Research Article

Exponentiated Gamma Distribution: Different Methods of Estimations

A. I. Shawky and R. A. Bakoban

Department of Statistics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

Correspondence should be addressed to A. I. Shawky, aishawky@yahoo.com

The exponentiated gamma (EG) distribution and Fisher information matrices for complete, Type I, and Type II censored observations are obtained. Asymptotic variances of the different estimators are derived. Also, we consider different estimators and compare their performance through Monte Carlo simulations.

1. Introduction

Gupta et al. [1] introduced the exponentiated gamma (EG) distribution. This model is flexible enough to accommodate both monotonic as well as nonmonotonic failure rates. The EG distribution has the distribution function (c.d.f.):

$$F(x; \theta, \lambda) = \left[1 - e^{-\lambda x}(\lambda x + 1)\right]^{\theta}, \quad \theta, \ \lambda, \ x > 0. \tag{1.1}$$

Therefore, EG distribution has the density function:

$$f(x; \theta, \lambda) = \theta \lambda^2 x e^{-\lambda x}\left[1 - e^{-\lambda x}(\lambda x + 1)\right]^{\theta-1}, \quad \theta, \ \lambda, \ x > 0, \tag{1.2}$$

the survival function

$$R(x; \theta, \lambda) = 1 - \left[1 - e^{-\lambda x}(\lambda x + 1)\right]^{\theta}, \quad \theta, \lambda, x > 0, \tag{1.3}$$

and the hazard function

$$h(x; \theta, \lambda) = \frac{\theta \lambda^2 \, x e^{-\lambda x} \left[1 - e^{-\lambda x} (\lambda x + 1)\right]^{\theta - 1}}{1 - \left[1 - e^{-\lambda x} (\lambda x + 1)\right]^{\theta}}, \quad \theta, \lambda, x > 0, \tag{1.4}$$

here θ, and λ are the shape and scale parameters, respectively. The two-parameter EG distribution will be denoted by $EG(\theta, \lambda)$. For details, see Bakoban [2], Coronel-Brizio et al. [3] and Shawky and Bakoban [4–10].

Computation of Fisher information for any particular distribution is quite important, see, for example, Zheng [11]. Fisher information matrix can be used to compute the asymptotic variances of the different functions of the estimators, for example, maximum likelihood estimators (MLEs). The problem is quite important when the data are censored. We compute Fisher information matrices of EG distribution for complete and censored samples. We then study the properties of the MLEs of EG distribution under complete and censored samples in great details. Also, we consider different estimators of EG distribution and study how the estimators of the different unknown parameter(s) behave for different sample sizes and for different parameter values. We mainly compare the MLEs, estimators based on percentiles (PCEs), least squares estimators (LSEs), weighted least squares estimators (WLSEs), method of moment estimators (MMEs) and the estimators based on the linear combinations of order statistics (LMEs) by using extensive simulation techniques. It is important to mention that many authors interested with estimating parameters of such distributions, for example, Wingo [12] derived MLEs of Burr XII distribution parameters under type II censoring. Estimations based on order statistics under complete and censored samples compared with MLE which in turn based on a complete sample were studied by Raqab [13] for the Burr X distribution. Also, Gupta and Kundu [14] presented the properties of MLE's for generalized exponential (GE) distribution, and they discussed other methods for GE in [15]. Kundu and Raqab [16] discussed the generalized Rayleigh distribution too. Hossain and Zimmer [17] compared several methods for estimating Weibull parameters with complete and censored samples. Surles and Padgett [18] considered the MLEs and discussed the asymptotic properties of these estimators for complete and censored samples from Burr X distribution.

The rest of the paper is organized as follows. In Section 2, we obtain the Fisher information matrices of EG distribution. In Section 3, we derive MLEs of EG distribution and study its properties. In Sections 4 to 7, we describe other methods of estimations. Simulation results and discussions are provided in Section 8.

