Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 202860, 16 pages
doi:10.1155/2012/202860

Research Article

An Alternative Regularization Method for Equilibrium Problems and Fixed Point of Nonexpansive Mappings

Shuo Sun

Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China

Correspondence should be addressed to Shuo Sun, sunshuo@tjpu.edu.cn

Received 16 December 2011; Accepted 26 December 2011

Academic Editor: Rudong Chen

We introduce a new regularization iterative algorithm for equilibrium and fixed point problems of nonexpansive mapping. Then, we prove a strong convergence theorem for nonexpansive mappings to solve a unique solution of the variational inequality and the unique sunny nonexpansive retraction. Our results extend beyond the results of S. Takahashi and W. Takahashi (2007), and many others.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. let C be a nonempty closed convex subset of H. Let ϕ be a bifunction of $C\times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $\phi : C\times C \to \mathbb{R}$ is to find $x\in C$ such that

$$\phi(x,y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $\mathrm{EP}(\phi)$. Given a mapping $T : C \to H$, let $\phi(x,y) = \langle Tx, y-x\rangle$ for all $x,y\in C$. Then, $z\in \mathrm{EP}(\phi)$ if and only if $\langle Tz, y-z\rangle \geq 0$ for all $y\in C$, that is, z is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.1) . Some methods have been proposed to solve the equilibrium problem; see, for instance, [1–6].

A mapping S of C into H is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x-y\| \quad \forall x,y \in C. \tag{1.2}$$

We denote by $F(S)$ the set of fixed points of S. The fixed point equation $Tx = x$ is ill-posed (it may fail to have a solution, nor uniqueness of solution) in general. Regularization therefore is needed. Contractions can be used to regularize nonexpansive mappings. In fact, the following regularization has widely been implemented ([7–9]). Fixing a point $u \in C$ and for each $t \in (0,1)$, one defines a contraction $T_t : C \to C$ by

$$T_t x = tu + (1-t)Tx, \quad x \in C. \tag{1.3}$$

In this paper we provide an alternative regularization method. Our idea is to shrink x first and then apply T to the convex combination of the shrunk x and the anchor u (this idea appeared implicitly in [10] where iterative methods for finding zeros of maximal monotone operators were investigated). In other words, we fix an anchor $u \in C$ and $t \in (0,1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = T(tu + (1-t)x), \quad x \in C. \tag{1.4}$$

Compared with (1.1), (1.4) looks slightly more compact in the sense that the mapping T is more directly involved in the regularization and thus may be more convenient in manipulations since the nonexpansivity of T is utilized first.

In 2000, Moudafi [11] proved the following strong convergence theorem.

Theorem 1.1 (Moudafi [11]). *Let C be a nonempty closed convex subset of a Hilbert space H and let S be a nonexpansive mapping of C into itself such that $F(S)$ is nonempty. Let f be a contraction of C into itself and let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \frac{1}{1+\varepsilon_n} S x_n + \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n) \tag{1.5}$$

for all $n \in N$, where $\{\varepsilon_n\} \subset (0,1)$ satisfies

$$\lim_{n\to\infty} \varepsilon_n = 0, \qquad \sum_{n=1}^{\infty} \varepsilon_n = \infty, \qquad \lim_{n\to\infty} \left| \frac{1}{\varepsilon_{n+1}} - \frac{1}{\varepsilon_n} \right| = 0. \tag{1.6}$$

Then, $\{x_n\}$ converges strongly to $z \in F(S)$, where $z = P_{F(S)} f(z)$ and $P_{F(S)}$ is the metric projection of H onto $F(S)$.

Such a method for approximation of fixed points is called the viscosity approximation method.

In 2007, S. Takahashi and W. Takahashi [5] introduced and considered the following iterative algorithm by the viscosity approximation method in the Hilbert space:

$$\begin{gathered} x_1 \in H, \\ \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S u_n, \end{gathered} \tag{1.7}$$

for all $n \in N$, where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy some appropriate conditions. Furthermore, they proved that $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S) \cap \mathrm{EP}(\phi)$, where $z = P_{F(S)\cap \mathrm{EP}(\phi)} f(z)$.

In this paper, motivated and inspired by the above results, we introduce an iterative scheme by the general iterative method for finding a common element of the set of solutions of (1.1) and the set of fixed points of a nonexpansive mapping in Hilbert space.

Let $S : C \to C$ be a nonexpansive mapping. Starting with an arbitrary $x_1, u \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{aligned} &\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ &x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) T(\alpha_n u + (1 - \alpha_n) u_n), \quad n \geq 1. \end{aligned} \tag{1.8}$$

We will prove in Section 3 that if the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ of parameters satisfy appropriate conditions, then the sequence $\{x_n\}$ and $\{u_n\}$ generated by (1.8) converges strongly to the unique solution of the variational inequality

$$\langle (I - f)z, x - z\rangle \geq 0, \quad \forall x \in F(S) \cap \mathrm{EP}(\phi), \tag{1.9}$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(S) \cap \mathrm{EP}(\phi)} \frac{1}{2}\langle x, x\rangle - h(x), \tag{1.10}$$

where h is a potential function for f and at the same time, the sequence $\{x_n\}$ and $\{u_n\}$ generated by (1.8) converges in norm to $Q(u)$, where $Q : C \to \mathrm{Fix}(T)$ is the sunny nonexpansive retraction.

2. Preliminaries

Throughout this paper, we consider H as the Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively, C is a nonempty closed convex subset of H. Consider a subset D of C and a mapping $Q : C \to D$. Then we say that

(i) Q is a retraction provided $Qx = x \ for \ x \in D$;

(ii) Q is a nonexpansive retraction provided Q is a retraction that is also nonexpansive;

(iii) Q is a sunny nonexpansive retraction provided Q is a nonexpansive retraction with the additional property: $Q(x + t(x - Qx)) = Qx$ whenever $x + t(x - Qx) \in C$, where $x \in C$ and $t \geq 0$.

