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Research Article

Reduction of Neighborhood-Based Generalized Rough Sets

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Rough set theory is a powerful tool for dealing with uncertainty, granularity, and incompleteness of knowledge in information systems. This paper discusses five types of existing neighborhood-based generalized rough sets. The concepts of minimal neighborhood description and maximal neighborhood description of an element are defined, and by means of the two concepts, the properties and structures of the third and the fourth types of neighborhood-based rough sets are deeply explored. Furthermore, we systematically study the covering reduction of the third and the fourth types of neighborhood-based rough sets in terms of the two concepts. Finally, two open problems proposed by Yun et al. (2011) are solved.

1. Introduction

Rough set theory was first proposed by Pawlak [1] for dealing with vagueness and granularity in information systems. It has been successfully applied to process control, economics, medical diagnosis, biochemistry, environmental science, biology, chemistry, psychology, conflict analysis, and other fields [2–10]. The further investigation into rough set theory and its extension will find new applications and new theories [11].

The classical rough set theory is based on equivalence relation. However, equivalence relation imposes restrictions and limitations on many applications [12–15]. Zakowski then established the covering-based rough set theory by exploiting coverings of a universe [16]. The covering generalized rough sets are an improvement of traditional rough set model to deal with more complex practical problems which the traditional one cannot handle. For covering models, two important theoretical issues must be explored. The first one is to present reasonable definitions of set approximations, and the second one is to develop reasonable algorithms for attribute reduct. The concept of attribute reduct can be viewed as the strongest and the most important result in rough set theory to distinguish itself from other theories.

However, the current processes covering generalized rough sets mainly focus on constructing approximation operations [16–24]. Little attention has been paid to attribute reduction of covering generalized rough sets [14, 19, 25]. In this paper, five types of special covering generalized rough sets, that is, neighborhood-based generalized rough sets [24, 26, 27] are elaborated, and the covering reduction is also examined and discussed in detail.

Zhu and Wang investigated the covering reduction of the first type of generalized rough sets [14]. Yang and Li constructed a unified reduction theory for the first, the second, and the fifth types of generalized rough sets [25]. This paper establishes the reduction theory for the third and the fourth types of neighborhood-based generalized rough sets in terms of the new concepts defined by us. This newly proposed theory can reduce redundant elements in a covering and then find the minimal coverings that induce the same neighborhood-based lower and upper approximation.

The remainder of this paper is organized as follows. In Section 2, we review the relevant concepts and properties of generalized rough sets. Section 3 defines the concepts of minimal neighborhood description and maximal neighborhood description related to an element, and the new characterizations of the third and the fourth types of neighborhood-based rough sets are given by means of the two concepts. In Section 4, we study the reduction issues of the third and the fourth types of neighborhood-based rough sets. In Section 5, two open problems proposed by Yun et al. in [28] are solved. This paper concludes in Section 6.

2. Preliminaries

In this section, we will briefly review basic concepts and results of the generalized rough sets. Let U be a nonempty set and $X \subseteq U$. In this paper, we denote by $\sim X$ the complement of X.

Definition 2.1 ([17] Covering). Let U be a universe of discourse and $\mathcal{C}$ a family of subsets of U. If no subsets in $\mathcal{C}$ is empty, and $\cup\mathcal{C} = U$, $\mathcal{C}$ is called a covering of U.

It is clear that a partition of U is a covering of U, so the concept of a covering is an extension of the concept of a partition. In the following discussion, unless stated to the contrary, the coverings are considered to be finite, that is, coverings consist of a finite number of sets in them.

Definition 2.2 ([17] Covering approximation space). Let U be a nonempty set and $\mathcal{C}$ a covering of U. The pair $(U, \mathcal{C})$ is called a covering approximation space.

Definition 2.3 ([17], Neighborhood). Let $\mathcal{C}$ be a covering of U and $x \in U$. $N_{\mathcal{C}}(x) = \cap\{K \in \mathcal{C} \mid x \in K\}$ is called the neighborhood of x. Generally, we omit the subscript $\mathcal{C}$ when there is no confusion.

By the above Definition, it is easy to see that for all $u \in N(x)$, $N(u) \subseteq N(x)$ and for all $x \in U$, $x \in N(x)$.

In this paper, we consider five pairs of dual approximation operators defined by means of neighborhoods.

Definition 2.4 ([24] Neighborhood-based approximation operations). Let $(U, \mathcal{C})$ be a covering approximation space. The five types of neighborhood-based approximation operations are defined as follows: for any $X \subseteq U$,

(1) $\underline{\mathcal{C}_1}(X) = \cup\{K \in \mathcal{C} \mid K \subseteq X\}$, $\overline{\mathcal{C}_1}(X) = \sim \underline{\mathcal{C}_1}(\sim X) = \sim (\cup\{K \in \mathcal{C} \mid K \subseteq \sim X\})$,

(2) $\underline{\mathcal{C}_2}(X) = \{x \in U \mid N(x) \subseteq X\}$, $\overline{\mathcal{C}_2}(X) = \{x \in U \mid N(x) \cap X \neq \emptyset\}$,

(3) $\underline{\mathcal{C}_3}(X) = \{x \in U \mid \exists u\ (u \in N(x) \wedge N(u) \subseteq X)\}$,

$\overline{\mathcal{C}_3}(X) = \{x \in U \mid \text{for all } u\ (u \in N(x) \rightarrow N(u) \cap X \neq \emptyset)\}$,

(4) $\underline{\mathcal{C}_4}(X) = \{x \in U \mid \text{for all } u\ (x \in N(u) \rightarrow N(u) \subseteq X)\}$, $\overline{\mathcal{C}_4}(X) = \cup\{N(x) \mid x \in U \wedge N(x) \cap X \neq \emptyset\}$,

(5) $\underline{\mathcal{C}_5}(X) = \{x \in U \mid \text{for all } u\ (x \in N(u) \rightarrow u \in X)\}$, $\overline{\mathcal{C}_5}(X) = \cup\{N(x) \mid x \in X\}$.

$\underline{\mathcal{C}_1}$, $\underline{\mathcal{C}_2}$, $\underline{\mathcal{C}_3}$, $\underline{\mathcal{C}_4}$, and $\underline{\mathcal{C}_5}$ are called the first, the second, the third, the fourth, and the fifth neighborhood-based lower approximation operations with respect to $\mathcal{C}$, respectively. $\overline{\mathcal{C}_1}$, $\overline{\mathcal{C}_2}$, $\overline{\mathcal{C}_3}$, $\overline{\mathcal{C}_4}$, and $\overline{\mathcal{C}_5}$ are called the first, the second, the third, the fourth, and the fifth neighborhood-based upper approximation operations with respect to $\mathcal{C}$, respectively.

This paper is concerned with the list of five definitions of approximations (Definition 2.4). In fact, the above definition can be extended. For definitions of dual approximations and many other approximations look at [29].

Note 1. In [24], $\overline{\mathcal{C}_1}(X)$ is denoted by $\cap\{\sim K \mid K \in \mathcal{C}, K \cap X = \emptyset\}$, that is, $\overline{\mathcal{C}_1}(X) = \cap\{\sim K \mid K \in \mathcal{C}, K \cap X = \emptyset\}$. This is not accurate. For example, let $U = \{x_1, x_2, x_3\}$, $C_1 = \{x_1, x_2\}$, $C_2 = \{x_3\}$, and $\mathcal{C} = \{C_1, C_2\}$. Clearly, $\mathcal{C}$ is a covering of U and $N(x_1) = \{x_1, x_2\} = N(x_2)$ and $N(x_3) = \{x_3\}$. Taking $X = \{x_1, x_3\}$, since $C_1 \cap X \neq \emptyset$ and $C_2 \cap X \neq \emptyset$, it follows that $\cap\{\sim K \mid K \in \mathcal{C}, K \cap X = \emptyset\} = \emptyset$. However, it is easy to see that $\sim \underline{\mathcal{C}_1}(\sim X) = U$. Hence, $\overline{\mathcal{C}_1}(X) \neq \sim \underline{\mathcal{C}_1}(\sim X)$. This contradicts the fact that $\overline{\mathcal{C}_1}$ and $\underline{\mathcal{C}_1}$ are dual with each other. In above definition, we denote $\overline{\mathcal{C}_1}(X)$ by $\sim (\cup\{K \in \mathcal{C} \mid K \subseteq \sim X\})$.

3. Minimal Neighborhood Description and Maximal Neighborhood Description

In this section, we define the concepts of minimal neighborhood description and maximal neighborhood description of an element. And we show that the two notions play essential roles in the studies of neighborhood-based rough sets.

Now we give the definitions of minimal neighborhood description and maximal neighborhood description related to an element.

Definition 3.1 (Minimal neighborhood description). Let $(U, \mathcal{C})$ be a covering approximation space and $x \in U$. The family of sets

$$N\mathrm{mid}_{\mathcal{C}}(x) = \left\{N(u) \mid u \in N(x) \wedge \left(\text{for all } y \in U \wedge N(y) \subseteq N(u) \Longrightarrow N(y) = N(u)\right)\right\} \tag{3.1}$$

is called the minimal neighborhood description of the element x. When there is no confusion, we omit the subscript $\mathcal{C}$.

By above definition, it is easy to see that every element in $N\mathrm{mid}(x)$ is a minimal neighborhood contained in $N(x)$.

Definition 3.2 (Maximal neighborhood description). Let $(U, \mathcal{C})$ be a covering approximation space and $x \in U$. The family of sets

$$N\mathrm{mad}_{\mathcal{C}}(x) = \{N(u) \mid x \in N(u) \wedge (\text{for all } y \in U \wedge N(u) \subseteq N(y) \Longrightarrow N(u) = N(y))\} \tag{3.2}$$

is called the maximal neighborhood description of the element x. When there is no confusion, we omit the subscript $\mathcal{C}$.

By above definition, it is easy to see that every element in $N\mathrm{mad}(x)$ is a maximal neighborhood containing x.

In order to describe an object, we need only the essential characteristics related to this object, not all the characteristics for this object. That are the purposes of the minimal neighborhood description and the maximal neighborhood description concepts.

For better understanding of Definitions 3.1 and 3.2, we illustrate them by the following example.

Example 3.3. Let $U = \{x_1, x_2, x_3, x_4\}$, $C_1 = \{x_1\}$, $C_2 = \{x_2\}$, $C_3 = \{x_1, x_2, x_3\}$, $C_4 = \{x_1, x_2, x_4\}$, and $\mathcal{C} = \{C_1, C_2, C_3, C_4\}$. Clearly, $\mathcal{C}$ is a covering of U. It is easy to check that $N(x_1) = \{x_1\}$, $N(x_2) = \{x_2\}$, $N(x_3) = \{x_1, x_2, x_3\}$, and $N(x_4) = \{x_1, x_2, x_4\}$. By Definitions 3.1 and 3.2, we can get that $N\mathrm{mid}(x_1) = \{N(x_1)\}$, $N\mathrm{mid}(x_2) = \{N(x_2)\}$, $N\mathrm{mid}(x_3) = \{N(x_1), N(x_2)\} = N\mathrm{mid}(x_4)$ and $N\mathrm{mad}(x_1) = \{N(x_3), N(x_4)\} = N\mathrm{mad}(x_2)$, $N\mathrm{mad}(x_3) = \{N(x_3)\}$, and $N\mathrm{mad}(x_4) = \{N(x_4)\}$.

Remark 3.4. Based on the above analysis, we know that every element in $N\mathrm{mid}_{\mathcal{C}}(x)$ is a neighborhood of covering approximation space $(U, \mathcal{C})$. Hence, for convenience, in this paper, we may use for all $N_{\mathcal{C}}(u) \in N\mathrm{mid}_{\mathcal{C}}(x)$ to express any element belonging to $N\mathrm{mid}_{\mathcal{C}}(x)$. Similarly, we may use for all $N_{\mathcal{C}}(u) \in N\mathrm{mad}_{\mathcal{C}}(x)$ to express any element belonging to $N\mathrm{mad}_{\mathcal{C}}(x)$.

3.1. The Third Type of Neighborhood-Based Rough Sets and the Minimal Neighborhood Description

In the following, we will employ the concept of minimal neighborhood description to characterize the third type of neighborhood-based rough sets. Firstly, we introduce a lemma.

Lemma 3.5. *Let $(U, \mathcal{C})$ be a covering approximation space and $x \in U$. Then for all $u \in N(x)$, there exists $N(z) \in N\mathrm{mid}(x)$ such that $N(z) \subseteq N(u)$.*

Proof. Since $\mathcal{C}$ is a finite covering of U, it follows from Definition 2.3 that the set $\{N(u) \mid u \in U\}$ has only finite elements. We will use this fact to prove the lemma.

Let $u \in N(x)$. Then $N(u) \subseteq N(x)$. Assume that for all $N(z) \in N\mathrm{mid}(x)$, $N(z) \nsubseteq N(u)$. Then $N(u) \notin N\mathrm{mid}(x)$, hence by Definition 3.1, $\exists u_1 \in U$, $N(u_1) \subset N(u)$. By $N(u_1) \subset N(u)$ and the assumption, we have $N(u_1) \notin N\mathrm{mid}(x)$. Clearly $u_1 \in N(x)$, so again by Definition 3.1, $\exists u_2 \in U$, $N(u_2) \subset N(u_1)$. Hence $N(u_2) \subseteq N(u)$. By the assumption, $N(u_2) \notin N\mathrm{mid}(x)$. Clearly, $u_2 \in N(x)$, so again by Definition 3.1, $\exists u_3 \in U$, $N(u_3) \subset N(u_2) \cdots$. Continue in this way, we have an infinite sequence $N(u), N(u_1), N(u_2), \ldots, N(u_m), \ldots$ in

$(U, \mathcal{C})$ such that $N(u) \supset N(u_1) \supset N(u_2) \supset \cdots \supset N(u_m) \supset \cdots$. But it is impossible since the set $\{N(u) \mid u \in U\}$ has only finite elements. This completes the proof. □

Remark 3.6. Since for all $x \in U$, $x \in N(x)$, it follows from Lemma 3.5 that there exists $N(z) \in N\mathrm{mid}(x)$ such that $N(z) \subseteq N(x)$. This implies that for all $x \in U$, $N\mathrm{mid}(x) \neq \emptyset$.