2. Fisher Information Matrix

2.1. Fisher Information Matrix for Complete Sample

Let X be a continuous random variable with the cumulative distribution function (c.d.f.) $F(x; \Omega)$ and the probability density function (p.d.f.) $f(x; \Omega)$. For the simplicity, we consider only two parameters θ and λ, although the results are true for any finite-dimensional vector. Under the standard regularity conditions (Gupta and Kundu [19]), the Fisher information

matrix for the parameter vector $\mathbf{\Omega} = (\theta, \lambda)$ based on an observation in terms of the expected values of the first and second derivatives of the log-likelihood function is

$$\mathbf{I}(\theta, \lambda) = \frac{1}{n} \begin{bmatrix} E\left(-\dfrac{\partial^2 \ln L(\theta, \lambda)}{\partial \theta^2}\right) & E\left(-\dfrac{\partial^2 \ln L(\theta, \lambda)}{\partial \theta \partial \lambda}\right) \\ E\left(-\dfrac{\partial^2 \ln L(\theta, \lambda)}{\partial \lambda \partial \theta}\right) & E\left(-\dfrac{\partial^2 \ln L(\theta, \lambda)}{\partial \lambda^2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \tag{2.1}$$

where for $i, j = 1, 2$,

$$a_{ij} = \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial \Omega_i} \ln f(x, \mathbf{\Omega})\right)\left(\frac{\partial}{\partial \Omega_j} \ln f(x, \mathbf{\Omega})\right) f(x, \mathbf{\Omega}) dx. \tag{2.2}$$

Now, we will derive Fisher information matrix of $EG(\theta, \lambda)$ under complete sample. It can be shown that

$$\ln f(x; \theta, \lambda) = \ln \theta + 2 \ln \lambda - \lambda x + \ln x + (\theta - 1) \ln\left[1 - e^{-\lambda x}(\lambda x + 1)\right]. \tag{2.3}$$

Differentiating (2.3) with respect to θ and λ respectively, we have

$$\frac{\partial \ln f(x; \theta, \lambda)}{\partial \theta} = \frac{1}{\theta} + \ln\left[1 - e^{-\lambda x}(\lambda x + 1)\right],$$

$$\frac{\partial \ln f(x; \theta, \lambda)}{\partial \lambda} = \frac{2}{\lambda} - x + (\theta - 1)\lambda x^2 e^{-\lambda x}\left[1 - e^{-\lambda x}(\lambda x + 1)\right]^{-1}. \tag{2.4}$$

Therefore, the second derivatives are

$$\frac{\partial^2 \ln f(x; \theta, \lambda)}{\partial \theta^2} = -\frac{1}{\theta^2},$$

$$\frac{\partial^2 \ln f(x; \theta, \lambda)}{\partial \lambda^2} = -\frac{2}{\lambda^2} - (\theta - 1)\lambda x^3 e^{-\lambda x}\left[1 - e^{-\lambda x}(\lambda x + 1)\right]^{-1}$$

$$\quad + (\theta - 1)x^2 e^{-\lambda x}\left[1 - e^{-\lambda x}(\lambda x + 1)\right]^{-1} - (\theta - 1)\lambda^2 x^4 e^{-\lambda x}\left[1 - e^{-\lambda x}(\lambda x + 1)\right]^{-2},$$

$$\frac{\partial^2 \ln f(x; \theta, \lambda)}{\partial \lambda \partial \theta} = \lambda x^2 e^{-\lambda x}\left[1 - e^{-\lambda x}(\lambda x + 1)\right]^{-1}. \tag{2.5}$$

Thus, the elements of Fisher information matrix for single observation from $EG(\theta, \lambda)$ are in the forms:

$$a_{11} = E\left(-\frac{\partial^2 \ln f(x; \theta, \lambda)}{\partial \theta^2}\right) = \frac{1}{\theta^2}, \tag{2.6}$$

$$a_{12} = a_{21} = E\left(-\frac{\partial^2 \ln f(x;\theta,\lambda)}{\partial\theta\partial\lambda}\right) = \frac{2 - \theta[2 + A_1(\theta)]}{\lambda(\theta - 1)}, \tag{2.7}$$

$$a_{22} = E\left(-\frac{\partial^2 \ln f(x;\theta,\lambda)}{\partial\lambda^2}\right) = \frac{\theta^2[A_2(\theta) - (4 + 2/\theta)A_1(\theta) - 2]}{\lambda^2(\theta - 2)}, \tag{2.8}$$

where

$$A_r(\theta) = \sum_{j=1}^{\infty}\sum_{k=0}^{j}(-1)^j\binom{\theta-1}{j}\binom{j}{k}\frac{\Gamma(r+k+2)}{(1+j)^{r+k+2}}, \qquad r = 1,\ 2,\ \ldots. \tag{2.9}$$