Let now $T : C \to C$ be a nonexpansive mapping. For a fixed anchor $u \in C$ and each $t \in (0, 1)$ recall that $z_t \in C$ is the unique fixed point of the contraction $C \ni x \mapsto T(tu + (1 - t)x)$. Namely, $z_t \in C$ is the unique solution in C to the fixed point equation

$$z_t = T(tu + (1 - t)z_t). \tag{2.1}$$

In the Hilbert space (either uniformly smooth or reflexive with a weakly continuous duality map), then z_t is strongly convergent should it is bounded as $t \to 0^+$.

We also know that for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{2.2}$$

holds for every $y \in H$ with $x \neq y$, (we usually call it Opial's condition); see [12, 13] for more details.

For solving the equilibrium problem for a bifunction $\phi : C \times C \to \mathbb{R}$, let us assume that ϕ satisfies the following conditions:

(A_1) $\phi(x,x) = 0$, for all $x \in C$;

(A_2) ϕ is monotone, that is, $\phi(x,y) + \phi(y,x) \leq 0$, for all $x, y \in C$;

(A_3) For each $x, y, z \in C$, $\lim_{t\downarrow 0} \phi(tz + (1-t)x, y) \leq \phi(x,y)$;

(A_4) For each $x \in C$, the function $y \mapsto \phi(x,y)$ is convex and lower semicontinuous.

The following lemma appeared implicitly in [14].

Lemma 2.1 (see [14]). *Let C be a nonempty closed convex subset of H and let $\phi : C \times C \to \mathbb{R}$ be a bifunction satisfying (A_1)–(A_4). Let $r > 0$ and $x \in H$, then, there exists $z \in C$ such that*

$$\phi(z,y) + \frac{1}{r}\langle y - z,\ z - x\rangle \geq 0, \quad \forall y \in C. \tag{2.3}$$

Lemma 2.2 (see [6]). *Assume that $\phi : C \times C \to \mathbb{R}$ satisfies (A_1)–(A_4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : \phi(z,y) + \frac{1}{r}\langle y - z,\ z - x\rangle \geq 0, \forall y \in C \right\}, \tag{2.4}$$

for all $z \in H$. Then, the following hold:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y,\ x - y\rangle; \tag{2.5}$$

(3) $F(T_r) = \mathrm{EP}(\phi)$;

(4) $\mathrm{EP}(\phi)$ *is closed and convex.*

Lemma 2.3 (see [15]). *Let $\{a_n\} \subset [0,\infty)$, $\{b_n\} \subset [0,\infty)$ and $\{c_n\} \subset [0,1)$ be sequences of real numbers such that*

$$\begin{gathered} a_{n+1} \leq (1 - c_n)a_n + b_n, \quad \forall n \in N, \\ \sum_{n=1}^{\infty} c_n = \infty, \qquad \sum_{n=1}^{\infty} b_n < \infty \quad \left(or \ \limsup_{n\to\infty} \frac{b_n}{c_n} \leq 0 \right), \end{gathered} \tag{2.6}$$

then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4 (see [9]). *Suppose that X is a smooth Banach space. Then a retraction $Q : C \to D$ is sunny nonexpansive if and only if*

$$\langle x - Qx, \ J(y - Qx) \rangle \leq 0, \quad x \in C, \ y \in D. \tag{2.7}$$

Lemma 2.5. *Let X be a uniformly smooth Banach space, C a nonempty closed convex subset of X, and $T : C \to C$ a nonexpansive mapping. Let z_t be defined by* (2.1)*. Then (z_t) remains bounded as $t \to 0$ if and only if $\mathrm{Fix}(T) \neq$. Moreover, if $\mathrm{Fix}(T) \neq$, then (z_t) converges in norm, as $t \to 0^+$, to a fixed point of T; and if one sets*

$$Q(u) := \lim_{t \to 0} z_t, \tag{2.8}$$

then Q defines the unique sunny nonexpansive retraction from C onto $\mathrm{Fix}(T)$.

Lemma 2.6. *In the Hilbert space, the following inequalities always hold*

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, \ x + y \rangle$;

(ii) $\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2$.

3. Main Results

Theorem 3.1. *Let C be a nonempty closed convex subset of H, $\phi : C \times C \to \infty$ be a bifunction satisfying (A_1)–(A_4) and $T : C \to C$ be a nonexpansive mapping of C into H such that $F(T) \cap \mathrm{EP}(\phi) \neq \emptyset$. Let f be a contraction of H into itself with $\alpha \in (0,1)$, initially give an arbitrary element $x_1 \in H$ and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by* (1.8)*, where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy the following conditions:*

(I) $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(II) $\liminf_n r_n > 0$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(III) $\lim_{n \to \infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(IV) $\lim_{n \to \infty} (\alpha_n / \beta_n) = 0$.

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(T) \cap \mathrm{EP}(\phi)$, where $z = P_{F(T) \cap \mathrm{EP}(\phi)} f(z)$ and converge in norm to $Q(u)$, where $Q : C \to \mathrm{Fix}(T)$ is the sunny nonexpansive retraction.

Proof. Let $Q = P_{F(S) \cap \mathrm{EP}(\phi)}$. Then Qf is a contraction of H into itself. In fact, there exists $a \in [0,1)$ such that $\|f(x) - f(y)\| \leq a\|x - y\|$ for all $x, y \in H$. So, we have that

$$\|Qf(x) - Qf(y)\| \leq \|f(x) - f(y)\| \leq a\|x - y\| \tag{3.1}$$

for all $x, y \in H$. So, Qf is a contraction of H into itself. Since H is complete, there exists a unique element $z \in H$ such that $z = Qf(z)$, such a $z \in H$ is an element of C. We divide the proof into serval steps.