Theorem 3.7. *Let $(U, \mathcal{C})$ be a covering approximation space. Then for $X \subseteq U$, $\underline{C_3}(X) = \{x \in U \mid \exists N(u) \in N\mathrm{mid}(x), N(u) \subseteq X\}$, $\overline{C_3}(X) = \{x \in U \mid$ for all $N(u) \in N\mathrm{mid}(x), N(u) \cap X \neq \emptyset\}$.*

Proof. Let $X \subseteq U$. We first show that $\underline{C_3}(X) = \{x \in U \mid \exists N(u) \in N\mathrm{mid}(x), N(u) \subseteq X\}$. For all $x \in \underline{C_3}(X)$, by the part (3) of Definition 2.4, we have that $\exists u \in N(x)$, $N(u) \subseteq X$. By Lemma 3.5, there exists $N(z) \in N\mathrm{mid}(x)$ such that $N(z) \subseteq N(u)$. This implies that $N(z) \subseteq X$. It follows that $x \in \{x \in U \mid \exists N(u) \in N\mathrm{mid}(x), N(u) \subseteq X\}$. Thus $\underline{C_3}(X) \subseteq \{x \in U \mid \exists N(u) \in N\mathrm{mid}(x), N(u) \subseteq X\}$. On the other hand, by Definitions 3.1 and 2.4, it is obvious that $\underline{C_3}(X) \supseteq \{x \in U \mid \exists N(u) \in N\mathrm{mid}(x), N(u) \subseteq X\}$. Hence $\underline{C_3}(X) = \{x \in U \mid \exists N(u) \in N\mathrm{mid}(x), N(u) \subseteq X\}$.

We have proved that $\underline{C_3}(X) = \{x \in U \mid \exists N(u) \in N\mathrm{mid}(x), N(u) \subseteq X\}$. By Definition 2.4, we know that $\underline{C_3}$ and $\overline{C_3}$ are dual with each other. Thus $\overline{C_3}(X) = \sim \underline{C_3}(\sim X) = \sim \{x \in U \mid \exists N(u) \in N\mathrm{mid}(x), N(u) \subseteq \sim X\} = \sim \{x \in U \mid \exists N(u) \in N\mathrm{mid}(x), N(u) \cap X = \emptyset\} = \{x \in U \mid$ for all $N(u) \in N\mathrm{mid}(x), N(u) \cap X \neq \emptyset\}$, that is, $\overline{C_3}(X) = \{x \in U \mid$ for all $N(u) \in N\mathrm{mid}(x), N(u) \cap X \neq \emptyset\}$. □

The above theorem establishes the relationship between the third type of neighborhood-based rough sets and the notion of minimal neighborhood description. In order to study further the third type of neighborhood-based rough sets, we will explore the properties of minimal neighborhood description.

Proposition 3.8. *Let $(U, \mathcal{C})$ be a covering approximation space and $x \in U$. Then for all $N(u) \in N\mathrm{mid}(x)$, $N(u) \subseteq N(x)$.*

Proof. For all $N(u) \in N\mathrm{mid}(x)$, by Definition 3.1, it is clear that $u \in N(x)$. Thus by Definition 2.3, we conclude that $N(u) \subseteq N(x)$. □

Proposition 3.9. *Let $(U, \mathcal{C})$ be a covering approximation space, $x \in U$ and $N(u) \in N\mathrm{mid}(x)$. Then for all $z \in N(u)$, $N(u) = N(z)$.*

Proof. Let $z \in N(u)$. Then by Definition 2.3, we have that $N(z) \subseteq N(u)$. By $N(u) \in N\mathrm{mid}(x)$ and Definition 3.1, it is clear that $N(u) = N(z)$. □

The above proposition shows that every element in $N\mathrm{mid}(x)$ is a minimal one.

Proposition 3.10. *Let $(U, \mathcal{C})$ be a covering approximation space, $x \in U$ and $N(u) \in N\mathrm{mid}(x)$. If $u \in N(z)$ for $z \in U$, then $N(u) \in N\mathrm{mid}(z)$.*

Proof. Let $z \in U$ and $u \in N(z)$. Suppose that $N(u) \notin N\mathrm{mid}(z)$. Then by Definition 3.1, we have that there exists $y \in U$ such that $N(y) \subset N(u)$. Thus $y \in N(u)$. By $N(u) \in N\mathrm{mid}(x)$ and Proposition 3.9, this implies that $N(y) = N(u)$, which contradicts the fact that $N(y) \subset N(u)$. Hence $N(u) \in N\mathrm{mid}(z)$. □

Proposition 3.11. *Let $(U, \mathcal{C})$ be a covering approximation space and $x \in U$. Then for all $N(u) \in N\mathrm{mid}(x)$, $N\mathrm{mid}(u) = \{N(u)\}$.*

Proof. Let $N(u) \in N\mathrm{mid}(x)$. Clearly, $u \in N(u)$, it follows from Proposition 3.10 that $N(u) \in N\mathrm{mid}(u)$ and so $\{N(u)\} \subseteq N\mathrm{mid}(u)$. On the other hand, for all $N(z) \in N\mathrm{mid}(u)$, by Proposition 3.8, we conclude that $N(z) \subseteq N(u)$ and so $z \in N(u)$. It follows from Proposition 3.9 that $N(z) = N(u)$ and thus $N(z) \in \{N(u)\}$. Hence $N\mathrm{mid}(u) \subseteq \{N(u)\}$. In summary, $N\mathrm{mid}(u) = \{N(u)\}$. □

In the following, we will use the above properties to study the third type of neighborhood-based generalized rough sets.

The following example shows that two distinct coverings can generate the same neighborhood-based lower and upper approximation (the third type of neighborhood-based rough sets).

Example 3.12 (Two different coverings generate the same the third type of neighborhood-based rough sets). Let $U = \{x, y, z\}$, $C_1 = \{x\}$, $C_2 = \{x, y\}$, $C_3 = \{x, z\}$, $C_4 = \{x, y, z\}$, $\mathcal{B} = \{C_1, C_4\}$, and $\mathcal{C} = \{C_1, C_2, C_3, C_4\}$. Clearly, $\mathcal{B}$ and $\mathcal{C}$ are two different coverings of U. Then by Definition 2.3, we can get that $N_{\mathcal{B}}(x) = \{x\}$, $N_{\mathcal{B}}(y) = \{x, y, z\} = N_{\mathcal{B}}(z)$ and $N_{\mathcal{C}}(x) = \{x\}$, $N_{\mathcal{C}}(y) = \{x, y\}$, $N_{\mathcal{C}}(z) = \{x, z\}$. By Definition 3.1, it is easy to check that $N\mathrm{mid}_{\mathcal{B}}(x) = N\mathrm{mid}_{\mathcal{B}}(y) = N\mathrm{mid}_{\mathcal{B}}(z) = \{N_{\mathcal{B}}(x)\} = \{\{x\}\}$ and $N\mathrm{mid}_{\mathcal{C}}(x) = N\mathrm{mid}_{\mathcal{C}}(y) = N\mathrm{mid}_{\mathcal{C}}(z) = \{N_{\mathcal{C}}(x)\} = \{\{x\}\}$. Hence $N\mathrm{mid}_{\mathcal{B}}(x) = N\mathrm{mid}_{\mathcal{C}}(x)$, $N\mathrm{mid}_{\mathcal{B}}(y) = N\mathrm{mid}_{\mathcal{C}}(y)$, and $N\mathrm{mid}_{\mathcal{B}}(z) = N\mathrm{mid}_{\mathcal{C}}(z)$. Thus by Theorem 3.7, it is easy to see that for all $X \subseteq U$, $\underline{\mathcal{B}_3}(X) = \underline{\mathcal{C}_3}(X)$, and $\overline{\mathcal{B}_3}(X) = \overline{\mathcal{C}_3}(X)$.

Now we present the conditions under which two coverings generate the same the third type of neighborhood-based rough sets.

Theorem 3.13. *Let $\mathcal{B}$ and $\mathcal{C}$ be two coverings of a nonempty set U. Then for all $X \subseteq U$, $\underline{\mathcal{B}_3}(X) = \underline{\mathcal{C}_3}(X)$ and $\overline{\mathcal{B}_3}(X) = \overline{\mathcal{C}_3}(X)$ if and only if for all $x \in U$, $N\mathrm{mid}_{\mathcal{B}}(x) = N\mathrm{mid}_{\mathcal{C}}(x)$.*

In order to prove the theorem, we first introduce a lemma.

Lemma 3.14. *Let $(U, \mathcal{C})$ be a covering approximation space. Then for all $y \in U$,*

$$\overline{\mathcal{C}_3}(\{y\}) = \{x \in U \mid N\mathrm{mid}(x) = \{N(y)\}\}. \tag{3.3}$$

Proof. By Theorem 3.7, we have that $\overline{\mathcal{C}_3}(\{y\}) = \{x \in U \mid \text{for all } N(u) \in N\mathrm{mid}(x), N(u) \cap \{y\} \neq \emptyset\} = \{x \in U \mid \text{for all } N(u) \in N\mathrm{mid}(x), y \in N(u)\}$. By Proposition 3.9, we conclude that $\{x \in U \mid \text{for all } N(u) \in N\mathrm{mid}(x), y \in N(u)\} = \{x \in U \mid \text{for all } N(u) \in N\mathrm{mid}(x), \text{ and } N(y) = N(u)\} = \{x \in U \mid N\mathrm{mid}(x) = \{N(y)\}\}$. Thus $\overline{\mathcal{C}_3}(\{y\}) = \{x \in U \mid N\mathrm{mid}(x) = \{N(y)\}\}$. □

Now we prove our theorem.

Proof. The sufficiency follows directly from Theorem 3.7.

Conversely, let $x \in U$. For all $N_{\mathcal{B}}(u) \in N\mathrm{mid}_{\mathcal{B}}(x)$.

(i) We first show that $N_{\mathcal{C}}(u) \in N\mathrm{mid}_{\mathcal{C}}(u)$. By Proposition 3.11, we have that $N\mathrm{mid}_{\mathcal{B}}(u) = \{N_{\mathcal{B}}(u)\}$. Thus by Lemma 3.14, $u \in \overline{\mathcal{B}_3}(\{u\})$. Applying the condition for all $X \subseteq U$, $\overline{\mathcal{B}_3}(X) = \overline{\mathcal{C}_3}(X)$, we can have that $\overline{\mathcal{B}_3}(\{u\}) = \overline{\mathcal{C}_3}(\{u\})$. Thus $u \in \overline{\mathcal{C}_3}(\{u\})$. It follows from Lemma 3.14 that $N\mathrm{mid}_{\mathcal{C}}(u) = \{N_{\mathcal{C}}(u)\}$. Therefore, $N_{\mathcal{C}}(u) \in N\mathrm{mid}_{\mathcal{C}}(u)$.

(ii) We will show that $N_{\mathcal{B}}(u) = N_{\mathcal{C}}(u)$. For all $v \in N_{\mathcal{B}}(u)$, by Proposition 3.9, we have that $N_{\mathcal{B}}(v) = N_{\mathcal{B}}(u)$. By Lemma 3.14, this implies that $\overline{\mathcal{B}_3}(\{v\}) = \overline{\mathcal{B}_3}(\{u\})$. In addition, applying the condition for all $X \subseteq U$, $\overline{\mathcal{B}_3}(X) = \overline{\mathcal{C}_3}(X)$, we can get that $\overline{\mathcal{B}_3}(\{u\}) = \overline{\mathcal{C}_3}(\{u\})$ and $\overline{\mathcal{B}_3}(\{v\}) = \overline{\mathcal{C}_3}(\{v\})$. It follows that $\overline{\mathcal{C}_3}(\{v\}) = \overline{\mathcal{C}_3}(\{u\})$. By (i) and Proposition 3.11, $N\mathrm{mid}_{\mathcal{C}}(u) = \{N_{\mathcal{C}}(u)\}$. Thus by Lemma 3.14, we have that $u \in \overline{\mathcal{C}_3}(\{u\})$ and so $u \in \overline{\mathcal{C}_3}(\{v\})$. According to Lemma 3.14, this implies that $N\mathrm{mid}_{\mathcal{C}}(u) = \{N_{\mathcal{C}}(v)\}$. It follows from Proposition 3.8 that $N_{\mathcal{C}}(v) \subseteq N_{\mathcal{C}}(u)$ and thus $v \in N_{\mathcal{C}}(u)$. Therefore, $N_{\mathcal{B}}(u) \subseteq N_{\mathcal{C}}(u)$. On the other hand, the proof of $N_{\mathcal{B}}(u) \supseteq N_{\mathcal{C}}(u)$ is similar to that of $N_{\mathcal{B}}(u) \subseteq N_{\mathcal{C}}(u)$. Therefore $N_{\mathcal{B}}(u) = N_{\mathcal{C}}(u)$.

(iii) We will show that $N_{\mathcal{C}}(u) \in N\mathrm{mid}_{\mathcal{C}}(x)$. By the condition, we have that $\underline{\mathcal{B}_3}(N_{\mathcal{B}}(u)) = \underline{\mathcal{C}_3}(N_{\mathcal{B}}(u))$. By (ii), we have that $N_{\mathcal{C}}(u) = N_{\mathcal{B}}(u)$. Thus $\underline{\mathcal{B}_3}(N_{\mathcal{B}}(u)) = \underline{\mathcal{C}_3}(N_{\mathcal{C}}(u))$. By Theorem 3.7, it is clear that $x \in \underline{\mathcal{B}_3}(N_{\mathcal{B}}(u))$. Thus $x \in \underline{\mathcal{C}_3}(N_{\mathcal{C}}(u))$. By Theorem 3.7, this implies that $\exists N_{\mathcal{C}}(z) \in N\mathrm{mid}_{\mathcal{C}}(x)$ such that $N_{\mathcal{C}}(z) \subseteq N_{\mathcal{C}}(u)$ and so $z \in N_{\mathcal{C}}(u)$. By (i) and Proposition 3.9, we can conclude that $N_{\mathcal{C}}(z) = N_{\mathcal{C}}(u)$. This implies that $N_{\mathcal{C}}(u) \in N\mathrm{mid}_{\mathcal{C}}(x)$.

By (ii) and (iii), we have that $N_{\mathcal{B}}(u) \in N\mathrm{mid}_{\mathcal{C}}(x)$. Thus $N\mathrm{mid}_{\mathcal{B}}(x) \subseteq N\mathrm{mid}_{\mathcal{C}}(x)$. In the same way, we can prove that $N\mathrm{mid}_{\mathcal{B}}(x) \supseteq N\mathrm{mid}_{\mathcal{C}}(x)$. It follows that $N\mathrm{mid}_{\mathcal{B}}(x) = N\mathrm{mid}_{\mathcal{C}}(x)$. This completes the proof of the necessity. □

For the covering $\mathcal{C}$ of U, since the lower approximation $\underline{\mathcal{C}_3}$ and the upper approximation $\overline{\mathcal{C}_3}$ are dual, they determine each other. That is to say, for two coverings $\mathcal{B}$ and $\mathcal{C}$, $\underline{\mathcal{B}_3} = \underline{\mathcal{C}_3}$ if and only if $\overline{\mathcal{B}_3} = \overline{\mathcal{C}_3}$. From the above analysis and Theorem 3.13, we can obtain the following two corollaries.