Moreover, Fisher information matrix for a complete sample of size n from $\mathrm{EG}(\theta,\lambda)$ is simply $n\mathbf{I}(\theta,\lambda)$.

2.2. Fisher Information Matrix under Type II Censoring

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from $F(x;\mathbf{\Omega})$. In life-time analysis, n items are on test. The test continues until the rth smallest outcome is observed, $1 < r < n$. Thus, we observe the smallest r-order statistics from $F(x;\mathbf{\Omega})$, denoted by $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{r:n}$, which are called the Type II censored data.

Denote $X_{r:n}$ by $X_{[np]:n}$, where $[np]$ is the integer part of np, $0 < p < 1$. Thus, $r/n \to p$ as $n \to \infty$. Denote Fisher information matrix in $X_{1:n},\ X_{2:n}, \ldots, X_{[np]:n}$ by $I_{1\cdots[np]:n}(\mathbf{\Omega})$ (see, Zheng [11]), where $\mathbf{\Omega} = (\theta,\lambda)$ in the case of the $\mathrm{EG}(\theta,\lambda)$ distribution, and define

$$I_{[0,\,p]}(\mathbf{\Omega}) = \lim_{n\to\infty}\frac{1}{n}I_{1\cdots[np]:n}(\mathbf{\Omega}). \tag{2.10}$$

The estimates based on $X_{1:n},\ X_{2:n}, \ldots, X_{[np]:n}$, under suitable conditions, are asymptotically normal, where the asymptotic covariance matrix is the inverse of $I_{[0,\,p]}(\mathbf{\Omega})$.

Assuming the regularity conditions hold, the following expression for $I_{[0,\,p]}(\mathbf{\Omega})$, where $\mathbf{\Omega}$ is any finite-dimensional vector, can be expressed as

$$I_{[0,\,p]}(\mathbf{\Omega}) = \int_{-\infty}^{v_p}\left[\frac{\partial}{\partial\mathbf{\Omega}}\ln h(x;\mathbf{\Omega})\right]\left[\frac{\partial}{\partial\mathbf{\Omega}}\ln h(x;\mathbf{\Omega})\right]^T f(x;\mathbf{\Omega})dx, \tag{2.11}$$

where v_p is the pth percentile of $F(x;\mathbf{\Omega})$, T denotes the transpose, and $h(x;\mathbf{\Omega})$ is the hazard function.

If there is no censoring ($p = 1$), then (2.11) becomes the usual Fisher information in a single variable (2.1).

In the following, we use (2.11) to obtain Fisher information matrix under Type II censoring for the $EG(\theta, \lambda)$ distribution. For $\mathbf{\Omega} = (\theta, \lambda)$, denote Fisher information matrix $I_{[0,\,p]}(\mathbf{\Omega})$ as

$$I_{[0,\,p]}(\mathbf{\Omega}) = \begin{bmatrix} I_{[0,\,p]}(\theta) & I_{[0,\,p]}(\theta, \lambda) \\ I_{[0,\,p]}(\theta, \lambda) & I_{[0,\,p]}(\lambda) \end{bmatrix}, \tag{2.12}$$

where $I_{[0,\,p]}(\theta)$, $I_{[0,\,p]}(\lambda)$, and $I_{[0,\,p]}(\theta, \lambda)$ can be obtained by (2.11).

