem (1.9) and the following periodic boundary value problem:

$$
\begin{aligned}
x^{(n)}(t) &= f\left(t, x(t), x'(t), \dots, x^{(n-1)}(t)\right), \quad t \in [0,T], \\
x^{(i)}(0) &= x^{(i)}(T), \quad i = 0,1,\dots,n-1.
\end{aligned}
\tag{2.77}
$$

We omit the details since they are similar to Theorems 2.3 and 2.4.

## Acknowledgment

The first author is supported by the Science Foundation of Educational Committee of Hunan Province and both authors by the National Natural Science Foundation of China.


## References


[1] A. Cabada and R. L. Pouso, *Existence results for the problem* $(\phi(u'))' = f(t,u,u')$ *with nonlinear boundary conditions*, Nonlinear Anal. Ser. A: Theory Methods **35** (1999), no. 2, 221–231.

[2] W. Feng and J. R. L. Webb, *Solvability of three point boundary value problems at resonance*, Nonlinear Anal. Ser. A: Theory Methods **30** (1997), no. 6, 3227–3238.

[3] R. E. Gaines and J. L. Mawhin, *Coincidence Degree, and Nonlinear Differential Equations*, Lecture Notes in Mathematics, vol. 568, Springer-Verlag, Berlin, 1977.

[4] C. P. Gupta, S. K. Ntouyas, and P. C. Tsamatos, *Solvability of an m-point boundary value problem for second order ordinary differential equations*, J. Math. Anal. Appl. **189** (1995), no. 2, 575–584.

[5] V. A. Il'in and E. I. Mojseev, *Nonlocal boundary-value problem of the first kind for a Sturm-Liouville operator in its differential and finite-difference aspects*, J. Differential Equations **23** (1987), no. 7, 803–811.

[6] I. Kiguradze, *On periodic solutions of nth order ordinary differential equations*, Nonlinear Anal. Ser. A: Theory Methods **40** (2000), no. 1–8, 309–321.

[7] H.-W. Knobloch, *Eine neue Methode zur Approximation periodischer Lösungen nicht-linearer Differentialgleichungen zweiter Ordnung*, Math. Z. **82** (1963), 177–197.

[8] B. Liu, *Solvability of multi-point boundary value problem at resonance. II*, Appl. Math. Comput. **136** (2003), no. 2-3, 353–377.

[9] ———, *Solvability of multi-point boundary value problem at resonance. IV*, Appl. Math. Comput. **143** (2003), no. 2-3, 275–299.

[10] B. Liu and J. Yu, *Solvability of multi-point boundary value problem at resonance. III*, Appl. Math. Comput. **129** (2002), no. 1, 119–143.

[11] ———, *Solvability of multi-point boundary value problems at resonance. I*, Indian J. Pure Appl. Math. **33** (2002), no. 4, 475–494.

[12] R. Manásevich and J. Mawhin, *Periodic solutions for nonlinear systems with p-Laplacian-like operators*, J. Differential Equations **145** (1998), no. 2, 367–393.

[13] F. I. Njoku and P. Omari, *Singularly perturbed higher order periodic boundary value problems*, J. Math. Anal. Appl. **289** (2004), no. 2, 639–649.

[14] K. Schmitt, *Periodic solutions of nonlinear second order differential equations*, Math. Z. **98** (1967), 200–207.

[15] S. Sędziwy and R. Srzednicki, *On periodic solutions of certain nth order differential equations*, J. Math. Anal. Appl. **196** (1995), no. 2, 666–675.

[16] X. Yang, *Upper and lower solutions for periodic problems*, Appl. Math. Comput. **137** (2003), no. 2-3, 413–422.



Yuji Liu: Department of Mathematics, Hunan Institute of Science and Technology, Yueyang, Hunan 414000, China

*Current address*: Department of Applied Mathematics, Beijing Institute of Technology, Beijing 1000811, China

*E-mail address*: liuyuji888@sohu.com

Weigao Ge: Department of Applied Mathematics, Beijing Institute of Technology, Beijing 100081, China

*E-mail address*: gew@bit.edu.cn