Corollary 3.15. *Let $\mathcal{B}$ and $\mathcal{C}$ be two coverings of a nonempty set U. Then for all $X \subseteq U$, $\underline{\mathcal{B}_3}(X) = \underline{\mathcal{C}_3}(X)$ if and only if for all $y \in U$, $N\mathrm{mid}_{\mathcal{B}}(y) = N\mathrm{mid}_{\mathcal{C}}(y)$.*

Corollary 3.16. *Let $\mathcal{B}$ and $\mathcal{C}$ be two coverings of a nonempty set U. Then for all $X \subseteq U$, $\overline{\mathcal{B}_3}(X) = \overline{\mathcal{C}_3}(X)$ if and only if for all $y \in U$, $N\mathrm{mid}_{\mathcal{B}}(y) = N\mathrm{mid}_{\mathcal{C}}(y)$.*

Theorem 3.13 is an important result for studying the covering reduction of the third type of neighborhood-based rough sets. In Section 4.1, we will present the concept of reduct based on this theorem for the third type of rough set model.

3.2. The Fourth Type of Neighborhood-Based Rough Sets and the Maximal Neighborhood Description

In this subsection, we will study the relationship between the fourth type of neighborhood-based rough sets and the notion of maximal neighborhood description. For this purpose, we first explore the properties of maximal neighborhood description.

Proposition 3.17. *Let $(U, \mathcal{C})$ be a covering approximation space and $x \in U$. Then for all $N(u) \in N\mathrm{mad}(x)$, $N(x) \subseteq N(u)$.*

Proof. Let $N(u) \in N\mathrm{mad}(x)$. By Definition 3.2, we have that $x \in N(u)$. Thus by Definition 2.3, $N(x) \subseteq N(u)$. □

Proposition 3.18. *Let $(U, \mathcal{C})$ be a covering approximation space and $x, y \in U$. If $x \in N(y)$, then there exists $N(u) \in N\mathrm{mad}(x)$ such that $N(y) \subseteq N(u)$.*

Proof. Since $\mathcal{C}$ is a finite covering of U, it follows from Definition 2.3 that the set $\{N(u) \mid u \in U\}$ has only finite elements. We will use this fact to prove the proposition.

Let $x \in N(y)$. Assume that for all $N(u) \in N\mathrm{mad}(x)$, $N(y) \nsubseteq N(u)$. Then $N(y) \notin N\mathrm{mad}(x)$, hence by Definition 3.2, $\exists u_1 \in U$, $N(y) \subset N(u_1)$. By the assumption, we have that $N(u_1) \notin N\mathrm{mad}(x)$. Clearly, $x \in N(u_1)$, so again by Definition 3.2, $\exists u_2 \in U$, $N(u_1) \subset N(u_2)$. Thus $N(y) \subseteq N(u_2)$. By the assumption, $N(u_2) \notin N\mathrm{mad}(x)$. Clearly, $x \in N(u_2)$, so again by Definition 3.2, $\exists u_3 \in U$, $N(u_2) \subset N(u_3) \cdots$. Continue in this way, we have an infinite sequence $N(y), N(u_1), N(u_2), \ldots, N(u_m), \ldots$ in $(U, \mathcal{C})$ such that $N(y) \subset N(u_1) \subset N(u_2) \subset \cdots \subset N(u_m) \subset \cdots$. But it is impossible since the set $\{N(u) \mid u \in U\}$ has only finite elements. This completes the proof. □

By the above proposition, we can easily conclude the following result.

Corollary 3.19. *Let $(U, \mathcal{C})$ be a covering approximation space and $x, y \in U$. If $x \in N(y)$, then $N(y) \subseteq \cup N\mathrm{mad}(x)$.*

Remark 3.20. Since for all $x \in U$, $x \in N(x)$, it follows from Proposition 3.18 that there exists $N(u) \in N\mathrm{mad}(x)$ such that $N(x) \subseteq N(u)$. This implies that for all $x \in U$, $N\mathrm{mad}(x) \neq \emptyset$.

Proposition 3.21. *Let $(U, \mathcal{C})$ be a covering approximation space, $x \in U$, and $N(u) \in N\mathrm{mad}(x)$. If $y \in N(u)$ then $N(u) \in N\mathrm{mad}(y)$.*

Proof. Suppose that $N(u) \notin N\mathrm{mad}(y)$. Then by $y \in N(u)$ and Definition 3.2, there exists $y_0 \in U$ such that $N(u) \subset N(y_0)$, which contradicts with $N(u) \in N\mathrm{mad}(x)$. Thus $N(u) \in N\mathrm{mad}(y)$. □

Now we use the concept of maximal neighborhood description to characterize the fourth type of neighborhood-based rough sets.

Theorem 3.22. *Let $(U, \mathcal{C})$ be a covering approximation space. Then for $X \subseteq U$,*

$$\underline{\mathcal{C}_4}(X) = \{x \in U \mid \cup N\mathrm{mad}(x) \subseteq X\}, \qquad \overline{\mathcal{C}_4}(X) = \cup\{\cup N\mathrm{mad}(x) \mid x \in X\}. \tag{3.4}$$

Proof. Let $X \subseteq U$. We first show that $\underline{\mathcal{C}_4}(X) = \{x \in U \mid \cup N\mathrm{mad}(x) \subseteq X\}$. For all $x \in \underline{\mathcal{C}_4}(X)$, we will prove that $x \in \{x \in U \mid \cup N\mathrm{mad}(x) \subseteq X\}$. For all $N(u) \in N\mathrm{mad}(x)$, by Definition 3.2, $x \in N(u)$. Since $x \in \underline{\mathcal{C}_4}(X)$, it follows from the part (4) of Definition 2.4 that $N(u) \subseteq X$. Thus $\cup N\mathrm{mad}(x) \subseteq X$. This implies that $x \in \{x \in U \mid \cup N\mathrm{mad}(x) \subseteq X\}$. Thus $\underline{\mathcal{C}_4}(X) \subseteq \{x \in U \mid \cup N\mathrm{mad}(x) \subseteq X\}$. On the other hand, for all $x \in \{x \in U \mid \cup N\mathrm{mad}(x) \subseteq X\}$, we can get that $\cup N\mathrm{mad}(x) \subseteq X$. Further, for all $u \in U$ and $x \in N(u)$, by Corollary 3.19, we have that $N(u) \subseteq \cup N\mathrm{mad}(x)$. Thus $N(u) \subseteq X$. It follows from the part (4) of Definition 2.4 that $x \in \underline{\mathcal{C}_4}(X)$ and so $\{x \in U \mid \cup N\mathrm{mad}(x) \subseteq X\} \subseteq \underline{\mathcal{C}_4}(X)$. In summary, $\underline{\mathcal{C}_4}(X) = \{x \in U \mid \cup N\mathrm{mad}(x) \subseteq X\}$.

Now we show that $\overline{\mathcal{C}_4}(X) = \cup\{\cup N\mathrm{mad}(x) \mid x \in X\}$. For all $y \in \overline{\mathcal{C}_4}(X)$, by Definition 2.4, there exists $z \in U$ such that $N(z) \cap X \neq \emptyset$ and $y \in N(z)$. Taking $x \in N(z) \cap X$, by Corollary 3.19, we have that $N(z) \subseteq \cup N\text{mad}(x)$ and $x \in X$. Thus $N(z) \subseteq \cup\{\cup N\text{mad}(x) \mid x \in X\}$ and so $y \in \cup\{\cup N\text{mad}(x) \mid x \in X\}$. Therefore, $\overline{C_4}(X) \subseteq \cup\{\cup N\text{mad}(x) \mid x \in X\}$. On the other hand, for all $y \in \cup\{\cup N\text{mad}(x) \mid x \in X\}$, there exists $x \in X$ such that $y \in \cup N\text{mad}(x)$. Thus $\exists N(u) \in N\text{mad}(x)$, s.t. $y \in N(u)$. By Definition 3.2, $x \in N(u)$. Hence there exists $N(u)$ such that $x \in N(u) \cap X \neq \emptyset$ and $y \in N(u)$. It follows from Definition 2.4 that $y \in \overline{C_4}(X)$ and so $\cup\{\cup N\text{mad}(x) \mid x \in X\} \subseteq \overline{C_4}(X)$. In summary, $\overline{C_4}(X) = \cup\{\cup N\text{mad}(x) \mid x \in X\}$.

This completes the proof of theorem. □

The following example shows that two different coverings can induce the same the fourth type of neighborhood-based lower and upper approximation operations.

Example 3.23 (Two different coverings generate the same the fourth type of neighborhood-based rough sets). Let $U = \{x_1, x_2, x_3, x_4\}$, $C_1 = \{x_1\}$, $C_2 = \{x_1, x_2\}$, $C_3 = \{x_3, x_4\}$, $\mathcal{B} = \{C_1, C_2, C_3\}$, and $\mathcal{C} = \{C_2, C_3\}$. Clearly, $\mathcal{B}$ and $\mathcal{C}$ are two different coverings of U. By Definition 2.3, it clear that $N_{\mathcal{B}}(x_1) = \{x_1\}$, $N_{\mathcal{B}}(x_2) = \{x_1, x_2\}$, $N_{\mathcal{B}}(x_3) = \{x_3, x_4\} = N_{\mathcal{B}}(x_4)$, and $N_{\mathcal{C}}(x_1) = \{x_1, x_2\} = N_{\mathcal{C}}(x_2)$, $N_{\mathcal{C}}(x_3) = \{x_3, x_4\} = N_{\mathcal{C}}(x_4)$. It is easy to check that $N\text{mad}_{\mathcal{B}}(x_1) = N\text{mad}_{\mathcal{B}}(x_2) = \{N_{\mathcal{B}}(x_2)\} = \{x_1, x_2\}$, $N\text{mad}_{\mathcal{B}}(x_3) = N\text{mad}_{\mathcal{B}}(x_4) = \{N_{\mathcal{B}}(x_3)\} = \{x_3, x_4\}$ and $N\text{mad}_{\mathcal{C}}(x_1) = N\text{mad}_{\mathcal{C}}(x_2) = \{N_{\mathcal{C}}(x_2)\} = \{x_1, x_2\}$, $N\text{mad}_{\mathcal{C}}(x_3) = N\text{mad}_{\mathcal{C}}(x_3) = \{N_{\mathcal{C}}(x_4)\} = \{x_3, x_4\}$. Thus for all $x \in U$, $N\text{mad}_{\mathcal{B}}(x) = N\text{mad}_{\mathcal{C}}(x)$ and so for all $x \in U$, $\cup N\text{mad}_{\mathcal{B}}(x) = \cup N\text{mad}_{\mathcal{C}}(x)$. It follows by Theorem 3.22 that $\underline{\mathcal{B}_4} = \underline{C_4}$ and $\overline{\mathcal{B}_4} = \overline{C_4}$.

In the following, we study the conditions for two coverings generating the same fourth type of neighborhood-based lower and upper approximation operations. Firstly, we present two lemmas.

Lemma 3.24. *Let $\mathcal{C}$ be a covering of a nonempty set U and $x, y \in U$. If $N(u) \in N\text{mad}(x)$ and $y \notin N(u)$, then $u \notin \cup N\text{mad}(y)$.*

Proof. Suppose that $u \in \cup N\text{mad}(y)$. Then $\exists N(v) \in N\text{mad}(y)$, s.t. $u \in N(v)$. Thus by Definition 2.3, $N(u) \subseteq N(v)$. Since $N(u) \in N\text{mad}(x)$, it follows from Definition 3.2 that $N(u) = N(v)$. By $N(v) \in N\text{mad}(y)$ and Definition 3.2, it is clear that $y \in N(v)$. Thus $y \in N(u)$, which contradicts the condition $y \notin N(u)$. Therefore, $u \notin \cup N\text{mad}(y)$. □

Lemma 3.25. *Let $\mathcal{B}$ and $\mathcal{C}$ be two coverings of a nonempty set U. And $\mathcal{B}$ and $\mathcal{C}$ satisfy the condition that for all $x \in U$, $\cup N\text{mad}_{\mathcal{B}}(x) = \cup N\text{mad}_{\mathcal{C}}(x)$. If for $x \in U$, $N_{\mathcal{C}}(u) \in N\text{mad}_{\mathcal{C}}(x)$, $N_{\mathcal{B}}(v) \in N\text{mad}_{\mathcal{B}}(x)$, and $N_{\mathcal{C}}(u) \subseteq N_{\mathcal{B}}(v)$, then $N_{\mathcal{C}}(u) = N_{\mathcal{B}}(v)$.*

Proof. Suppose that $N_{\mathcal{C}}(u) \subset N_{\mathcal{B}}(v)$. Then, $u \in N_{\mathcal{B}}(v)$. Taking $w \in N_{\mathcal{B}}(v) - N_{\mathcal{C}}(u)$, that is, $w \in N_{\mathcal{B}}(v)$, $w \notin N_{\mathcal{C}}(u)$. Thus by $N_{\mathcal{B}}(v) \in N\text{mad}_{\mathcal{B}}(x)$ and Proposition 3.21, we have that $N_{\mathcal{B}}(v) \in N\text{mad}_{\mathcal{B}}(w)$ and so $u \in \cup N\text{mad}_{\mathcal{B}}(w)$. On the other hand, since $w \notin N_{\mathcal{C}}(u)$ and $N_{\mathcal{C}}(u) \in N\text{mad}_{\mathcal{C}}(x)$, it follows from Lemma 3.24 that $u \notin \cup N\text{mad}_{\mathcal{C}}(w)$. Thus $\cup N\text{mad}_{\mathcal{C}}(w) \neq \cup N\text{mad}_{\mathcal{B}}(w)$, which contradicts the condition for all $x \in U$, $\cup N\text{mad}_{\mathcal{B}}(x) = \cup N\text{mad}_{\mathcal{C}}(x)$. Therefore, $N_{\mathcal{C}}(u) = N_{\mathcal{B}}(v)$. □

Now we present the conditions under which the two different coverings generate the same fourth type of neighborhood-based upper approximation operation.

Theorem 3.26. *Let $\mathcal{B}$ and $\mathcal{C}$ be two coverings of a nonempty set U. Then the following assertions are equivalent:*

(1) *for all* $X \subseteq U$, $\overline{\mathcal{B}_4}(X) = \overline{C_4}(X)$,

(2) *for all* $x \in U$, $\cup N\mathrm{mad}_{\mathcal{B}}(x) = \cup N\mathrm{mad}_{\mathcal{C}}(x)$,

(3) *for all* $x \in U$, $N\mathrm{mad}_{\mathcal{B}}(x) = N\mathrm{mad}_{\mathcal{C}}(x)$,

(4) $\cup\{N\mathrm{mad}_{\mathcal{B}}(x) \mid x \in U\} = \cup\{N\mathrm{mad}_{\mathcal{C}}(x) \mid x \in U\}$.