Step 1. $\{x_n\}$ and $\{u_n\}$ are all bounded. Let $p \in F(T) \cap \mathrm{EP}(\phi)$, Then from $u_n = T_{r_n} x_n$, we have

$$\|u_n - p\| = \|T_{r_n} x_n - T_{r_n} p\| \leq \|x_n - p\| \tag{3.2}$$

for all $n \in N$. Put $y_n = \alpha_n u + (1-\alpha_n)u_n$, so $\{x_{n+1}\}$ can be rewritten as

$$\begin{aligned}
x_{n+1} &= \beta_n f(x_n) + (1-\beta_n)Ty_n, \\
\|y_n - p\| &= \|\alpha_n u + (1-\alpha_n)u_n - p\| \\
&= \|\alpha_n(u-p) + (1-\alpha_n)(u_n - p)\| \\
&\le \alpha_n\|u-p\| + (1-\alpha_n)\|u_n - p\|.
\end{aligned} \tag{3.3}$$

Therefore, from (3.2) we get

$$\|y_n - p\| \le \alpha_n\|u-p\| + (1-\alpha_n)\|x_n - p\| \le \max\{\|u-p\|, \|x_n - p\|\}. \tag{3.4}$$

If $\|x_n - p\| \le \|u-p\|$, then $\{x_n\}$ is bounded. So, we assume that $\|x_n - p\| \ge \|u-p\|$.

Therefore $\|y_n - p\| \le \|x_n - p\|$,

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\beta_n f(x_n) + (1-\beta_n)Ty_n - p\| \\
&= \|\beta_n(f(x_n) - p) + (1-\beta_n)(Ty_n - p)\| \\
&\le \beta_n\|f(x_n) - p\| + (1-\beta_n)\|y_n - p\| \\
&\le \beta_n\alpha\|x_n - p\| + (1-\beta_n)\|x_n - p\| + \beta_n\|f(p) - p\| \\
&= (1-(1-\alpha)\beta_n)\|x_n - p\| + \beta_n(1-\alpha)\frac{\|f(p)-p\|}{1-\alpha} \\
&\le \max\left\{\|x_n - p\|, \left\|\frac{\|f(p)-p\|}{1-\alpha}\right\|\right\}.
\end{aligned} \tag{3.5}$$

So, by induction, we have

$$\|x_n - p\| \le \max\left\{\|x_1 - p\|\ ,\ \left\|\frac{\|f(p)-p\|}{1-\alpha}\right\|\right\}, \tag{3.6}$$

hence $\{x_n\}$ is bounded. we also obtain that $\{u_n\}$, $\{Tu_n\}$, $\{Tx_n\}$, $\{f(x_n)\}$ and $\{y_n\}$ are bounded.

Step 2. $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$,

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\beta_n f(x_n) + (1-\beta n)Ty_n - \beta_{n-1}f(x_{n-1}) - (1-\beta_{n-1})Ty_{n-1}\| \\
&= \|\beta_n(f(x_n) - f(x_{n-1})) + (\beta_n - \beta_{n-1})f(x_{n-1}) + Ty_n - Ty_{n-1} \\
&\quad -\beta_n(Ty_n - Ty_{n-1}) - (\beta_n - \beta_{n-1})Ty_{n-1}\| \\
&\le \beta_n\alpha\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|f(x_{n-1}) - Ty_{n-1}\| \\
&\quad + (1-\beta_n)\|y_n - y_{n-1}\|,
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
\|y_n - y_{n-1}\| &= \|\alpha_n u + (1-\alpha_n)u_n - \alpha_{n-1}u - (1-\alpha_{n-1})u_{n-1}\| \\
&= \|(\alpha_n - \alpha_{n-1})u + u_n - u_{n-1} - \alpha_n(u_n - u_{n-1}) - (\alpha_n - \alpha_{n-1})u_{n-1}\| \\
&\le |\alpha_n - \alpha_{n-1}|\|u - u_{n-1}\| + (1-\alpha_n)\|u_n - u_{n-1}\|.
\end{aligned} \tag{3.8}$$

On the other hand, from $u_n = T_{r_n} x_n$ and $u_{n+1} = T_{r_{n+1}} x_{n+1}$, we have

$$\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n,\ u_n - x_n\rangle \geq 0, \quad \forall y \in C, \tag{3.9}$$

$$\phi(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1},\ u_{n+1} - x_{n+1}\rangle \geq 0, \quad \forall y \in C. \tag{3.10}$$

Putting $y = u_{n+1}$ in (3.9) and $y = u_n$ in (3.10), we have

$$\begin{gathered}\phi(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n,\ u_n - x_n\rangle \geq 0, \\ \phi(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1},\ u_{n+1} - x_{n+1}\rangle \geq 0.\end{gathered} \tag{3.11}$$

So, from (A_2) we have

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0, \tag{3.12}$$

and hence

$$\left\langle u_{n+1} - u_n,\ u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \right\rangle \geq 0. \tag{3.13}$$

Without loss of generality, let us assume that there exists a real number b such that $r_n > b > 0$ for all $n \in N$. Then, we have

$$\begin{aligned}\|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n,\ x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1}) \right\rangle \\ &\leq \|u_{n+1} + u_n\| \left\{ \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \right\},\end{aligned} \tag{3.14}$$

and hence

$$\begin{aligned}\|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{1}{b}|r_{n+1} - r_n|L,\end{aligned} \tag{3.15}$$

where $L = \sup\{\|u_n - x_n\| : n \in N\}$. Then we obtain

$$\|u_n - u_{n-1}\| \leq \|x_n - x_{n-1}\| + \frac{1}{b}|r_n - r_{n-1}|L. \tag{3.16}$$

So, put (3.8) and (3.16) into (3.7) we have

$$\begin{aligned}
\|x_{n+1}-x_n\| &\le \beta_n\alpha\|x_n-x_{n-1}\| + \left|\beta_n-\beta_{n-1}\right| \left\|f(x_{n-1})-Ty_{n-1}\right\| \\
&\quad + (1-\beta_n)\left(|\alpha_n-\alpha_{n-1}|\|u-u_{n-1}\| + (1-\alpha_n)\left(\|x_{n+1}-x_n\| + \frac{1}{b}|r_{n+1}-r_n|L\right)\right) \\
&\le (1-(1-\alpha)\beta_n)\|x_n-x_{n-1}\| + \frac{1}{b}|r_{n+1}-r_n|L \\
&\quad + |\alpha_n-\alpha_{n-1}|K_1 + \left|\beta_n-\beta_{n-1}\right|K_2,
\end{aligned} \tag{3.17}$$

where $K_1 := \sup\{\|u-u_n\|,\ \forall n \ge 1\}$ is a constant; $K_2 := \sup\{\|f(x_{n-1})\| + \|Ty_{n-1}\|,\ \forall n \ge 1\}$ is a constant.