It can be shown that

$$\ln h(x; \theta, \lambda) = \ln \theta + 2 \ln \lambda + \ln x - \lambda x + (\theta - 1) \ln\left[1 - e^{-\lambda x}(\lambda x + 1)\right]$$
$$- \ln\left\{1 - \left[1 - e^{-\lambda x}(\lambda x + 1)\right]^{\theta}\right\}. \tag{2.13}$$

Differentiating (2.13) with respect to θ and λ, respectively, we have

$$\frac{\partial \ln h(x; \theta, \lambda)}{\partial \theta} = \frac{1}{\theta} + \frac{\ln\left[1 - e^{-\lambda x}(\lambda x + 1)\right]}{1 - \left[1 - e^{-\lambda x}(\lambda x + 1)\right]^{\theta}},$$

$$\frac{\partial \ln h(x; \theta, \lambda)}{\partial \lambda} = \frac{2}{\lambda} - \frac{x(1 - e^{-\lambda x})}{1 - e^{-\lambda x}(\lambda x + 1)} + \frac{\theta \lambda x^2 e^{-\lambda x}}{\left[1 - e^{-\lambda x}(\lambda x + 1)\right]\left\{1 - \left[1 - e^{-\lambda x}(\lambda x + 1)\right]^{\theta}\right\}}. \tag{2.14}$$

Thus, it is easily to see, for $p = \left[1 - e^{-\lambda v_p}(\lambda v_p + 1)\right]^{\theta}$, that the elements of $I_{[0,\,p]}(\mathbf{\Omega})$ from $EG(\theta, \lambda)$ are

$$I_{[0,\,p]}(\theta) = \theta \lambda^2 \int_0^{v_p} \left[\frac{\partial}{\partial \theta} \ln h(x; \theta, \lambda)\right]^2 x e^{-\lambda x}\left[1 - e^{-\lambda x}(\lambda x + 1)\right]^{\theta - 1} dx$$

$$= \theta \int_0^{\lambda v_p} \left[\frac{1}{\theta} + \frac{\ln\left[1 - e^{-y}(y + 1)\right]}{1 - \left[1 - e^{-y}(y + 1)\right]^{\theta}}\right]^2 y e^{-y}\left[1 - e^{-y}(y + 1)\right]^{\theta - 1} dy \tag{2.15}$$

$$= \frac{1}{\theta^2} \int_0^p \left[1 + \frac{\ln x}{1 - x}\right]^2 dx = \frac{1}{\theta^2}\left[p + \frac{p}{1 - p}(\ln p)^2\right],$$

$$I_{[0,\,p]}(\lambda) = \theta \lambda^2 \int_0^{v_p} \left[\frac{\partial}{\partial \lambda} \ln h(x; \theta, \lambda)\right]^2 x e^{-\lambda x}\left[1 - e^{-\lambda x}(\lambda x + 1)\right]^{\theta - 1} dx$$

$$= \frac{\theta}{\lambda^2} \int_0^{\lambda v_p} \left[2 - \frac{y(1 - e^{-y})}{1 - e^{-y}(y + 1)} + \frac{\theta y^2 e^{-y}}{\left[1 - e^{-y}(y + 1)\right]\left\{1 - \left[1 - e^{-y}(y + 1)\right]^{\theta}\right\}}\right]^2 \tag{2.16}$$

$$\times y e^{-y}\left[1 - e^{-y}(y + 1)\right]^{\theta - 1} dy,$$

$$
\begin{aligned}
I_{[0,\,p]}(\theta,\lambda) &= \theta\lambda^2 \int_0^{v_p} \left[\frac{\partial}{\partial\theta}\ln h(x;\theta,\lambda)\right]\left[\frac{\partial}{\partial\lambda}\ln h(x;\theta,\lambda)\right] xe^{-\lambda x}\left[1-e^{-\lambda x}(\lambda x+1)\right]^{\theta-1} dx \\
&= \frac{\theta}{\lambda}\int_0^{\lambda v_p}\left[2-\frac{y(1-e^{-y})}{1-e^{-y}(y+1)}+\frac{\theta y^2 e^{-y}}{[1-e^{-y}(y+1)]\{1-[1-e^{-y}(y+1)]^\theta\}}\right] \\
&\quad \times\left[\frac{1}{\theta}+\frac{\ln[1-e^{-y}(y+1)]}{1-[1-e^{-y}(y+1)]^\theta}\right]ye^{-y}\left[1-e^{-y}(y+1)\right]^{\theta-1} dy.
\end{aligned}
\tag{2.17}
$$

It follows, by (2.15), that the percentage of Fisher information about θ, $I_{[0,\,p]}(\theta)/a_{11}$, is independent of θ, and thus $I_{[0,\,p]}(\theta)$ is a decreasing function of θ.