Proof. (1)⇒(2) for all $x \in U$, by Theorem 3.22, we have that $\overline{\mathcal{B}_4}(\{x\}) = \cup N\mathrm{mad}_{\mathcal{B}}(x)$ and $\overline{\mathcal{C}_4}(\{x\}) = \cup N\mathrm{mad}_{\mathcal{C}}(x)$. Since for all $X \subseteq U$, $\overline{\mathcal{B}_4}(X) = \overline{\mathcal{C}_4}(X)$, it follows that $\overline{\mathcal{B}_4}(\{x\}) = \overline{\mathcal{C}_4}(\{x\})$. Thus $\cup N\mathrm{mad}_{\mathcal{B}}(x) = \cup N\mathrm{mad}_{\mathcal{C}}(x)$.

(2)⇒(1) It follows directly from Theorem 3.22.

(2)⇒(3) Let $x \in U$. By assertion (2), $\cup N\mathrm{mad}_{\mathcal{B}}(x) = \cup N\mathrm{mad}_{\mathcal{C}}(x)$. Thus, for all $N_{\mathcal{B}}(u) \in N\mathrm{mad}_{\mathcal{B}}(x)$, there exists $N_{\mathcal{C}}(v) \in N\mathrm{mad}_{\mathcal{C}}(x)$ such that $u \in N_{\mathcal{C}}(v)$. Suppose that $N_{\mathcal{B}}(u) \neq N_{\mathcal{C}}(v)$. Then by Lemma 3.25, we have that $N_{\mathcal{C}}(v) \nsubseteq N_{\mathcal{B}}(u)$. Thus there exists $w \in N_{\mathcal{C}}(v)$ such that $w \notin N_{\mathcal{B}}(u)$. By Proposition 3.21, $w \in N_{\mathcal{C}}(v)$ implies $N_{\mathcal{C}}(v) \in N\mathrm{mad}_{\mathcal{C}}(w)$ and so $u \in \cup N\mathrm{mad}_{\mathcal{C}}(w)$. In addition, by $w \notin N_{\mathcal{B}}(u)$ and Lemma 3.24, we have that $u \notin \cup N\mathrm{mad}_{\mathcal{B}}(w)$. Thus $\cup N\mathrm{mad}_{\mathcal{C}}(w) \neq \cup N\mathrm{mad}_{\mathcal{B}}(w)$, which is a contradiction with the assertion (2). Therefore, $N_{\mathcal{B}}(u) = N_{\mathcal{C}}(v)$ and so $N_{\mathcal{B}}(u) \in N\mathrm{mad}_{\mathcal{C}}(x)$. Thus $N\mathrm{mad}_{\mathcal{B}}(x) \subseteq N\mathrm{mad}_{\mathcal{C}}(x)$. In the same way, we can prove that $N\mathrm{mad}_{\mathcal{B}}(x) \supseteq N\mathrm{mad}_{\mathcal{C}}(x)$. Thus $N\mathrm{mad}_{\mathcal{B}}(x) = N\mathrm{mad}_{\mathcal{C}}(x)$.

(3)⇒(2) It is obvious.

(3)⇒(4) It is obvious.

(4)⇒(3) Let $x \in U$. For all $N_{\mathcal{B}}(u) \in N\mathrm{mad}_{\mathcal{B}}(x)$, clearly, $N_{\mathcal{B}}(u) \in \cup\{N\mathrm{mad}_{\mathcal{B}}(x) \mid x \in U\}$. Since $\cup\{N\mathrm{mad}_{\mathcal{B}}(x) \mid x \in U\} = \cup\{N\mathrm{mad}_{\mathcal{C}}(x) \mid x \in U\}$, there exists $N_{\mathcal{C}}(v) \in \cup\{N\mathrm{mad}_{\mathcal{C}}(x) \mid x \in U\}$ such that $N_{\mathcal{B}}(u) = N_{\mathcal{C}}(v)$. By $N_{\mathcal{B}}(u) \in N\mathrm{mad}_{\mathcal{B}}(x)$ and Definition 3.2, we have that $x \in N_{\mathcal{B}}(u)$. Thus $x \in N_{\mathcal{C}}(v)$. Since $N_{\mathcal{C}}(v) \in \cup\{N\mathrm{mad}_{\mathcal{C}}(x) \mid x \in U\}$, it follows that there exists $z \in U$ such that $N_{\mathcal{C}}(v) \in N\mathrm{mad}_{\mathcal{C}}(z)$. Thus again by Proposition 3.21, we can have that $N_{\mathcal{C}}(v) \in N\mathrm{mad}_{\mathcal{C}}(x)$. It follows from $N_{\mathcal{B}}(u) = N_{\mathcal{C}}(v)$ that $N_{\mathcal{B}}(u) \in N\mathrm{mad}_{\mathcal{C}}(x)$. Hence $N\mathrm{mad}_{\mathcal{B}}(x) \subseteq N\mathrm{mad}_{\mathcal{C}}(x)$. In the same way, we can prove that $N\mathrm{mad}_{\mathcal{B}}(x) \supseteq N\mathrm{mad}_{\mathcal{C}}(x)$. Thus $N\mathrm{mad}_{\mathcal{B}}(x) = N\mathrm{mad}_{\mathcal{C}}(x)$. □

For the covering $\mathcal{C}$ of U, since the lower approximation $\underline{\mathcal{C}_4}$ and the upper approximation $\overline{\mathcal{C}_4}$ are dual, they determine each other. That is, for two coverings $\mathcal{B}$ and $\mathcal{C}$, $\underline{\mathcal{B}_4} = \underline{\mathcal{C}_4}$ if and only if $\overline{\mathcal{B}_4} = \overline{\mathcal{C}_4}$. From the above analysis and Theorem 3.26, we can obtain the following results.

Corollary 3.27. *Let $\mathcal{B}$ and $\mathcal{C}$ be two coverings of a nonempty set U. Then the following assertions are equivalent:*

(1) *for all* $X \subseteq U$, $\underline{\mathcal{B}_4}(X) = \underline{\mathcal{C}_4}(X)$,

(2) *for all* $x \in U$, $\cup N\mathrm{mad}_{\mathcal{B}}(x) = \cup N\mathrm{mad}_{\mathcal{C}}(x)$,

(3) *for all* $x \in U$, $N\mathrm{mad}_{\mathcal{B}}(x) = N\mathrm{mad}_{\mathcal{C}}(x)$,

(4) $\cup\{N\mathrm{mad}_{\mathcal{B}}(x) \mid x \in U\} = \cup\{N\mathrm{mad}_{\mathcal{C}}(x) \mid x \in U\}$.

Now we present the conditions under which the two different coverings generate the same the fourth type of neighborhood-based rough sets.

Theorem 3.28. *Let $\mathcal{B}$ and $\mathcal{C}$ be two coverings of a nonempty set U. Then the following assertions are equivalent:*

(1) *for all* $X \subseteq U$, $\underline{\mathcal{B}_4}(X) = \underline{\mathcal{C}_4}(X)$ *and* $\overline{\mathcal{B}_4}(X) = \overline{\mathcal{C}_4}(X)$,

(2) *for all* $x \in U$, $\cup N\mathrm{mad}_{\mathcal{B}}(x) = \cup N\mathrm{mad}_{\mathcal{C}}(x)$,

(3) *for all* $y \in U$, $N\mathrm{mad}_{\mathcal{B}}(x) = N\mathrm{mad}_{\mathcal{C}}(x)$,

(4) $\cup\{N\mathrm{mad}_{\mathcal{B}}(x) \mid x \in U\} = \cup\{N\mathrm{mad}_{\mathcal{C}}(x) \mid x \in U\}$.

Proof. It follows directly from Theorem 3.26 and Corollary 3.27. □

The above theorem is an important result for studying the covering reduction of the fourth type of neighborhood-based rough sets. In Section 4.2, we will present the concept of reduct based on this theorem for the fourth type of rough set model.

4. Reduction of the Third and the Fourth Types of Neighborhood-Based Rough Sets

Examples 3.12 and 3.23 show that for a covering, it could still be a covering by dropping some of its members. Furthermore, the resulting new covering might still produce the same neighborhood-based lower and upper approximations. Hence, a covering may have "redundant" members, and a procedure is needed to find its "smallest" covering that induces the same neighborhood-based lower and upper approximations. This technique can be used to reduce the redundant information in data mining.

In this section, we will investigate the reduction issues about the third and the fourth types of neighborhood-based generalized rough sets. Since for a covering it could not be a covering by dropping some of its members, we need to extend the concepts of neighborhood, minimal neighborhood description, and maximal neighborhood description to a general family of subsets of a universe case so as to reasonably explore the covering reduction of neighborhood-based rough sets.

Let U be a nonempty set called the universe of discourse. The class of all subsets of U will be denoted by $P(U)$. Naturally, we present the definitions of generalization of neighborhood, minimal neighborhood description, and maximal neighborhood description.

Definition 4.1 (Neighborhood). Let U be a universe, $\mathcal{C} \subseteq P(U)$ and $x \in U$. $N_{\mathcal{C}}(x) = \cap\{K \in \mathcal{C} \mid x \in K\}$ is called the neighborhood of x. Generally, we omit the subscript $\mathcal{C}$ when there is no confusion.

Definition 4.2 (Minimal neighborhood description). Let U be a universe, $\mathcal{C} \subseteq P(U)$ and $x \in U$. The family of sets $N\mathrm{mid}_{\mathcal{C}}(x) = \{N(u) \mid u \in N(x) \wedge (\text{for all } y \in U \wedge N(y) \subseteq N(u) \Rightarrow N(y) = N(u))\}$ is called the minimal neighborhood description of the element x. When there is no confusion, we omit the subscript $\mathcal{C}$.

Definition 4.3 (Maximal neighborhood description). Let U be a universe, $\mathcal{C} \subseteq P(U)$ and $x \in U$. The family of sets $N\mathrm{mad}_{\mathcal{C}}(x) = \{N(u) \mid x \in N(u) \wedge (\text{for all } y \in U \wedge N(u) \subseteq N(y) \Rightarrow N(u) = N(y))\}$ is called the maximal neighborhood description of the element x. When there is no confusion, we omit the subscript $\mathcal{C}$.

It is easy to see that when $\mathcal{C}$ is a covering of U, Definitions 4.1, 4.2, and 4.3 are coincident with Definitions 2.3, 3.1, and 3.2, respectively.

4.1. Reduction of the Third Type of Neighborhood-Based Rough Sets

Throughout this subsection, we always assume that the rough set model which is discussed by us is the third type of neighborhood-based generalized rough sets. So the definitions of a reducible element, an irreducible covering, and a reduct are all based on the third type of neighborhood-based rough sets.

A reduct should be able to preserve the original classification power provided by the initial covering. In order to present a reasonablenotion of reduct, we first give the definition of a reducible element of a covering.

Definition 4.4 (A reducible element about the third type of lower and upper approximation operations). Let $\mathcal{C}$ be a covering of a universe U and $K \in \mathcal{C}$. If for all $x \in U$, $N\mathrm{mid}_{\mathcal{C}}(x) = N\mathrm{mid}_{\mathcal{C}-\{K\}}(x)$, we say that K is a reducible element of $\mathcal{C}$. Otherwise, K is an irreducible element of $\mathcal{C}$.

Definition 4.5 (Irreducible covering about the third type of lower and upper approximation operations). Let $\mathcal{C}$ be a covering of a universe U. If every element of $\mathcal{C}$ is an irreducible element, we say that $\mathcal{C}$ is irreducible. Otherwise, $\mathcal{C}$ is reducible.

Definition 4.6 (Reduct about the third type of lower and upper approximation operations). Let $\mathcal{C}$ be a covering of a universe U and $\mathcal{B} \subseteq \mathcal{C}$. If $\mathcal{B}$ is an irreducible covering and for all $x \in U$, $N\mathrm{mid}_{\mathcal{B}}(x) = N\mathrm{mid}_{\mathcal{C}}(x)$, we say that $\mathcal{B}$ is a reduct of $\mathcal{C}$. Let $\mathrm{red}_3(\mathcal{C}) = \{\mathcal{B} \mid \mathcal{B}$ is a reduct of $\mathcal{C}\}$.

In the following, we will illustrate that, for a covering, the reduct always exists and is not unique. Further, we will show that every reduct and the initial covering induce the same lower and upper approximation operations.

Firstly, we give an important proposition.

Proposition 4.7. *Let $\mathcal{C}$ be a covering of a universe U and $\mathcal{B} \subseteq \mathcal{C}$. If $\mathcal{B}$ and $\mathcal{C}$ satisfy the condition that for all $x \in U$, $N\mathrm{mid}_{\mathcal{B}}(x) = N\mathrm{mid}_{\mathcal{C}}(x)$, then $\mathcal{B}$ is a covering of U.*

Proof. Suppose that $\mathcal{B}$ is not a covering of U. Then $\cup\mathcal{B} \subset U$. Taking $x_0 \in U - \cup\mathcal{B}$, by Definition 4.1, we have that $N_{\mathcal{B}}(x_0) = \emptyset$. Thus by Definition 4.2, $N\mathrm{mid}_{\mathcal{B}}(x_0) = \emptyset$. On the other hand, since $\mathcal{C}$ is a covering of U, it follows from Remark 3.6 that $N\mathrm{mid}_{\mathcal{C}}(x_0) \neq \emptyset$. Thus $N\mathrm{mid}_{\mathcal{B}}(x_0) \neq N\mathrm{mid}_{\mathcal{C}}(x_0)$, which contradicts with the conditions for all $x \in U$, $N\mathrm{mid}_{\mathcal{B}}(x) = N\mathrm{mid}_{\mathcal{C}}(x)$. This completes the proof. □

Corollary 4.8. *Let $\mathcal{C}$ be a covering of a universe U and $K \in \mathcal{C}$. If K is a reducible element of $\mathcal{C}$, then $\mathcal{C} - \{K\}$ is still a covering of U.*

Proof. It comes directly from Definition 4.4 and Proposition 4.7. □

The following theorem shows that for a covering, there is at least one reduct.