Using Lemma 2.3 and conditions (I), (II), (III) we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.18}$$

From (3.15) and $|r_{n+1}-r_n| \to 0$, we have

$$\lim_{n\to\infty}\|u_{n+1}-u_n\| = 0. \tag{3.19}$$

Since $y_n = \alpha_n u + (1-\alpha_n)u_n,\ x_n = \beta_n f(x_{n-1}) + (1-\beta_n)Ty_{n-1}$, we have

$$\left\|y_n-u_n\right\| = \alpha_n\|u-u_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{3.20}$$

$$\begin{aligned}
\|x_n-Tu_n\| &\le \left\|x_n-Ty_{n-1}\right\| + \left\|Ty_{n-1}-Tu_{n-1}\right\| + \|Tu_{n-1}-Tu_n\| \\
&\le \beta_{n-1}\left\|f(x_{n-1})+Ty_{n-1}\right\| + \left\|y_{n-1}-u_{n-1}\right\| + \|u_n-u_{n-1}\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.21}$$

For $p \in F(T) \cap \mathrm{EP}(\phi)$, we have

$$\begin{aligned}
\left\|u_n-p\right\|^2 &= \left\|T_{r_n}x_n - T_{r_n}p\right\|^2 \le \left\langle T_{r_n}x_n - T_{r_n}p,\ x_n-p\right\rangle \\
&= \left\langle u_n-p,\ x_n-p\right\rangle \\
&= \frac{1}{2}\left(\left\|u_n-p\right\|^2 + \left\|x_n-p\right\|^2 - \|u_n-x_n\|^2\right), \\
\left\|u_n-p\right\|^2 &\le \left\|x_n-p\right\|^2 - \|u_n-x_n\|^2.
\end{aligned} \tag{3.22}$$

Therefore, we have

$$
\begin{aligned}
\left\|x_{n+1}-p\right\|^2 &= \left\|(1-\beta_n)(Ty_n-p)+\beta_n(f(x_n)-p)\right\|^2 \\
&\le (1-\beta_n)^2\left\|Ty_n-p\right\|^2+2\beta_n\langle f(x_n)-p,\ x_{n+1}-p\rangle \\
&\le (1-\beta_n)^2\left(\left\|y_n-u_n\right\|+\left\|u_n-p\right\|\right)^2 \\
&\quad +2\beta_n\alpha\left\|x_n-p\right\|\left\|x_{n+1}-p\right\|+2\beta_n\left\|f(p)-p\right\|\left\|x_{n+1}-p\right\| \\
&= (1-\beta_n)^2\left\|u_n-p\right\|^2 \\
&\quad +\left\|y_n-u_n\right\|(1-\beta_n)^2\left(\left\|y_n-u_n\right\|+2\left\|u_n-p\right\|\right) \\
&\quad +2\beta_n\alpha\left\|x_n-p\right\|\left\|x_{n+1}-p\right\|+2\beta_n\left\|f(p)-p\right\|\left\|x_{n+1}-p\right\| \\
&= (1-\beta_n)^2\left(\left\|x_n-p\right\|^2-\|u_n-x_n\|^2\right) \\
&\quad +\left\|y_n-u_n\right\|(1-\beta_n)^2\left(\left\|y_n-u_n\right\|+2\left\|u_n-p\right\|\right) \\
&\quad +2\beta_n\alpha\left\|x_n-p\right\|\left\|x_{n+1}-p\right\|+2\beta_n\left\|f(p)-p\right\|\left\|x_{n+1}-p\right\| \\
&= \left(1-2\beta_n+\beta_n^2\right)\left\|x_n-p\right\|^2-(1-\beta_n)^2\|u_n-x_n\|^2 \\
&\quad +\left\|y_n-u_n\right\|(1-\beta_n)^2\left(\left\|y_n-u_n\right\|+2\left\|u_n-p\right\|\right) \\
&\quad +2\beta_n\alpha\left\|x_n-p\right\|\left\|x_{n+1}-p\right\|+2\beta_n\left\|f(p)-p\right\|\left\|x_{n+1}-p\right\|, \\
(1-\beta_n)^2\|u_n-x_n\|^2 &\le \left\|x_n-p\right\|^2-\left\|x_{n+1}-p\right\|^2 \\
&\quad +\beta_n\left(\beta_n\left\|x_n-p\right\|^2-2\left\|x_n-p\right\|^2\right)-(1-\beta_n)^2\|u_n-x_n\|^2 \\
&\quad +\left\|y_n-u_n\right\|(1-\beta_n)^2\left(\left\|y_n-u_n\right\|+2\left\|u_n-p\right\|\right) \\
&\quad +2\beta_n\alpha\left\|x_n-p\right\|\left\|x_{n+1}-p\right\|+2\beta_n\left\|f(p)-p\right\|\left\|x_{n+1}-p\right\| \\
&\le \|x_n-x_{n+1}\|\left\{\left\|x_n-p\right\|+\left\|x_{n+1}-p\right\|\right\} \\
&\quad +\beta_n\left(\beta_n\left\|x_n-p\right\|^2-2\left\|x_n-p\right\|^2\right)-(1-\beta_n)^2\|u_n-x_n\|^2 \\
&\quad +\left\|y_n-u_n\right\|(1-\beta_n)^2\left(\left\|y_n-u_n\right\|+2\left\|u_n-p\right\|\right) \\
&\quad +2\beta_n\alpha\left\|x_n-p\right\|\left\|x_{n+1}-p\right\|+2\beta_n\left\|f(p)-p\right\|\left\|x_{n+1}-p\right\|.
\end{aligned}
\tag{3.23}
$$