2.3. Fisher Information Matrix under Type I Censoring

If the observation of X is right censored at a fixed time point t, that is, one observe $\min(X,\,t)$, Fisher information for the parameter vector $\boldsymbol{\Omega}$ based on a censored observation is thus

$$
I_R^{(c)}(t,\theta) = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},
\tag{2.18}
$$

where, for $i,\,j=1,\,2$,

$$
b_{ij} = \int_0^t \left(\frac{\partial}{\partial\Omega_i}\ln h(x,\boldsymbol{\Omega})\right)\left(\frac{\partial}{\partial\Omega_j}\ln h(x,\boldsymbol{\Omega})\right) f(x,\boldsymbol{\Omega})dx.
\tag{2.19}
$$

The Fisher information matrix of $EG(\theta,\lambda)$ under Type I censoring can be similarly derived as shown in Type II censoring. For $\tilde{p} = [1-e^{-\lambda t}(\lambda t+1)]^\theta$,

$$
\begin{aligned}
b_{11} &= \frac{1}{\theta^2}\left[\tilde{p}+\frac{\tilde{p}}{1-\tilde{p}}(\ln\tilde{p})^2\right], \\
b_{22} &= \frac{\theta}{\lambda^2}\int_0^{\lambda t}\left[2-\frac{y(1-e^{-y})}{1-e^{-y}(y+1)}+\frac{\theta y^2 e^{-y}}{[1-e^{-y}(y+1)]\{1-[1-e^{-y}(y+1)]^\theta\}}\right]^2 \\
&\quad \times ye^{-y}\left[1-e^{-y}(y+1)\right]^{\theta-1} dy, \\
b_{12} = b_{21} &= \frac{\theta}{\lambda}\int_0^{\lambda t}\left[2-\frac{y(1-e^{-y})}{1-e^{-y}(y+1)}+\frac{\theta y^2 e^{-y}}{[1-e^{-y}(y+1)]\{1-[1-e^{-y}(y+1)]^\theta\}}\right] \\
&\quad \times\left[\frac{1}{\theta}+\frac{\ln[1-e^{-y}(y+1)]}{1-[1-e^{-y}(y+1)]^\theta}\right]ye^{-y}\left[1-e^{-y}(y+1)\right]^{\theta-1} dy.
\end{aligned}
\tag{2.20}
$$

3. Maximum Likelihood Estimators

3.1. Maximum Likelihood Estimators for Complete Sample

In this section, the maximum likelihood estimators (MLE's) of $EG(\theta, \lambda)$ are considered. First, we consider the case when both θ and λ are unknown. Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from $EG(\theta, \lambda)$, then the log–likelihood function is

$$
\ln L(\theta, \lambda) = n \ln \theta + 2n \ln \lambda - \lambda \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \ln x_i
$$

$$
+ (\theta - 1) \sum_{i=1}^{n} \ln\left[1 - e^{-\lambda x_i}(\lambda x_i + 1)\right].
$$

(3.1)

The normal equations become

$$
\frac{\partial \ln L(\theta, \lambda)}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \ln\left[1 - e^{-\lambda x_i}(\lambda x_i + 1)\right] = 0,
$$

(3.2)

$$
\frac{\partial \ln L(\theta, \lambda)}{\partial \lambda} = \frac{2n}{\lambda} - \sum_{i=1}^{n} x_i + (\theta - 1)\lambda \sum_{i=1}^{n} \left[1 - e^{-\lambda x_i}(\lambda x_i + 1)\right]^{-1} x_i^2 e^{-\lambda x_i} = 0.
$$