Theorem 4.9. *Let $\mathcal{C}$ be a covering of a universe U. Then there exists $\mathcal{B} \subseteq \mathcal{C}$ such that $\mathcal{B}$ is a reduct of $\mathcal{C}$.*

Proof. Suppose that for all $\mathcal{B} \subseteq \mathcal{C}$, $\mathcal{B}$ is not a reduct of $\mathcal{C}$. Then $\mathcal{C}$ is not a reduct of $\mathcal{C}$. Thus by Definition 4.6, $\mathcal{C}$ is a reducible covering. This implies that there exists $K_1 \in \mathcal{C}$ such that K_1 is a reducible element of $\mathcal{C}$. We write $\mathcal{B}_1 = \mathcal{C} - \{K_1\}$. By Definition 4.4, we have that for all $x \in U$,

$N\mathrm{mid}_{\mathcal{B}_1}(x) = N\mathrm{mid}_{\mathcal{C}}(x)$. Further, by Corollary 4.8, $\mathcal{B}_1$ is a covering of U, and clearly, $\mathcal{B}_1 \subset \mathcal{C}$. Thus by the assumption, $\mathcal{B}_1$ is not a reduct of $\mathcal{C}$. Since for all $x \in U$, $N\mathrm{mid}_{\mathcal{B}_1}(x) = N\mathrm{mid}_{\mathcal{C}}(x)$, it follows from Definition 4.6 that $\mathcal{B}_1$ is a reducible covering and so there exists $K_2 \in \mathcal{B}_1$ such that K_2 is a reducible element of $\mathcal{B}_1$. We write $\mathcal{B}_2 = \mathcal{B}_1 - \{K_2\}$. By Definition 4.4, we have that for all $x \in U$, $N\mathrm{mid}_{\mathcal{B}_2}(x) = N\mathrm{mid}_{\mathcal{B}_1}(x)$. By for all $x \in U$, $N\mathrm{mid}_{\mathcal{B}_1}(x) = N\mathrm{mid}_{\mathcal{C}}(x)$, this implies that for all $x \in U$, $N\mathrm{mid}_{\mathcal{B}_2}(x) = N\mathrm{mid}_{\mathcal{C}}(x)$. Further, by Corollary 4.8, $\mathcal{B}_2$ is a covering of U, and clearly, $\mathcal{B}_2 \subset \mathcal{B}_1 \subset \mathcal{C}$, that is, $\mathcal{B}_2 \subset \mathcal{C}$. Thus by the assumption, $\mathcal{B}_2$ is not a reduct of $\mathcal{C} \cdots$. Continue in this way, we have an infinite sequence $\mathcal{C}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m, \ldots$ such that $\mathcal{C} \supset \mathcal{B}_1 \supset \mathcal{B}_2 \supset \cdots \supset \mathcal{B}_m \supset \cdots$. But it is impossible since the covering $\mathcal{C}$ has only finite elements. This completes the proof. □

Actually, the proving process of the above theorem provides a procedure to compute the reduct of a covering of a universe.

The following result shows that the definition of reduct is reasonable.

Theorem 4.10. *Let $\mathcal{C}$ be a covering of a universe U. Then for all $\mathcal{B} \in \mathrm{red}_3(\mathcal{C})$, $\mathcal{B}$ and $\mathcal{C}$ generate the same the third neighborhood-based lower and upper approximations.*

Proof. It follows directly from Definition 4.6, Proposition 4.7, and Theorem 3.13. □

Lemma 4.11. *Let $\mathcal{C}$ be a covering of a universe U and $\mathcal{B} \subseteq \mathcal{C}$. $\mathcal{B}$ and $\mathcal{C}$ satisfy the condition that for all $x \in U$, $N\mathrm{mid}_{\mathcal{B}}(x) = N\mathrm{mid}_{\mathcal{C}}(x)$. Then for all $x \in U$, for all $N_{\mathcal{C}}(u) \in N\mathrm{mid}_{\mathcal{C}}(x)$, we have that $N_{\mathcal{C}}(u) = N_{\mathcal{B}}(u)$ and $N_{\mathcal{B}}(u) \in N\mathrm{mid}_{\mathcal{B}}(x)$.*

Proof. Let $x \in U$. Then by the condition, $N\mathrm{mid}_{\mathcal{B}}(x) = N\mathrm{mid}_{\mathcal{C}}(x)$. Thus for all $N_{\mathcal{C}}(u) \in N\mathrm{mid}_{\mathcal{C}}(x)$, there exists $N_{\mathcal{B}}(v) \in N\mathrm{mid}_{\mathcal{B}}(x)$ such that $N_{\mathcal{C}}(u) = N_{\mathcal{B}}(v)$. Hence $u \in N_{\mathcal{B}}(v)$ and by Proposition 4.7, $\mathcal{B}$ is a covering of U. Thus, applying Proposition 3.9, we have that $N_{\mathcal{B}}(v) = N_{\mathcal{B}}(u)$. It follows that $N_{\mathcal{C}}(u) = N_{\mathcal{B}}(u)$ and $N_{\mathcal{B}}(u) \in N\mathrm{mid}_{\mathcal{B}}(x)$. □

Proposition 4.12. *Let $\mathcal{C}$ be a covering of a universe U and $\mathcal{B} \subset \mathcal{C}$. $\mathcal{B}$ and $\mathcal{C}$ satisfy the condition that for all $x \in U$, $N\mathrm{mid}_{\mathcal{B}}(x) = N\mathrm{mid}_{\mathcal{C}}(x)$. Then for all $K \in \mathcal{C} - \mathcal{B}$, K is a reducible element of $\mathcal{C}$.*

Proof. Let $K \in \mathcal{C} - \mathcal{B}$ and $x \in U$. Then $\mathcal{B} \subseteq \mathcal{C} - \{K\} \subseteq \mathcal{C}$.

(i) We will show that $\mathcal{C} - \{K\}$ and $\mathcal{B}$ are two coverings of U. By the condition and Proposition 4.7, $\mathcal{B}$ is a covering of U. Clearly, $\mathcal{B} \subseteq \mathcal{C} - \{K\}$, thus $\mathcal{C} - \{K\}$ is also a covering of U.

By (i) and the condition, we know that $\mathcal{C}$, $\mathcal{C} - \{K\}$, and $\mathcal{B}$ are all coverings of U. Hence, in the following process of proof, we can use directly the concepts and conclusions obtained in Section 3.

(ii) We will prove that for all $z \in U$, for all $N_{\mathcal{C}}(u) \in N\mathrm{mid}_{\mathcal{C}}(z)$, $N_{\mathcal{C}}(u) = N_{\mathcal{C}-\{K\}}(u)$. For all $N_{\mathcal{C}}(u) \in N\mathrm{mid}_{\mathcal{C}}(z)$, by the condition and Lemma 4.11, we have that $N_{\mathcal{C}}(u) = N_{\mathcal{B}}(u)$. Since $\mathcal{B} \subseteq \mathcal{C} - \{K\} \subseteq \mathcal{C}$, it follows by Definition 2.3 that $N_{\mathcal{C}}(u) \subseteq N_{\mathcal{C}-\{K\}}(u) \subseteq N_{\mathcal{B}}(u) = N_{\mathcal{C}}(u)$. Thus $N_{\mathcal{C}}(u) = N_{\mathcal{C}-\{K\}}(u)$.

(iii) We will show that $N\mathrm{mid}_{\mathcal{C}}(x) \subseteq N\mathrm{mid}_{\mathcal{C}-\{K\}}(x)$. For all $N_{\mathcal{C}}(u) \in N\mathrm{mid}_{\mathcal{C}}(x)$, then by Definition 3.1, $u \in N_{\mathcal{C}}(x)$. By $N_{\mathcal{C}}(x) \subseteq N_{\mathcal{C}-\{K\}}(x)$, we have that $u \in N_{\mathcal{C}-\{K\}}(x)$. Thus suppose that $N_{\mathcal{C}-\{K\}}(u) \notin N\mathrm{mid}_{\mathcal{C}-\{K\}}(x)$. Then by Definition 3.1, there exists $z \in U$ such that $N_{\mathcal{C}-\{K\}}(z) \subset N_{\mathcal{C}-\{K\}}(u)$. By (ii), we have that $N_{\mathcal{C}}(u) = N_{\mathcal{C}-\{K\}}(u)$, thus, $N_{\mathcal{C}-\{K\}}(z) \subset N_{\mathcal{C}}(u)$. Clearly, $N_{\mathcal{C}}(z) \subseteq N_{\mathcal{C}-\{K\}}(z)$, thus $N_{\mathcal{C}}(z) \subset N_{\mathcal{C}}(u)$, which contradicts the fact that $N_{\mathcal{C}}(u) \in N\mathrm{mid}_{\mathcal{C}}(x)$. Thus $N_{\mathcal{C}-\{K\}}(u) \in N\mathrm{mid}_{\mathcal{C}-\{K\}}(x)$. It follows by (ii) that $N_{\mathcal{C}}(u) \in N\mathrm{mid}_{\mathcal{C}-\{K\}}(x)$. Thus $N\mathrm{mid}_{\mathcal{C}}(x) \subseteq N\mathrm{mid}_{\mathcal{C}-\{K\}}(x)$.

(iv) We will show that $N\mathrm{mid}_{\mathcal{C}-\{K\}}(x) \subseteq N\mathrm{mid}_{\mathcal{C}}(x)$. For all $N_{\mathcal{C}-\{K\}}(u) \in N\mathrm{mid}_{\mathcal{C}-\{K\}}(x)$. Since $\mathcal{C} - \{K\} \subseteq \mathcal{C}$, it is clear that $N_{\mathcal{C}}(u) \subseteq N_{\mathcal{C}-\{K\}}(u)$. By $u \in N_{\mathcal{C}}(u)$ and Lemma 3.5, there exists $N_{\mathcal{C}}(z) \in N\mathrm{mid}_{\mathcal{C}}(u)$ such that $N_{\mathcal{C}}(z) \subseteq N_{\mathcal{C}}(u)$. This implies that $N_{\mathcal{C}}(z) \subseteq N_{\mathcal{C}-\{K\}}(u)$ and so $z \in N_{\mathcal{C}-\{K\}}(u)$. Thus by Proposition 3.9, $N_{\mathcal{C}-\{K\}}(z) = N_{\mathcal{C}-\{K\}}(u)$. Further, by $N_{\mathcal{C}}(z) \in N\mathrm{mid}_{\mathcal{C}}(u)$ and (ii), we have that $N_{\mathcal{C}}(z) = N_{\mathcal{C}-\{K\}}(z)$. Thus $N_{\mathcal{C}}(z) = N_{\mathcal{C}-\{K\}}(u)$. In addition, by Lemma 4.11, $N_{\mathcal{C}}(z) = N_{\mathcal{B}}(z)$ and $N_{\mathcal{B}}(z) \in N\mathrm{mid}_{\mathcal{B}}(u)$. Thus $N_{\mathcal{B}}(z) = N_{\mathcal{C}-\{K\}}(u)$ and so $u \in N_{\mathcal{B}}(z)$. It follows by Proposition 3.9 that $N_{\mathcal{B}}(z) = N_{\mathcal{B}}(u)$ and so $N_{\mathcal{C}-\{K\}}(u) = N_{\mathcal{B}}(u)$ and $N_{\mathcal{B}}(u) \in N\mathrm{mid}_{\mathcal{B}}(u)$. By Definition 3.1, it is clear that $u \in N_{\mathcal{C}-\{K\}}(x)$. Since $B \subseteq \mathcal{C} - \{K\}$, it follows that $N_{\mathcal{C}-\{K\}}(x) \subseteq N_{\mathcal{B}}(x)$. Thus $u \in N_{\mathcal{B}}(x)$. It follows by Proposition 3.10 that $N_{\mathcal{B}}(u) \in N\mathrm{mid}_{\mathcal{B}}(x)$ and so $N_{\mathcal{C}-\{K\}}(u) \in N\mathrm{mid}_{\mathcal{B}}(x)$. Thus, by the condition $N\mathrm{mid}_{\mathcal{B}}(x) = N\mathrm{mid}_{\mathcal{C}}(x)$, we conclude that $N_{\mathcal{C}-\{K\}}(u) \in N\mathrm{mid}_{\mathcal{C}}(x)$. It follows that $N\mathrm{mid}_{\mathcal{C}-\{K\}}(x) \subseteq N\mathrm{mid}_{\mathcal{C}}(x)$.

By (iii) and (iv), we have that $N\mathrm{mid}_{\mathcal{C}-\{K\}}(x) = N\mathrm{mid}_{\mathcal{C}}(x)$. We have proved that for all $x \in U$, $N\mathrm{mid}_{\mathcal{C}-\{K\}}(x) = N\mathrm{mid}_{\mathcal{C}}(x)$. Thus by Definition 4.4, K is a reducible element of $\mathcal{C}$. □

Corollary 4.13. *Let $\mathcal{C}$ be a covering of a universe U, $K \in \mathcal{C}$ a reducible element of $\mathcal{C}$ and $K_1 \in \mathcal{C} - \{K\}$. If K_1 is an irreducible element of $\mathcal{C}$, then K_1 is an irreducible element of $\mathcal{C} - \{K\}$.*

Proof. Suppose that K_1 is a reducible element of $\mathcal{C} - \{K\}$. Then by Definition 4.4, for all $x \in U$, $N\mathrm{mid}_{\mathcal{C}-\{K\}}(x) = N\mathrm{mid}_{\mathcal{C}-\{K,K_1\}}(x)$. In addition, since K is a reducible element of $\mathcal{C}$, it follows from Definition 4.4 that for all $x \in U$, $N\mathrm{mid}_{\mathcal{C}}(x) = N\mathrm{mid}_{\mathcal{C}-\{K\}}(x)$. Thus for all $x \in U$, $N\mathrm{mid}_{\mathcal{C}}(x) = N\mathrm{mid}_{\mathcal{C}-\{K,K_1\}}(x)$. Clearly, $K_1 \in \mathcal{C} - (\mathcal{C} - \{K, K_1\})$, thus by Proposition 4.12, K_1 is a reducible element of $\mathcal{C}$, which contradicts the condition that K_1 is an irreducible element of $\mathcal{C}$. This completes the proof. □

The above proposition guarantees that omitting a reducible element in a covering will not make any current irreducible element reducible. Therefore, the set of all irreducible elements of $\mathcal{C}$ is constant. We denote this set by $\mathrm{cor}_3(\mathcal{C})$, that is,

$$\mathrm{cor}_3(\mathcal{C}) = \{K \mid K \text{ is an irreducible element of } \mathcal{C}\}. \tag{4.1}$$

The following result establishes the relationship between $\mathrm{cor}_3(\mathcal{C})$ and $\mathrm{red}_3(\mathcal{C})$.