By the above of what we have and the condition of $\lim_{n\to\infty}\beta_n=0$, we get $\lim_{n\to\infty}\|x_n-u_n\|=0$. Since $\|Tu_n-u_n\|\le\|Tu_n-x_n\|+\|x_n-u_n\|$, it follows that $\|Tu_n-u_n\|\ \to\ 0$.

Step 3. we show that

$$
\limsup_{n\to\infty}\langle f(z)-z,\ x_n-z\rangle\le 0,
\tag{3.24}
$$

where $z = P_{F(S)\cap \mathrm{EP}(\phi)} f(z)$. To show this inequality, we choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\limsup_{i\to\infty}\langle f(z)-z,\ x_{n_i}-z\rangle = \limsup_{n\to\infty}\langle f(z)-z,\ x_n-z\rangle. \tag{3.25}$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ which converges weakly to w. Without loss of generality, we can assume that $u_{n_i} \rightharpoonup w$. From $\|Tu_n - u_n\| \to 0$, we obtain $Tu_{n_i} \rightharpoonup w$. Let us show $w \in \mathrm{EP}(\phi)$. By $u_n = T_{r_n} x_n$, we have

$$\phi(u_n, y) + \frac{1}{r_n}\langle y-u_n,\ u_n-x_n\rangle \geq 0, \quad \forall y \in C. \tag{3.26}$$

From (A_2), we also have

$$\frac{1}{r_n}\langle y-u_n,\ u_n-x_n\rangle \geq \phi(y, u_n), \tag{3.27}$$

and hence

$$\left\langle y-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\right\rangle \geq \phi(y, u_{n_i}). \tag{3.28}$$

Since $u_{n_i} - x_{n_i}/r_{n_i} \to 0$ and $u_{n_i} \rightharpoonup w$, from (A_4) we have $0 \geq \phi(y, w)$ for all $y \in C$. For t with $0 < t < 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we have $y_t \in C$ and hence $\phi(y_t, w) \leq 0$. So, from (A_1) and A_4 we have

$$\begin{aligned} 0 &= \phi(y_t, y_t) \\ &\leq t\phi(y_t, y) + (1-t)\phi(y_t, w) \\ &\leq t\phi(y_t, y), \end{aligned} \tag{3.29}$$

and hence $0 \leq \phi(y_t, y)$. From (A_3), we have $0 \leq \phi(w, y)$ for all $y \in C$, and hence $w \in \mathrm{EP}(\phi)$. We will show that $w \in F(T)$. Assume that $w \notin F(T)$. Since $u_{n_i} \rightharpoonup w$ and $w \neq Tw$, from Opial's theorem we have

$$\begin{aligned} \liminf_{i\to\infty}\|u_{n_i}-w\| &< \liminf_{i\to\infty}\|u_{n_i}-Tw\| \\ &\leq \liminf_{i\to\infty}\{\|u_{n_i}-Tu_{n_i}\| + \|Tu_{n_i}-Tw\|\} \\ &\leq \liminf_{i\to\infty}\|u_{n_i}-w\|. \end{aligned} \tag{3.30}$$

This is a contradiction. So, we get $w \in F(T)$. Therefore, $w \in F(T) \cap \mathrm{EP}(\phi)$. Since $z = P_{F(T)\cap \mathrm{EP}(\phi)} f(z)$, we have

$$\begin{aligned} \limsup_{n\to\infty}\langle f(z)-z,\ x_n-z\rangle &= \lim_{i\to\infty}\langle f(z)-z,\ x_{n_i}-z\rangle \\ &= \langle f(z)-z,\ w-z\rangle \leq 0. \end{aligned} \tag{3.31}$$

From $x_{n+1} - z = (1-\beta_n)(Ty_n - z) + \beta_n(f(x_{\mathrm{n}}) - z)$, we have

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &\leq (1-\beta_n)^2 \left\|y_n - z\right\|^2 + 2\beta_n \langle f(x_n) - z,\ x_{n+1} - z\rangle \\
&\leq (1-\beta_n)^2 \|\alpha_n(u-z) + (1-\alpha_n)(x_n - z)\|^2 \\
&\quad + 2\beta_n\alpha\|x_n - z\|\|x_{n+1} - z\| + 2\beta_n\langle f(z) - z,\ x_{n+1} - z\rangle \\
&\leq (1-\beta_n)^2\left(\alpha_n\|u - z\|^2 + (1-\alpha_n)\|x_n - z\|^2\right) \\
&\quad + \beta_n\alpha\left(\|x_n - z\|^2 + \|x_{n+1} - z\|^2\right) + 2\beta_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\leq \left[(1-\beta_n)^2(1-\alpha_n) + \beta_n\alpha\right]\|x_n - z\|^2 \\
&\qquad + \beta_n\alpha\|x_{n+1} - z\|^2 + (1-\beta_n)^2\alpha_n\|u - z\|^2 + 2\beta_n\langle f(z) - z, x_{n+1} - z\rangle, \\
\|x_{n+1} - z\|^2 &\leq \frac{\left((1-\beta_n)^2 + \beta_n\alpha\right)}{1-\alpha\beta_n}\|x_n - z\|^2 \\
&\quad + \frac{(1-\beta_n)^2\alpha_n}{1-\alpha\beta_n}\|u - z\|^2 + \frac{2\beta_n}{1-\alpha\beta_n}\langle f(z) - z,\ \ x_{n+1} - z\rangle \\
&\leq \left[1 - \frac{2(1-\alpha)\beta_n}{1-\alpha\beta_n}\right]\|x_n - z\|^2 + \frac{\beta_n^2}{1-\alpha\beta_n}\|x_n - z\|^2 \\
&\quad + \frac{(1-\beta_n)^2\alpha_n}{1-\alpha\beta_n}\|u - z\|^2 + \frac{2\beta_n}{1-\alpha\beta_n}\langle f(z) - z,\ \ x_{n+1} - z\rangle \\
&\leq \left[1 - \frac{2(1-\alpha)\beta_n}{1-\alpha\beta_n}\right]\|x_n - z\|^2 + \frac{2(1-\alpha)\beta_n}{1-\alpha\beta_n} \\
&\quad \times \left\{\frac{\beta_n}{2(1-\alpha)}M + \frac{(1-\beta_n)^2\alpha_n}{2(1-\alpha)\beta_n}\|u - z\|^2 + \frac{1}{1-\alpha}\langle f(z) - z,\ x_{n+1} - z\rangle\right\} \\
&= (1-\delta_n)\|x_n - z\|^2 + \delta_n\zeta_n,
\end{aligned}
\tag{3.32}
$$