(3.3)

It follows, by (3.2), that the MLE of θ as a function of λ, say $\widehat{\theta}(\lambda)$, where

$$
\widehat{\theta}(\lambda) = \frac{-n}{\sum_{i=1}^{n} \ln\left[1 - e^{-\lambda x_i}(\lambda x_i + 1)\right]}.
$$

(3.4)

Substituting $\widehat{\theta}(\lambda)$ in (3.1), we obtain the profile log-likelihood of λ as

$$
g(\lambda) = \ln L\left(\widehat{\theta}(\lambda), \lambda\right)
$$

$$
= n \ln n - n \ln\left\{-\sum_{i=1}^{n} \ln\left[1 - e^{-\lambda x_i}(\lambda x_i + 1)\right]\right\}
$$

$$
- n + 2n \ln \lambda - \lambda \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n} \ln\left[1 - e^{-\lambda x_i}(\lambda x_i + 1)\right].
$$

(3.5)

Therefore, MLE of λ, say $\widehat{\lambda}_{\text{MLE}}$, can be obtained by maximizing (3.5) with respect to λ as follows:

$$\frac{\partial g(\lambda)}{\partial \lambda} = -\sum_{i=1}^{n} \frac{\lambda x_i^2 e^{-\lambda x_i}}{1 - e^{-\lambda x_i}(\lambda x_i + 1)} + \frac{2n}{\lambda} - \sum_{i=1}^{n} x_i$$

$$- \frac{n}{\sum_{i=1}^{n} \ln[1 - e^{-\lambda x_i}(\lambda x_i + 1)]} \sum_{i=1}^{n} \frac{\lambda x_i^2 e^{-\lambda x_i}}{1 - e^{-\lambda x_i}(\lambda x_i + 1)} = 0. \tag{3.6}$$

Once $\widehat{\lambda}_{\text{MLE}}$ is obtained, the MLE of θ, say $\widehat{\theta}_{\text{MLE}}$, can be obtained from (3.4) as $\widehat{\theta}_{\text{MLE}} = \widehat{\theta}(\widehat{\lambda}_{\text{MLE}})$.

Now, we state the asymptotic normality results to obtain the asymptotic variances of the different parameters. It can be stated as follows:

$$\left[\sqrt{n}\left(\widehat{\theta}_{\text{MLE}} - \theta\right), \ \sqrt{n}\left(\widehat{\lambda}_{\text{MLE}} - \lambda\right) \right] \longrightarrow N_2\left(\mathbf{0}, \mathbf{I}^{-1}(\theta, \lambda)\right), \tag{3.7}$$

where $\mathbf{I}(\theta, \lambda)$ is the information matrix (2.1) whose elements are given by (2.6), (2.7), and (2.8).

Now, consider the MLE of θ, when the scale parameter λ is known. Without loss of generality, we can take $\lambda = 1$. If λ is known, the MLE of θ, say $\widehat{\theta}_{\text{MLESCK}}$, is

$$\widehat{\theta}_{\text{MLESCK}} = \frac{-n}{\sum_{i=1}^{n} \ln[1 - e^{-x_i}(x_i + 1)]}. \tag{3.8}$$

It follows, by the asymptotic properties of the MLE, that

$$\widehat{\theta}_{\text{MLESCK}} \approx N\left(\theta, \frac{1}{na_{11}}\right), \tag{3.9}$$

where a_{11} is the single information about θ which is defined in (2.6).