Theorem 4.14. *Let $\mathcal{C}$ be a covering of a universe U. Then $\mathrm{cor}_3(\mathcal{C}) = \cap\mathrm{red}_3(\mathcal{C})$.*

Proof. Let $K \in \mathrm{cor}_3(\mathcal{C})$. Suppose that $K \notin \cap\mathrm{red}_3(\mathcal{C})$. Then there exists $\mathcal{B} \in \mathrm{red}_3(\mathcal{C})$ such that $K \notin \mathcal{B}$. Hence $K \in \mathcal{C} - \mathcal{B}$. By Definition 4.6 and Proposition 4.12, this implies that K is a reducible element of $\mathcal{C}$, which contradicts the fact that $K \in \mathrm{cor}_3(\mathcal{C})$. Hence $K \in \cap\mathrm{red}_3(\mathcal{C})$ and so $\mathrm{cor}_3(\mathcal{C}) \subseteq \cap\mathrm{red}_3(\mathcal{C})$. On the other hand, let $K \in \cap\mathrm{red}_3(\mathcal{C})$. Suppose that $K \notin \mathrm{cor}_3(\mathcal{C})$. Then K is a reducible element of $\mathcal{C}$. By Definition 4.4, we have that for all $x \in U$, $N\mathrm{mid}_{\mathcal{C}}(x) = N\mathrm{mid}_{\mathcal{C}-\{K\}}(x)$ and by Corollary 4.8, $\mathcal{C} - \{K\}$ is a covering of U. Thus by Theorem 4.9, there exists $\mathcal{B} \subseteq \mathcal{C} - \{K\}$ such that $\mathcal{B}$ is a reduct of $\mathcal{C} - \{K\}$. By Definition 4.6, this implies that $\mathcal{B}$ is an irreducible covering and for all $x \in U$, $N\mathrm{mid}_{\mathcal{B}}(x) = N\mathrm{mid}_{\mathcal{C}-\{K\}}(x)$. Thus for all $x \in U$, $N\mathrm{mid}_{\mathcal{B}}(x) = N\mathrm{mid}_{\mathcal{C}}(x)$ and $\mathcal{B}$ is an irreducible covering. It follows from Definition 4.6 that $\mathcal{B}$

is a reduct of $\mathcal{C}$. Since $\mathcal{B} \subseteq \mathcal{C} - \{K\}$, it follows that $K \notin \mathcal{B}$, which contradicts the fact that $K \in \cap\mathrm{red}_3(\mathcal{C})$. Thus $K \in \mathrm{cor}_3(\mathcal{C})$, and so $\cap\mathrm{red}_3(\mathcal{C}) \subseteq \mathrm{cor}_3(\mathcal{C})$. In summary, $\mathrm{cor}_3(\mathcal{C}) = \cap\mathrm{red}_3(\mathcal{C})$. □

The above result states that an element will not be reduced in any reduction procedure if and only if it is irreducible. Hence the irreducible elements will be reserved in any reduction procedure, that is to say, $\mathrm{cor}_3(\mathcal{C})$ is contained in any reduct of $\mathcal{C}$. So we can compute the reduct of $\mathcal{C}$ based on $\mathrm{cor}_3(\mathcal{C})$.

Example 4.15. Let $U = \{x_1, x_2, x_3, x_4\}$, $C_1 = \{x_1\}$, $C_2 = \{x_1, x_2, x_3\}$, $C_3 = \{x_1, x_2, x_4\}$, $C_4 = \{x_1, x_2, x_3, x_4\}$, and $\mathcal{C} = \{C_1, C_2, C_3, C_4\}$. Clearly, $\mathcal{C}$ is a covering of U.

By Definition 4.2, it easy to check that

$$N\mathrm{mid}_{\mathcal{C}}(x_1) = N\mathrm{mid}_{\mathcal{C}}(x_2) = N\mathrm{mid}_{\mathcal{C}}(x_3) = N\mathrm{mid}_{\mathcal{C}}(x_4) = \{\{x\}\}. \tag{4.2}$$

By Definition 4.2, we can get that

$$N\mathrm{mid}_{\mathcal{C}-\{C_1\}}(x_1) = N\mathrm{mid}_{\mathcal{C}-\{C_1\}}(x_2) = N\mathrm{mid}_{\mathcal{C}-\{C_1\}}(x_3) = N\mathrm{mid}_{\mathcal{C}-\{C_1\}}(x_4) = \{\{x_1, x_2\}\}. \tag{4.3}$$

Hence $\mathcal{C}$ and $\mathcal{C} - \{C_1\}$ do not satisfy the condition for all $x \in U$, $N\mathrm{mid}_{\mathcal{C}}(x) = N\mathrm{mid}_{\mathcal{C}-\{C_1\}}(x)$. It follows from Definition 4.4 that C_1 is an irreducible element of $\mathcal{C}$.

By Definition 4.2, we can get that

$$N\mathrm{mid}_{\mathcal{C}-\{C_2\}}(x_1) = N\mathrm{mid}_{\mathcal{C}-\{C_2\}}(x_2) = N\mathrm{mid}_{\mathcal{C}-\{C_2\}}(x_3) = N\mathrm{mid}_{\mathcal{C}-\{C_2\}}(x_4) = \{\{x_1\}\}. \tag{4.4}$$

Hence

$$\begin{aligned} N\mathrm{mid}_{\mathcal{C}}(x_1) &= N\mathrm{mid}_{\mathcal{C}-\{C_2\}}(x_1), \qquad & N\mathrm{mid}_{\mathcal{C}}(x_2) &= N\mathrm{mid}_{\mathcal{C}-\{C_2\}}(x_2), \\ N\mathrm{mid}_{\mathcal{C}}(x_3) &= N\mathrm{mid}_{\mathcal{C}-\{C_2\}}(x_3), \qquad & N\mathrm{mid}_{\mathcal{C}}(x_4) &= N\mathrm{mid}_{\mathcal{C}-\{C_2\}}(x_4). \end{aligned} \tag{4.5}$$

It follows from Definition 4.4 that C_2 is a reducible element of $\mathcal{C}$. In the same way, we can check that C_3 and C_4 are all reducible elements of $\mathcal{C}$.

Hence $\mathrm{cor}_3(\mathcal{C}) = \{C_1\}$.

Since $\{C_1, C_2\}$ and $\{C_1, C_3\}$ are not coverings of U, it follows from Corollaries 4.8 and 4.13 that every element of $\{C_1, C_2, C_3\}$ is an irreducible. Thus $\{C_1, C_2, C_3\}$ is a reduct of $\mathcal{C}$.

For $\{C_1, C_2, C_4\}$, by Definition 4.2, it is easy to check that

$$\begin{aligned} N\mathrm{mid}_{\{C_1,C_2,C_4\}}(x_1) &= N\mathrm{mid}_{\{C_1,C_2,C_4\}-\{C_2\}}(x_1), \qquad & N\mathrm{mid}_{\{C_1,C_2,C_4\}}(x_2) &= N\mathrm{mid}_{\{C_1,C_2,C_4\}-\{C_2\}}(x_2), \\ N\mathrm{mid}_{\{C_1,C_2,C_4\}}(x_3) &= N\mathrm{mid}_{\{C_1,C_2,C_4\}-\{C_2\}}(x_3), \qquad & N\mathrm{mid}_{\{C_1,C_2,C_4\}}(x_4) &= N\mathrm{mid}_{\{C_1,C_2,C_4\}-\{C_2\}}(x_4). \end{aligned} \tag{4.6}$$

It follows from Definition 4.4 that C_2 is a reducible element of $\{C_1, C_2, C_4\}$. Further, for $\{C_1, C_4\}$, by Corollaries 4.8 and 4.13, it is clear that C_1 and C_4 are all irreducible elements of $\{C_1, C_4\}$. Thus it is clear that $\{C_1, C_4\}$ is a reduct of $\mathcal{C}$. A similar analysis to $\{C_1, C_3, C_4\}$, we can also get that $\{C_1, C_4\}$ is a reduct of $\mathcal{C}$.

To sum up, $\mathcal{C}$ has two reducts that are $\{C_1, C_2, C_3\}$ and $\{C_1, C_4\}$. It is easy to see that $\mathrm{cor}_3(\mathcal{C}) = \{C_1\} = \{C_1, C_2, C_3\} \cap \{C_1, C_4\} = \cap\mathrm{red}_3(\mathcal{C})$.

The above example also illustrates that for a covering, the reduct is not unique.

Remark 4.16. Let $\mathcal{C}$ be a covering of a universe U. For $\mathcal{B} \in \mathrm{red}_3(\mathcal{C})$, for all $K \in \mathcal{B}$, by Definitions 4.4, 4.5, and 4.6, it is easy to see that $\mathcal{B}$ and $\mathcal{B} - \{K\}$ do not satisfy the condition for all $x \in U$, $N\mathrm{mid}_{\mathcal{B}}(x) = N\mathrm{mid}_{\mathcal{C}}(x)$. Thus by Theorem 3.13, we know that $\mathcal{B}$ and $\mathcal{B} - \{K\}$ cannot induce the same lower and upper approximation operations. This illustrates that for all $\mathcal{B} \in \mathrm{red}_3(\mathcal{C})$, $\mathcal{B}$ is a smallest covering that induces the same the third type of neighborhood-based rough sets.

4.2. Reduction of the Third Type of Neighborhood-Based Rough Sets

Throughout this subsection, we always assume that the rough set model which is discussed by us is the fourth type of neighborhood-based generalized rough sets. So the definitions of a reducible element, an irreducible covering and a reduct, are all based on the fourth type of neighborhood-based rough sets.

Notation 1. Let $\mathcal{C}$ be a covering of U. We write $NM_{\mathcal{C}} = \cup\{N\mathrm{mad}_{\mathcal{C}}(x) \mid x \in U\}$.

By Theorem 3.28, we know that if $NM_{\mathcal{C}} = NM_{\mathcal{B}}$, then $\mathcal{B}$ and $\mathcal{C}$ generate the same the forth type of neighborhood-based lower and upper approximation operations. So we can give the following definition of a reducible element.

Definition 4.17 (A reducible element about the fourth type of lower and upper approximation operations). Let $\mathcal{C}$ be a covering of a universe U and $K \in \mathcal{C}$. If $NM_{\mathcal{C}} = NM_{\mathcal{C}-\{K\}}$, we say that K is a reducible element of $\mathcal{C}$. Otherwise, K is an irreducible element of $\mathcal{C}$.

Definition 4.18 (Irreducible covering about the fourth type of lower and upper approximation operations). Let $\mathcal{C}$ be a covering of a universe U. If every element of $\mathcal{C}$ is an irreducible element, we say that $\mathcal{C}$ is irreducible. Otherwise, $\mathcal{C}$ is reducible.

Definition 4.19 (Reduct about the fourth type of lower and upper approximation operations). Let $\mathcal{C}$ be a covering of a universe U and $\mathcal{B} \subseteq \mathcal{C}$. If $\mathcal{B}$ is irreducible and $NM_{\mathcal{B}} = NM_{\mathcal{C}}$, we say that $\mathcal{B}$ is a reduct of $\mathcal{C}$. Let $\mathrm{red}_4(\mathcal{C}) = \{\mathcal{B} \mid \mathcal{B}$ is a reduct of $\mathcal{C}\}$.

The following proposition is basic.

Proposition 4.20. *Let $\mathcal{C}$ be a covering of a universe U and $\mathcal{B} \subseteq \mathcal{C}$. If $\mathcal{B}$ and $\mathcal{C}$ satisfy the condition that $NM_{\mathcal{B}} = NM_{\mathcal{C}}$, then $\mathcal{B}$ is a covering of U.*

Proof. Suppose that $\mathcal{B}$ is not a covering of U. Then $\cup\mathcal{B} \subset U$. Taking $x_0 \in U - \cup\mathcal{B}$, by Definition 4.1, we have that for all $u \in U$, $x_0 \notin N_{\mathcal{B}}(u)$. Thus for all $N_{\mathcal{B}}(u) \in NM_{\mathcal{B}}$, $x_0 \notin N_{\mathcal{B}}(u)$. On the other hand, since $\mathcal{C}$ is a covering of U, it follows from Definition 3.2 that $\cup NM_{\mathcal{C}} = U$. Thus there exists $N_C(u) \in NM_{\mathcal{C}}$ such that $x_0 \in N_C(u)$. Hence $N_C(u) \notin NM_{\mathcal{B}}$ and so $NM_{\mathcal{B}} \neq NM_{\mathcal{C}}$, which is a contradiction with the condition $NM_{\mathcal{B}} = NM_{\mathcal{C}}$. Thus $\mathcal{B}$ is a covering of U. $\square$

Corollary 4.21. *Let $\mathcal{C}$ be a covering of a universe U and $K \in \mathcal{C}$. If K is a reducible element of $\mathcal{C}$, then $\mathcal{C} - \{K\}$ is still a covering of U.*

Proof. It comes directly from Definition 4.17 and Proposition 4.20. □

In the following, we will illustrate that for a covering, the reduct always exists.

Theorem 4.22. *Let $\mathcal{C}$ be a covering of a universe U. Then there exists $\mathcal{B} \subseteq \mathcal{C}$ such that $\mathcal{B}$ is a reduct of $\mathcal{C}$.*

Proof. The proof is similar to that of Theorem 4.9. □

Now we show that every reduct and the initial covering induce the same lower and upper approximation operations.

Theorem 4.23. *Let $\mathcal{C}$ be a covering of a universe U. Then for all $\mathcal{B} \in \mathrm{red}_4(\mathcal{C})$, $\mathcal{B}$ and $\mathcal{C}$ generate the same the fourth type of neighborhood-based lower and upper approximations.*

Proof. By Definition 4.19 and Proposition 4.20, $\mathcal{B}$ is a covering of U. Thus by Definition 4.19 and Theorem 3.28, for all $\mathcal{B} \in \mathrm{red}_4(\mathcal{C})$, $\mathcal{B}$ and $\mathcal{C}$ generate the same the fourth type of neighborhood-based lower and upper approximations. □

Lemma 4.24. *Let $\mathcal{C}$ be a covering of a universe U and $\mathcal{B} \subseteq \mathcal{C}$. $\mathcal{B}$ and $\mathcal{C}$ satisfy the condition that $NM_{\mathcal{B}} = NM_{\mathcal{C}}$. Then for all $N_{\mathcal{C}}(u) \in NM_{\mathcal{C}}$, we have that $N_{\mathcal{C}}(u) = N_{\mathcal{B}}(u)$ and $N_{\mathcal{B}}(u) \in NM_{\mathcal{B}}$.*

Proof. By the conditions and Proposition 4.20, we know that $\mathcal{B}$ and $\mathcal{C}$ are all coverings of U. Hence, in the following process of proof, we can use directly the concepts and conclusions obtained in Section 3.