where $M = \sup\{\|x_n - z\|^2 : n \in \mathbb{N}\}$, $\delta_n = 2(1-\alpha)\beta_n/1-\alpha\beta_n$ and $\zeta_n := \beta_n/2(1-\alpha)M + (1-\beta_n)^2\alpha_n/2(1-\alpha)\beta_n\|u - z\|^2 + 1/1-\alpha\langle f(z) - z,\ x_{n+1} - z\rangle$. It is easy to see that $\delta_n \to 0$, $\Sigma_{n=1}^{\infty}\delta_n = \infty$ and $\limsup_{n\to\infty}\zeta_n/\delta_n \leq 0$ by (3.31) and the conditions. Hence, by Lemma 2.3, the sequence $\{x_n\}$ converges strongly to z.

If z_t is definition as (2.1), then, from Lemma 2.5, we have $\|z_t - q\| \to 0$ as $t \to 0$, and if we set $Q(u) := \lim_{t\to 0} z_t$, then Q defines the unique sunny nonexpansive retraction from C onto $\mathrm{Fix}(T)$. So, if we replace t with α_n, the corollary still holds. and it is that $z_n = T(\alpha_n u + (1-\alpha_n)z_n)$ is a fixed point sequence and $\|z_n - q\| \to 0$ as $n \to \infty$, and if we set $Q(u) := \lim_{n\to\infty} z_n$, then Q defines the unique sunny nonexpansive retraction from C onto $\mathrm{Fix}(T)$. In the iterative algorithm of Theorem 3.1, we can take z_n to replace Ty_n in particular. Then, we have $x_{n+1} = \beta_n f(x_n) + (1-\beta_n)z_n$, so $\|x_{n+1} - z_n\| = \beta_n\|f(x_n) - z_n\| \to 0$ as $n \to \infty$. By the uniqueness of limit, we have $z = q$, that is, $z = Q(u)$, where Q defines the unique sunny nonexpansive retraction from C onto $\mathrm{Fix}(T)$. □

Remark. We notice that $u_n = T_{r_n} x_n$ has not influence on x_n, $u_n \to z = P_{F(T)\cap \mathrm{EP}(\phi)} f(z)$.

As direct consequences of Theorem 3.1, we obtain corollary.

Corollary 3.2. *Let C be a nonempty closed convex subset of H, $S : C \to C$ be a nonexpansive mapping of C into H such that $F(S) \neq \emptyset$. Let f be a contraction of H into itself and let $\{x_n\}$ and $\{u_n\}$ be sequences generated initially by an arbitrary elements $x_1 \in H$ and then by*

$$x_{n+1} = \beta_n f(x_n) + (1-\beta_n) S(\alpha_n u + (1-\alpha_n) P_C x_n) \tag{3.33}$$

for all $n \in N$, where $\{\alpha_n\} \subset (0, \infty)$ satisfies the following conditions:

(I) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(II) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;

(III) $\lim_{n\to\infty} \alpha_n / \beta_n = 0$.

Then, the sequences $\{x_n\}$ converge strongly to $z \in F(S)$, where $z = P_{F(S)} f(z)$.

Proof. Put $\phi(x, y) = 0$, for all $x, y \in C$ and $r_n = 1$, for all $n \in \mathbb{N}$ in Theorem 3.1.

Then we have $u_n = P_C x_n$. So, from Theorem 3.1, the sequence x_n generated by $x_1 \in H$ and

$$x_{n+1} = \beta_n f(x_n) + (1-\beta_n) S(\alpha_n u + (1-\alpha_n) P_C x_n) \tag{3.34}$$

for all $n \in \mathbb{N}$ converges strongly to $z \in F(S)$, where $z = P_{F(S)} f(z)$. □

4. Application for Zeros of Maximal Monotone Operators

We adapt in this section the iterative algorithm (3.1) to find zeros of maximal monotone operators and find $\mathrm{EP}(\phi)$. Let us recall that an operator A with domain $D(A)$ and range $R(A)$ in a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ is said to be monotone if the graph of A,

$$G(T) := \{(x, y) \in H \times H : x \in D(T),\ y \in Tx\} \tag{4.1}$$

is a monotone set. Namely,

$$\langle x - x',\ y - y' \rangle \geq 0, \qquad (x, y), (x', y') \in G(A). \tag{4.2}$$

A monotone operator A is said to be maximal monotone of the graph $G(T)$ is not properly contained in the graph of any other monotone defined in H. See Brezis [16] for more details on maximal monotone operators.

In this section we always assume that A is maximal monotone and the set of zeros of A, $N(A) = \{x \in D(A) : 0 \in Ax\}$, is nonempty so that the metric projection $P_{N(A)}$ from H onto $N(A)$ is well-defined.

One of the major problems in the theory of maximal operators is to find a point in the zero set $N(A)$ because various problems arising from economics, convex programming, and other applied areas can be formulated as finding a zero of maximal monotone operators. The proximal point algorithm (PPA) of Rockafellar [17] is commonly recognized as the most powerful algorithm in finding a zero of maximal monotone operators. This (PPA) generates, starting with any initial guess $x_0 \in H$, a sequence $\{x_n\}$ according to the inclusion:

$$x_n + e_n \in x_{n+1} + c_n A(x_{n+1}), \tag{4.3}$$

where $\{e_n\}$ is a sequence of errors and $\{c_n\}$ is a sequence of positive regularization parameters. Equivalently, we can write

$$x_{n+1} = J_{c_n}^{A}(x_n + e_n), \tag{4.4}$$

where for $c > 0$, J_c^A denotes the resolvent of A,

$$J_c^A = (I + cA)^{-1}, \tag{4.5}$$

with I being the identity operator on the space H.

Rockafellar [17] proved the weak convergence of his algorithm (4.4) provided the regularization sequence $\{c_n\}$ remains bounded away from zero and the error sequence $\{e_n\}$ satisfies the condition

$$\sum_{n=0}^{\infty} \|e_n\| < \infty. \tag{4.6}$$

The aim of this section is to combine algorithm (3.1) with algorithm (4.4). Our algorithm generates a sequence $\{x_n\}$ and $\{u_n\}$ be sequences generated initially by an arbitrary elements $x_1 \in H$ and then by

$$\begin{gathered} \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n,\ u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) J_{c_n}^{A}(\alpha_n u + (1 - \alpha_n) u_n + e_n), \end{gathered} \tag{4.7}$$

where α_n and c_n are sequences of positive real numbers. Furthermore, we prove that $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in N(A) \cap \mathrm{EP}(\phi)$, where $z = P_{N(A)\cap \mathrm{EP}(\phi)} f(z)$.

Before stating the convergence theorem of the algorithm (4.7), we list some properties of maximal monotone operators.

Proposition 4.1. *Let A be a maximal monotone operator in H and let $J_c^A = (I + cA)^{-1}$ denote the resolvent, where $c > 0$,*

(a) *J_c^A is nonexpansive for all $c > 0$;*

(b) *$N(A) = F(J_c)$ for all $c > 0$;*

(c) *For* $c > c' > 0$, $\|J_{c'}^A - x\| \leq 2\|J_c^A - x\|$ *for* $x \in H$;

(d) *(The Resolvent Identity) For* $\lambda,\ \mu > 0$, *there holds the identity:*

$$J_\lambda^T x = J_\mu^T\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_\lambda^T x\right), \quad x \in H. \tag{4.8}$$

Theorem 4.2. *Let* C *be a nonempty closed convex subset of* H, $\phi : C \times C \to \infty$ *be a bifunction satisfying* (A_1)*–*(A_4) *and* A *be a maximal monotone operator such that* $N(A) \cap \mathrm{EP}(\phi) \neq \emptyset$. *Let* f *be a contraction of* H *into itself and let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated initially by an arbitrary elements* $x_0 \in H$ *and then by*

$$\begin{gathered}\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n,\ u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) J_{c_n}^A(\alpha_n u + (1 - \alpha_n)u_n + e_n),\end{gathered} \tag{4.9}$$

for all $n \in N$, *where* $\{\alpha_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0, \infty)$ *satisfy the following conditions:*

(I) $\lim_{n\to\infty}\alpha_n = 0$, $\Sigma_{n=1}^\infty \alpha_n = \infty$ *and* $\Sigma_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;

(II) $\liminf_n r_n > 0$ *and* $\Sigma_{n=1}^\infty |r_{n+1} - r_n| < \infty$;

(III) $\lim_{n\to\infty}(c_{n+1}/c_n) = 1$;

(IV) $\lim_{n\to\infty}\beta_n = 0$, $\Sigma_{n=1}^\infty \beta_n = \infty$, $\Sigma_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$, *and* $\lim_{n\to\infty}(\alpha_n/\beta_n) = 0$.

Then, the sequences $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $z \in N(A) \cap \mathrm{EP}(\phi)$, *where* $z = P_{N(A)\cap \mathrm{EP}(\phi)} f(z)$.

Proof. Below we write $J_c = J_c^A$ for simplicity. Setting

$$w_n = \alpha_n u + (1 - \alpha_n)u_n + e_n, \qquad y_n = J_{c_n} w_n, \tag{4.10}$$

we rewrite x_{n+1} of (4.7) as

$$x_{n+1} = \beta_n f(x_n) + (1 - \beta_n) J_{c_n} w_n = \beta_n f(x_n) + (1 - \beta_n) y_n. \tag{4.11}$$

Because the proof is similar to Theorem 3.1, here we just give the main steps as follows:

(1) $\{x_n\}$ is bounded;

(2) $\|x_{n+1} - x_n\| \to 0$, as $n \to 0$;

(3) $\|u_n - J_{c_n} u_n\| \to 0$, as $n \to 0$;

(4) $\|x_n - u_n\| \to 0$, as $n \to 0$;

(5) $\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle \leq 0$;