Let $N_{\mathcal{C}}(u) \in NM_{\mathcal{C}}$. Then by $NM_{\mathcal{B}} = NM_{\mathcal{C}}$, there exists $N_{\mathcal{B}}(v) \in NM_{\mathcal{B}}$ such that $N_{\mathcal{C}}(u) = N_{\mathcal{B}}(v)$ and so $u \in N_{\mathcal{B}}(v)$. By Definition 2.3, it is clear that $N_{\mathcal{B}}(u) \subseteq N_{\mathcal{B}}(v)$. In addition, since $\mathcal{B} \subseteq \mathcal{C}$, it follows from Definition 2.3 that $N_{\mathcal{C}}(u) \subseteq N_{\mathcal{B}}(u)$. By $N_{\mathcal{C}}(u) = N_{\mathcal{B}}(v)$, we have that $N_{\mathcal{B}}(v) \subseteq N_{\mathcal{B}}(u)$. By $N_{\mathcal{B}}(v) \in NM_{\mathcal{B}}$, Notation 1 and Definition 3.2, this implies $N_{\mathcal{B}}(v) = N_{\mathcal{B}}(u)$. Thus $N_{\mathcal{C}}(u) = N_{\mathcal{B}}(u)$ and $N_{\mathcal{B}}(u) \in NM_{\mathcal{B}}$. □

Proposition 4.25. *Let $\mathcal{C}$ be a covering of a universe U and $\mathcal{B} \subset \mathcal{C}$. $\mathcal{B}$ and $\mathcal{C}$ satisfy the condition that $NM_{\mathcal{B}} = NM_{\mathcal{C}}$. Then for all $K \in \mathcal{C} - \mathcal{B}$, K is a reducible element of $\mathcal{C}$.*

Proof. Let $K \in \mathcal{C} - \mathcal{B}$, then $\mathcal{B} \subseteq \mathcal{C} - \{K\} \subseteq \mathcal{C}$.

(i) We will show that $\mathcal{C} - \{K\}$ and $\mathcal{B}$ are two coverings of U. By the condition and Proposition 4.20, $\mathcal{B}$ is a covering of U. Clearly, $\mathcal{B} \subseteq \mathcal{C} - \{K\}$, thus $\mathcal{C} - \{K\}$ is also a covering of U.

By (i) and the condition, we know that $\mathcal{C}$, $\mathcal{C} - \{K\}$ and $\mathcal{B}$ are all coverings of U. Hence, in the following process of proof, we can use directly the concepts and conclusions obtained in Section 3.

(ii) We will show that $NM_{\mathcal{C}} \subseteq NM_{\mathcal{C}-\{K\}}$. For all $N_{\mathcal{C}}(u) \in NM_{\mathcal{C}}$, then by Lemma 4.24, $N_{\mathcal{C}}(u) = N_{\mathcal{B}}(u)$ and $N_{\mathcal{B}}(u) \in NM_{\mathcal{B}}$. Since $\mathcal{B} \subseteq \mathcal{C} - \{K\} \subseteq \mathcal{C}$, it follows from Definition 2.3 that $N_{\mathcal{C}}(u) \subseteq N_{\mathcal{C}-\{K\}}(u) \subseteq N_{\mathcal{B}}(u)$. Thus by $N_{\mathcal{C}}(u) = N_{\mathcal{B}}(u)$, we have that $N_{\mathcal{C}-\{K\}}(u) = N_{\mathcal{B}}(u)$. Suppose that $N_{\mathcal{C}-\{K\}}(u) \notin NM_{\mathcal{C}-\{K\}}$. Then $N_{\mathcal{C}-\{K\}}(u) \notin N\mathrm{mad}_{\mathcal{C}-\{K\}}(u)$. Thus by Definition 3.2, there exists $z \in U$ such that

$N_{\mathcal{C}-\{K\}}(u) \subset N_{\mathcal{C}-\{K\}}(z)$ and so $N_{\mathcal{B}}(u) \subset N_{\mathcal{C}-\{K\}}(z)$, which contradicts the fact that $N_{\mathcal{B}}(u) \in NM_{\mathcal{B}}$. Thus $N_{\mathcal{C}-\{K\}}(u) \in NM_{\mathcal{C}-\{K\}}$. Combining $N_{\mathcal{C}}(u) = N_{\mathcal{B}}(u)$ and $N_{\mathcal{C}-\{K\}}(u) = N_{\mathcal{B}}(u)$, we can get that $N_{\mathcal{C}}(u) = N_{\mathcal{C}-\{K\}}(u)$ and so $N_{\mathcal{C}}(u) \in NM_{\mathcal{C}-\{K\}}$. Thus, $NM_{\mathcal{C}} \subseteq NM_{\mathcal{C}-\{K\}}$.

(iii) We will show that $NM_{\mathcal{C}-\{K\}} \subseteq NM_{\mathcal{C}}$, for all $N_{\mathcal{C}-\{K\}}(u) \in NM_{\mathcal{C}-\{K\}}$. By $\mathcal{B} \subseteq \mathcal{C}-\{K\}$, it is clear that $N_{\mathcal{C}-\{K\}}(u) \subseteq N_{\mathcal{B}}(u)$. By $u \in N_{\mathcal{B}}(u)$ and Proposition 3.18, there exists $N_{\mathcal{B}}(z) \in N\mathrm{mad}_{\mathcal{B}}(u)$ such that $N_{\mathcal{B}}(u) \subseteq N_{\mathcal{B}}(z)$. This implies $N_{\mathcal{C}-\{K\}}(u) \subseteq N_{\mathcal{B}}(z)$. By Notation 1, it is clear that $N_{\mathcal{B}}(z) \in NM_{\mathcal{B}}$. Since $NM_{\mathcal{B}} = NM_{\mathcal{C}}$, it follows that there exists $N_{\mathcal{C}}(v) \in NM_{\mathcal{C}}$ such that $N_{\mathcal{B}}(z) = N_{\mathcal{C}}(v)$. Thus $N_{\mathcal{C}-\{K\}}(u) \subseteq N_{\mathcal{C}}(v)$. Clearly, $N_{\mathcal{C}}(v) \subseteq N_{\mathcal{C}-\{K\}}(v)$, hence $N_{\mathcal{C}-\{K\}}(u) \subseteq N_{\mathcal{C}}(v) \subseteq N_{\mathcal{C}-\{K\}}(v)$. In addition, since $N_{\mathcal{C}-\{K\}}(u) \in NM_{\mathcal{C}-\{K\}}$, it follows from Notation 1 and Definition 3.2 that $N_{\mathcal{C}-\{K\}}(u) = N_{\mathcal{C}-\{K\}}(v)$. This implies that $N_{\mathcal{C}-\{K\}}(u) = N_{\mathcal{C}}(v)$. It follows by $N_{\mathcal{C}}(v) \in NM_{\mathcal{C}}$ that $N_{\mathcal{C}-\{K\}}(u) \in NM_{\mathcal{C}}$. Thus $NM_{\mathcal{C}-\{K\}} \subseteq NM_{\mathcal{C}}$.

By (ii) and (iii), we have that $NM_{\mathcal{C}-\{K\}} = NM_{\mathcal{C}}$. Thus by Definition 4.17, K is a reducible element of $\mathcal{C}$. □

Corollary 4.26. *Let $\mathcal{C}$ be a covering of a universe U, $K \in \mathcal{C}$ a reducible element of $\mathcal{C}$ and $K_1 \in \mathcal{C} - \{K\}$. If K_1 is an irreducible element of $\mathcal{C}$, then K_1 is an irreducible element of $\mathcal{C} - \{K\}$.*

Proof. Suppose that K_1 is a reducible element of $\mathcal{C} - \{K\}$. Then by Definition 4.17, $NM_{\mathcal{C}-\{K\}} = NM_{\mathcal{C}-\{K,K_1\}}$. Since K is a reducible element of $\mathcal{C}$, it follows from Definition 4.17 that $NM_{\mathcal{C}} = NM_{\mathcal{C}-\{K\}}$. Thus $NM_{\mathcal{C}} = NM_{\mathcal{C}-\{K,K_1\}}$. Clearly, $K_1 \in \mathcal{C}-(\mathcal{C}-\{K, K_1\})$, thus by Proposition 4.25, K_1 is a reducible element of $\mathcal{C}$, which contradicts the condition that K_1 is an irreducible element of $\mathcal{C}$. This completes the proof. □

The above proposition guarantees that omitting a reducible element in a covering will not make any current irreducible element reducible. Therefore, for a covering $\mathcal{C}$, the set of all irreducible elements is constant. We denote this set by $\mathrm{cor}_4(\mathcal{C})$, that is,

$$\mathrm{cor}_4(\mathcal{C}) = \{K \mid K \text{ is an irreducible element of } \mathcal{C}\}. \tag{4.7}$$

Theorem 4.27. *Let $\mathcal{C}$ be a covering of a universe U. Then* $\mathrm{cor}_4(\mathcal{C}) = \cap\mathrm{red}_4(\mathcal{C})$.

Proof. The proof is similar to that of Theorem 4.14. □

The above result states that an element will not be reduced in any reduction procedure if and only if it is irreducible. Hence the irreducible elements will be reserved in any reduction procedure, that is to say, $\mathrm{cor}_4(\mathcal{C})$ is contained in any reduct of $\mathcal{C}$. So we can compute the reduct of $\mathcal{C}$ based on $\mathrm{cor}_4(\mathcal{C})$.

Example 4.28. Let $U = \{x_1, x_2, x_3, x_4\}$, $C_1 = \{x_1, x_2, x_3\}$, $C_2 = \{x_1, x_2, x_4\}$, $C_3 = \{x_1, x_2\}$, $C_4 = \{x_3\}$, $C_5 = \{x_4\}$, and $\mathcal{C} = \{C_1, C_2, C_3, C_4, C_5\}$. Clearly, $\mathcal{C}$ is a covering of U.

By Definition 4.3 and Notation 1, it is easy to see that $NM_{\mathcal{C}} = \{\{x_1, x_2\}, \{x_3\}, \{x_4\}\}$.

Since $NM_{\mathcal{C}-\{C_1\}} = \{\{x_1, x_2\}, \{x_3\}, \{x_4\}\}$, it follows that $NM_{\mathcal{C}} = NM_{\mathcal{C}-\{C_1\}}$. Thus C_1 is a reducible element of $\mathcal{C}$. In the same way, we can check that C_2 and C_3 are reducible elements of $\mathcal{C}$.

Since $NM_{\mathcal{C}-\{C_4\}} = \{\{x_1, x_2, x_3\}, \{x_4\}\}$, it follows that $NM_{\mathcal{C}} \neq NM_{\mathcal{C}-\{C_4\}}$. Thus C_4 is an irreducible element of $\mathcal{C}$.

Since $NM_{\mathcal{C}-\{C_5\}} = \{\{x_1, x_2, x_4\}, \{x_3\}\}$, it follows that $NM_{\mathcal{C}} \neq NM_{\mathcal{C}-\{C_5\}}$. Thus C_5 is an irreducible element of $\mathcal{C}$.

Hence $\mathrm{cor}_4(\mathcal{C}) = \{C_4, C_5\}$.

It is easy to check that $NM_{\{C_1,C_2,C_4,C_5\}} \neq NM_{\{C_2,C_4,C_5\}}$ and $NM_{\{C_1,C_2,C_4,C_5\}} \neq NM_{\{C_1,C_4,C_5\}}$. Thus C_1 and C_2 are irreducible elements of $\{C_1, C_2, C_4, C_5\}$. By Corollary 4.26, C_4 and C_5 are also irreducible elements of $\{C_1, C_2, C_4, C_5\}$. Hence $\{C_1, C_2, C_4, C_5\}$ is an irreducible covering of U. It follows that $\{C_1, C_2, C_4, C_5\}$ is a reduct of $\mathcal{C}$.

It is easy to check that $NM_{\{C_1,C_3,C_4,C_5\}} = NM_{\{C_3,C_4,C_5\}}$ and $NM_{\{C_1,C_3,C_4,C_5\}} \neq NM_{\{C_1,C_4,C_5\}}$. Thus C_1 is a reducible element of $\{C_1, C_3, C_4, C_5\}$, and C_3 is an irreducible element of $\{C_1, C_3, C_4, C_5\}$. Further, by Corollary 4.26, $\{C_3, C_4, C_5\}$ is an irreducible covering. Further, it is easy to verify that $NM_{\mathcal{C}} = NM_{\{C_3,C_4,C_5\}}$. Hence $\{C_3, C_4, C_5\}$ is a reduct of $\mathcal{C}$. A similar analysis to $\{C_2, C_3, C_4, C_5\}$, we can also get that $\{C_3, C_4, C_5\}$ is a reduct of $\mathcal{C}$.

To sum up, $\mathcal{C}$ has two reducts that are $\{C_1, C_2, C_3, C_4\}$ and $\{C_3, C_4, C_5\}$. It is easy to see that $\mathrm{cor}_4(\mathcal{C}) = \{C_3, C_4\} = \{C_1, C_2, C_3, C_4\} \cap \{C_3, C_4, C_5\} = \cap \mathrm{red}_4(\mathcal{C})$.

Remark 4.29. Let $\mathcal{C}$ be a covering of a universe U. For $\mathcal{B} \in \mathrm{red}_4(\mathcal{C})$, for all $K \in \mathcal{B}$, by Definitions 4.17, 4.18, and 4.19, it is easy to see that $\mathcal{B}$ and $\mathcal{B} - \{K\}$ do not satisfy the condition for all $x \in U$, $N\mathrm{mid}_{\mathcal{B}}(x) = N\mathrm{mid}_{\mathcal{C}}(x)$. Thus by Theorem 3.28, we know that $\mathcal{B}$ and $\mathcal{B} - \{K\}$ cannot induce the same lower and upper approximation operations. This illustrates that for all $\mathcal{B} \in \mathrm{red}_4(\mathcal{C})$, $\mathcal{B}$ is a smallest covering that induces the same the fourth type of neighborhood-based rough sets.

5. The Two Open Problems

In [28], Yun et al. proposed two open problems how to give sufficient and necessary conditions for $\{N(x) \mid x \in U\}$ to form a partition of U by using only a single covering approximation operator C_i $(i = 1, 4)$. That is to say, the first one is how to characterize the conditions for $\{N(x) \mid x \in U\}$ to form a partition by applying the first type of generalized approximation operator, and the second one is how to characterize the conditions for $\{N(x) \mid x \in U\}$ to form a partition by applying the fourth type of generalized approximation operator. In this section, we present some conditions under which $\{N(x) \mid x \in U\}$ forms a partition of U. As a result, the two open problems are solved (see Theorems 5.3 and 5.4).