(6) $x_n, u_n \to z$, as $n \to z$. □

5. Application for Optimization Problem

In this section, we study a kind of optimization problem by using the result of this paper. That is, we will give an iterative algorithm of solution for the following optimization problem with nonempty set of solutions

$$\begin{gathered} \min h(x), \\ x \in C, \end{gathered} \tag{5.1}$$

where $h(x)$ is a convex and lower semicontinuous functional defined on a closed convex subset C of a Hilbert space H. We denoted by B the set of solutions of (5.1). Let ϕ be a bifunction from $C \times C$ to R defined by $\phi(x,y) = h(y) - h(x)$. We consider the following equilibrium problem, that is, to find $x \in C$ such that

$$\phi(x,y) \geq 0, \quad \forall y \in C. \tag{5.2}$$

It is obvious that $\mathrm{EP}(\phi) = B$, where $\mathrm{EP}(\phi)$ denotes the set of solutions of equilibrium problem (5.2). In addition, it is easy to see that $\phi(x,y)$ satisfies the conditions (A_1)–(A_4) in the Section 2. Therefore, from Theorem 3.1, we know that the following iterative algorithm:

$$\begin{gathered} h(y) - h(u_n) + \frac{1}{r_n}\langle y - u_n,\ u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n) T(\alpha_n u + (1-\alpha_n) u_n), \end{gathered} \tag{5.3}$$

for any initial guess x_1, converges strongly to a solution $z = P_B f(z)$ of optimization problem (5.1), where $\{\alpha_n\} \subset [0,1]$, $\{\beta_n\} \subset [0,1]$, and $\{r_n\} \subset [0,\infty)$ satisfy

$$\begin{gathered} \lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \\ \lim_{n\to\infty} \beta_n = 0, \qquad \sum_{n=1}^{\infty} \beta_n = \infty, \qquad \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \qquad \lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0, \\ \liminf_{n\to\infty} r_n > 0, \qquad \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty. \end{gathered} \tag{5.4}$$

For a special case, we pick $f(x) = \eta$, for all $\eta \in H$, and $r_n = 1$, $\beta_n = 1/2$ and $\alpha_n = 0$ for all $n \geq 1$, then $x_{n+1} = (1/2)Tu_n + (1/2)\eta$, from (5.3), we get the special iterative algorithm as follows:

$$\begin{gathered} h(y) - h(u_n) + \left\langle y - u_n,\ u_n - \left(\frac{1}{2}\eta + \frac{1}{2}Tu_n\right)\right\rangle \geq 0, \quad \forall y \in C,\ n \geq 2, \\ h(y) - h(u_1) + \left\langle y - u_1,\ u_1 - \left(\frac{1}{2}\eta + \frac{1}{2}Tu_1\right)\right\rangle \geq 0, \quad \forall y \in C. \end{gathered} \tag{5.5}$$

Then $\{u_n\}$ converges strongly to a solution $z = P_B \eta$ of optimization problem (5.1).

In fact, z is the minimum norm point from η onto the B, furthermore, if $\eta = 0$, then z is the minimum norm point on the B.

Acknowledgment

This work was supported by the National Natural Science Foundation of China under Grants (11071270), and (10771050).

References

[1] S. D. Flåm and A. S. Antipin, "Equilibrium programming using proximal-like algorithms," *Mathematical Programming*, vol. 78, no. 1, pp. 29–41, 1997.
[2] A. Moudafi and M. Théra, "Proximal and dynamical approaches to equilibrium problems," in *Lecture Note in Economics and Mathematical Systems*, vol. 477, pp. 187–201, Springer-Verlag, New York, NY, USA, 1999.
[3] A. Tada and W. Takahashi, "Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem," *Journal of Optimization Theory and Applications*, vol. 133, no. 3, pp. 359–370, 2007.
[4] Y. Su, M. Shang, and X. Qin, "An iterative method of solution for equilibrium and optimization problems," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 69, no. 8, pp. 2709–2719, 2008.
[5] S. Takahashi and W. Takahashi, "Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 1, pp. 506–515, 2007.
[6] P. L. Combettes and S. A. Hirstoaga, "Equilibrium programming in Hilbert spaces," *Journal of Nonlinear and Convex Analysis*, vol. 6, no. 1, pp. 117–136, 2005.
[7] F. E. Browder, "Existence and approximation of solutions of nonlinear variational inequalities," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 56, pp. 1080–1086, 1966.
[8] F. E. Browder, "Convergence of approximants to fixed points of nonexpansive non-linear mappings in Banach spaces," *Archive for Rational Mechanics and Analysis*, vol. 24, pp. 82–90, 1967.
[9] R. E. Bruck, Jr., "Nonexpansive projections on subsets of Banach spaces," *Pacific Journal of Mathematics*, vol. 47, pp. 341–355, 1973.
[10] H.-K. Xu, "A regularization method for the proximal point algorithm," *Journal of Global Optimization*, vol. 36, no. 1, pp. 115–125, 2006.
[11] A. Moudafi, "Viscosity approximation methods for fixed-points problems," *Journal of Mathematical Analysis and Applications*, vol. 241, no. 1, pp. 46–55, 2000.
[12] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, Japan, 2000.
[13] Z. Opial, "Weak convergence of the sequence of successive approximations for nonexpansive mappings," *Bulletin of the American Mathematical Society*, vol. 73, pp. 591–597, 1967.
[14] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, no. 1–4, pp. 123–145, 1994.
[15] H. K. Xu, "An iterative approach to quadratic optimization," *Journal of Optimization Theory and Applications*, vol. 116, no. 3, pp. 659–678, 2003.
[16] H. Brezis, *Operateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert*, North-Holland Publishing, Amsterdam, Holland, 1973.
[17] R. T. Rockafellar, "Monotone operators and the proximal point algorithm," *SIAM Journal on Control and Optimization*, vol. 14, no. 5, pp. 877–898, 1976.


Advances in
Operations Research
Advances in
Decision Sciences
Mathematical Problems
in Engineering
Algebra
Journal of
Probability and Statistics
The Scientific
World Journal
International Journal of
Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of
Combinatorics
Advances in
Mathematical Physics
Journal of
Complex Analysis
Journal of
Mathematics
International Journal of
Stochastic Analysis
Abstract and
Applied Analysis
Discrete Dynamics in
Nature and Society
International
Journal of
Mathematics and
Mathematical
Sciences
Journal of
Discrete Mathematics
Journal of
Function Spaces
Journal of
Applied Mathematics
Journal of
Optimization