Lemma 5.1. *Let $\mathcal{C}$ be a covering of a universe U. If $\{N(x) \mid x \in U\}$ forms a partition of U, then for all $x \in U$, $N(x) \cap (\cup\{K \mid K \in \mathcal{C},\ x \notin K\}) = \emptyset$.*

Proof. Let $x \in U$. Suppose that $N(x) \cap (\cup\{K \mid K \in \mathcal{C},\ x \notin K\}) \neq \emptyset$. Then we choose $u \in N(x) \cap (\cup\{K \mid K \in \mathcal{C},\ x \notin K\})$, that is, $u \in N(x)$ and $u \in \cup\{K \mid K \in \mathcal{C},\ x \notin K\}$. Thus there exists $K \in \{K \mid K \in \mathcal{C},\ x \notin K\}$ such that $u \in K$. By Definition 2.3, this implies that $N(u) \subseteq K$. On the other hand, since $\{N(x) \mid x \in U\}$ forms a partition of U, it follows from $u \in N(x)$ that $N(u) = N(x)$ and so $x \in N(u)$. Thus $x \in K$. This is a contradiction with $K \in \{K \mid K \in \mathcal{C},\ x \notin K\}$. Therefore, $N(x) \cap (\cup\{K \mid K \in \mathcal{C},\ x \notin K\}) = \emptyset$. □

Lemma 5.2. *Let $\mathcal{C}$ be a covering of a universe U and $x, z \in U$. If $N(z) \subseteq N(x)$, then $\{K \in \mathcal{C} \mid z \notin K\} \subseteq \{K \in \mathcal{C} \mid x \notin K\}$.*

Proof. Let $K \in \{K \in \mathcal{C} \mid z \notin K\}$. Suppose that $K \notin \{K \in \mathcal{C} \mid x \notin K\}$. Then $x \in K$. By Definition 2.3, this implies that $N(x) \subseteq K$. Since $N(z) \subseteq N(x)$, it follows that $N(z) \subseteq K$ and so $z \in K$, which contradicts the fact that $K \in \{K \in \mathcal{C} \mid z \notin K\}$. Thus $K \in \{K \in \mathcal{C} \mid x \notin K\}$. Hence $\{K \in \mathcal{C} \mid z \notin K\} \subseteq \{K \in \mathcal{C} \mid x \notin K\}$. □

Theorem 5.3. *Let $\mathcal{C}$ be a covering of a universe U. Then $\{N(x) \mid x \in U\}$ forms a partition of U if and only if for each $x \in U$, $\overline{\mathcal{C}_1}(\{x\}) = N(x)$.*

Proof. Let $x \in U$. By Lemma 5.1, we have that $N(x) \cap (\cup\{K \mid K \in \mathcal{C},\ x \notin K\}) = \emptyset$. This implies that $N(x) \subseteq \sim (\cup\{K \mid K \in \mathcal{C},\ x \notin K\})$. In addition, by the part (1) of Definition 2.4, we have that $\sim (\cup\{K \mid K \in \mathcal{C},\ x \notin K\}) = \sim (\cup\{K \mid K \in \mathcal{C}, K \subseteq \sim \{x\}\}) = \overline{\mathcal{C}_1}(\{x\})$. Consequently, $N(x) \subseteq \overline{\mathcal{C}_1}(\{x\})$. On the other hand, for all $y \in \overline{\mathcal{C}_1}(\{x\})$, then $y \in \sim (\cup\{K \mid K \in \mathcal{C}, K \subseteq \sim \{x\}\})$. Thus $y \notin \cup\{K \mid K \in \mathcal{C}, K \subseteq \sim \{x\}\}$, that is, $y \notin \cup\{K \mid K \in \mathcal{C}, x \notin K\}$. This implies that for all $K \in \mathcal{C}, x \notin K \Rightarrow y \notin K$. Thus for all $K \in \mathcal{C}, y \in K \Rightarrow x \in K$. This implies that $\{K \mid K \in \mathcal{C}, y \in K\} \subseteq \{K \mid K \in \mathcal{C}, x \in K\}$ and so $\cap\{K \mid K \in \mathcal{C}, y \in K\} \supseteq \cap\{K \mid K \in \mathcal{C}, x \in K\}$, that is, $N(y) \supseteq N(x)$. Since $\{N(x) \mid x \in U\}$ forms a partition of U, it follows that $N(y) = N(x)$ and so $y \in N(x)$. Thus $\overline{\mathcal{C}_1}(\{x\}) \subseteq N(x)$. In summary, $\overline{\mathcal{C}_1}(\{x\}) = N(x)$. This completes the proof of the necessity.

Conversely, suppose that $\{N(x) \mid x \in U\}$ is not a partition of U. Then there exist $N(x), N(y) \in \{N(x) \mid x \in U\}$ such that $N(x) \neq N(y)$ and $N(x) \cap N(y) \neq \emptyset$. Taking $z \in N(x) \cap N(y)$, then $N(z) \subseteq N(x) \cap N(y)$. Clearly, $N(z) \neq N(x)$ or $N(z) \neq N(y)$. Without loss of generality, we may assume that $N(z) \neq N(x)$, then $N(z) \subset N(x)$. Thus, by Lemma 5.2, we have that $\{K \in \mathcal{C} \mid z \notin K\} \subseteq \{K \in \mathcal{C} \mid x \notin K\}$. Thus $\sim (\cup\{K \in \mathcal{C} \mid z \notin K\}) \supseteq \sim (\cup\{K \in \mathcal{C} \mid x \notin K\})$ and so $\sim (\cup\{K \in \mathcal{C} \mid K \subseteq \sim \{z\}\}) \supseteq \sim (\cup\{K \in \mathcal{C} \mid K \subseteq \sim \{x\}\})$. It follows from the part (1) of Definition 2.4 that $\overline{\mathcal{C}_1}(\{z\}) \supseteq \overline{\mathcal{C}_1}(\{x\})$. By the condition, we have that $\overline{\mathcal{C}_1}(\{z\}) = N(z)$ and $\overline{\mathcal{C}_1}(\{x\}) = N(x)$. Thus $N(z) \supseteq N(x)$, which contradicts with $N(z) \subset N(x)$. Hence $\{N(x) \mid x \in U\}$ forms a partition of U. □

In fact, the above theorem establishes the relationship between the first type of generalized rough sets and the other types of neighborhood-based rough sets.

Theorem 5.4. *Let $\mathcal{C}$ be a covering of a universe U. Then $\{N(x) \mid x \in U\}$ forms a partition of U if and only if for each $x \in U$, $\overline{\mathcal{C}_4}(N(x)) = N(x)$.*

Proof. The necessity is obvious.

Conversely, suppose that $\{N(x) \mid x \in U\}$ is not a partition of U. Then there exist $N(x), N(y) \in \{N(x) \mid x \in U\}$ such that $N(x) \neq N(y)$ and $N(x) \cap N(y) \neq \emptyset$. Taking $z \in N(x) \cap N(y)$, then $N(z) \subseteq N(x) \cap N(y)$. Clearly, $N(z) \neq N(x)$ or $N(z) \neq N(y)$. Without loss of generality, we may assume that $N(z) \neq N(x)$, then $N(z) \subset N(x)$. Thus $x \notin N(z)$ (Otherwise, by Definition 2.3, $N(x) \subseteq N(z)$). In addition, clearly, $N(x) \cap N(z) \neq \emptyset$; therefore, by the part (4) of Definition 2.4, we have that $x \in \overline{\mathcal{C}_4}(N(z))$. Thus $\overline{\mathcal{C}_4}(N(z)) \neq N(z)$, which contradicts the condition for all $x \in U$, $\overline{\mathcal{C}_4}(N(x)) = N(x)$. Thus $\{N(x) \mid x \in U\}$ forms a partition of U. □

6. Conclusions

This paper defines the concepts of minimal neighborhood description and maximal neighborhood description in neighborhood-based rough set models. We give the new characterizations of the third and the fourth types of neighborhood-based rough sets. By means of these new characterizations, we explore the covering reduction of two types of neighborhood-based rough sets and have shown that the reduct of a covering is the minimal covering that generates the same lower and upper approximations. Clearly, the notions of minimal neighborhood description and maximal neighborhood description play essential roles in the studies of the reduction issues of the third and the fourth types of neighborhood-based rough sets. In fact, the two concepts are the essential characteristics related to the neighborhood-based rough sets. In particular, the notion of maximal neighborhood description is very useful. A similar notion was also discussed in [30]. In the future, we will further study neighborhood-based rough sets by means of these concepts.

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References

[1] Z. Pawlak, "Rough sets," *International Journal of Computer and Information Sciences*, vol. 11, no. 5, pp. 341–356, 1982.

[2] W. Li, N. Zhong, Y. Y. Yao, J. Liu, and C. Liu, "Developing intelligent applications in social e-mail networks," in *Rough Sets and Current Trends in Computing*, S. Greco, Y. Hata, S. Hirano et al., Eds., vol. 4259 of *Lecture Notes in Computer Science*, pp. 776–785, Springer, Berlin, Germany, 2006.

[3] S. K. Pal, B. U. Shankar, and P. Mitra, "Granular computing, rough entropy and object extraction," *Pattern Recognition Letters*, vol. 26, no. 16, pp. 2509–2517, 2005.

[4] Z. Pawlak, *Rough Sets: Theoretical Aspects of Reasoning About Data*, Kluwer Academic Publishers, Boston, Mass, USA, 1991.

[5] L. Polkowski and A. Skowron, Eds., *Rough Sets and Current Trends in Computing*, vol. 1424, Springer, Berlin, Germany, 1998.

[6] Y. Y. Yao and N. Zhong, "Potential applications of granular computing in knowledge discovery and data mining," in *Proceedings of the World Multiconference on Systemics, Cybernetics and Informatics (WMSCI '99)*, pp. 573–580, 1999.

[7] D. S. Yeung, D. Chen, E. C. C. Tsang, J. W. T. Lee, and W. Xizhao, "On the generalization of fuzzy rough sets," *IEEE Transactions on Fuzzy Systems*, vol. 13, no. 3, pp. 343–361, 2005.

[8] N. Zhong, J. Dong, and S. Ohsuga, "Using rough sets with heuristics for feature selection," *Journal of Intelligent Information Systems*, vol. 16, no. 3, pp. 199–214, 2001.

[9] N. Zhong, Y. Yao, and M. Ohshima, "Peculiarity oriented multidatabase mining," *IEEE Transactions on Knowledge and Data Engineering*, vol. 15, no. 4, pp. 952–960, 2003.

[10] W. Zhu and F.-Y. Wang, "Covering based granular computing for conflict analysis," in *Intelligence and Security Informatics*, S. Mehrotra, D. D. Zeng, H. Chen et al., Eds., vol. 3975 of *Lecture Notes in Computer Science*, pp. 566–571, 2006.

[11] Z. Pawlak and A. Skowron, "Rough sets: some extensions," *Information Sciences*, vol. 177, no. 1, pp. 28–40, 2007.

[12] A. Skowron and J. Stepaniuk, "Tolerance approximation spaces," *Fundamenta Informaticae*, vol. 27, no. 2-3, pp. 245–253, 1996.

[13] R. Slowinski and D. Vanderpooten, "A generalized definition of rough approximations based on similarity," *IEEE Transactions on Knowledge and Data Engineering*, vol. 12, no. 2, pp. 331–336, 2000.

[14] W. Zhu and F.-Y. Wang, "Reduction and axiomization of covering generalized rough sets," *Information Sciences*, vol. 152, pp. 217–230, 2003.

[15] W. Zhu and F. Wang, "Binary relation based rough set," in *Fuzzy Systems and Knowledge Discovery*, L. Wang, L. Jiao, G. Shi et al., Eds., vol. 4223 of *Lecture Notes in Computer Science*, pp. 276–285, 2006.

[16] W. Żakowski, "Approximations in the space (u, π)," *Demonstratio Mathematica*, vol. 16, no. 3, pp. 761–769, 1983.

[17] Z. Bonikowski, E. Bryniarski, and U. Wybraniec-Skardowska, "Extensions and intentions in the rough set theory," *Information Sciences*, vol. 107, no. 1–4, pp. 149–167, 1998.

[18] W. Zhu, "Topological approaches to covering rough sets," *Information Sciences*, vol. 177, no. 6, pp. 1499–1508, 2007.

[19] E. C. C. Tsang, C. Degang, and D. S. Yeung, "Approximations and reducts with covering generalized rough sets," *Computers & Mathematics with Applications*, vol. 56, no. 1, pp. 279–289, 2008.

[20] W.-H. Xu and W.-X. Zhang, "Measuring roughness of generalized rough sets induced by a covering," *Fuzzy Sets and Systems*, vol. 158, no. 22, pp. 2443–2455, 2007.

[21] T.-J. Li, Y. Leung, and W.-X. Zhang, "Generalized fuzzy rough approximation operators based on fuzzy coverings," *International Journal of Approximate Reasoning*, vol. 48, no. 3, pp. 836–856, 2008.

[22] T.-J. Li and W.-X. Zhang, "Fuzzy rough approximations on two universes of discourse," *Information Sciences*, vol. 178, pp. 829–906, 2008.

[23] W. Zhu and F. Y. Wang, "On three types of covering-based rough sets," *IEEE Transactions on Knowledge and Data Engineering*, vol. 19, no. 8, pp. 1131–1143, 2007.

[24] K. Qin, Y. Gao, and Z. Pei, "On covering rough sets," in *Rough Sets and Knowledge Technology*, J. T. Yao, P. Lingras, W. Z. Wu et al., Eds., vol. 4481 of *Lecture Notes in Computer Science*, pp. 34–41, 2007.

[25] T. Yang and Q. Li, "Reduction about approximation spaces of covering generalized rough sets," *International Journal of Approximate Reasoning*, vol. 51, no. 3, pp. 335–345, 2010.

[26] Y. Y. Yao, "Relational interpretations of neighborhood operators and rough set approximation operators," *Information Sciences*, vol. 111, no. 1–4, pp. 239–259, 1998.

[27] Y. Y. Yao, "On generalizing rough set theory," in *Proceedings of the 9th International Conference (RSFDGrC '03)*, vol. 2639 of *Rough Sets, Fuzzy Sets, Data Mining, and Granular Computing*, pp. 44–51, Lecture Notes in Artificial Intelligence, 2003.

[28] Z. Yun, X. Ge, and X. Bai, "Axiomatization and conditions for neighborhoods in a covering to form a partition," *Information Sciences*, vol. 181, no. 9, pp. 1735–1740, 2011.

[29] J. W. Grzymalabusse and W. Rzasa, "Definability of approximations for a generalization of the indiscernibility relation," in *Proceedings of the IEEE Symposium on Foundations of Computational Intelligence (FOCI '07)*, pp. 65–72, USA, April 2007.

[30] Y. Y. Yao and B. X. Yao, "Covering based roughset approximations," *Information Science*. In press.