Now, note that if X_i's are independently and identically distributed EG(θ, 1), then $-\theta \sum_{i=1}^{n} \ln[1 - e^{-x_i}(x_i + 1)]$ follows G(n, 1). Therefore, for $n > 2$,

$$E\left(\widehat{\theta}_{\text{MLESCK}}\right) = \frac{n}{n-1}\theta, \qquad \text{Var}\left(\widehat{\theta}_{\text{MLESCK}}\right) = \frac{n^2}{(n-1)^2(n-2)}\theta^2. \tag{3.10}$$

Using (3.8), an unbiased estimate of θ can be obtained by

$$\widehat{\theta}_{\text{USCK}} = \frac{n-1}{n}\widehat{\theta}_{\text{MLESCK}} = -\frac{n-1}{\sum_{i=1}^{n} \ln[1 - e^{-x_i}(x_i + 1)]}, \tag{3.11}$$

where

$$\text{Var}\left(\widehat{\theta}_{\text{USCK}}\right) = \frac{\theta^2}{n-2}. \tag{3.12}$$

Let us consider the MLE of λ when the shape parameter θ is known. For known θ the MLE of λ, say $\widehat{\lambda}_{\text{MLESHK}}$, can be obtained by numerical solving of the following equation:

$$\frac{2n}{\lambda} - \sum_{i=1}^{n} x_i + (\theta - 1)\lambda \sum_{i=1}^{n} \left[1 - e^{-\lambda x_i}(\lambda x_i + 1)\right]^{-1} x_i^2 e^{-\lambda x_i} = 0. \tag{3.13}$$

It follows, by the asymptotic properties of the MLE, that

$$\widehat{\lambda}_{\text{MLESHK}} \approx N\left(\lambda, \ \frac{1}{n a_{22}}\right), \tag{3.14}$$

where a_{22} is the single information about λ which is defined in (2.8).

3.2. Maximum Likelihood Estimators under Censored Samples

Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{r:n}$ be the Type II censored data, then the likelihood function is given (Lawless [20]) by

$$L(\boldsymbol{\Omega}) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} f(x_i; \boldsymbol{\Omega})[1 - F(x_{r:n})]^{n-r}. \tag{3.15}$$

Next, let (X_i, t), $i = 1, 2, \ldots, n$, be independent, then $x_i = \min(X_i, t)$ are the Type I censored data, thus the likelihood function is given (Lawless [20]) by

$$L(\boldsymbol{\Omega}) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} f(x_i; \boldsymbol{\Omega})[1 - F(t)]^{n-r}. \tag{3.16}$$

We now turn to the computationally more complicated case of censored data. Type I and Type II censorings will be considered simultaneously, since they give the same form of likelihood function above. We will deal with the MLE under Type II censoring from $\text{EG}(\theta, \lambda)$ and it is the same for Type I censoring.

In life testing, under the Type II censoring from $\text{EG}(\theta, \lambda)$, the log likelihood function is

$$\ln L(\theta, \lambda) = \ln\left(\frac{n!}{(n-r)!}\right) + r \ln \theta + 2r \ln \lambda - \lambda \sum_{i=1}^{r} x_i + \sum_{i=1}^{r} \ln x_i$$

$$+ (\theta - 1)\sum_{i=1}^{r} \ln\left[1 - e^{-\lambda x_i}(\lambda x_i + 1)\right] + (n-r)\ln\left\{1 - \left[1 - e^{-\lambda x_i}(\lambda x_i + 1)\right]^{\theta}\right\}. \tag{3.17}$$

The normal equations become

$$\frac{\partial \ln L(\theta, \lambda)}{\partial \theta} = \frac{r}{\theta} + \ln u - (n - r)\left[v^{-\theta} - 1\right]^{-1} \ln v = 0, \tag{3.18}$$

$$\frac{\partial \ln L(\theta, \lambda)}{\partial \lambda} = \frac{2r}{\lambda} - \sum_{i=1}^{n} x_i + (\theta - 1)\lambda \sum_{i=1}^{n} \left[1 - e^{-\lambda x_i}(\lambda x_i + 1)\right]^{-1} x_i^2 e^{-\lambda x_i}$$

$$+ (n - r)\theta \lambda x_{r:n}^2 e^{-\lambda x_{r:n}} \left[v\left(v^{-\theta} - 1\right)\right]^{-1} = 0, \tag{3.19}$$

where

$$u = \prod_{i=1}^{r} \left[1 - e^{-\lambda x_i}(\lambda x_i + 1)\right], \quad v = 1 - e^{-\lambda x_r}(\lambda x_r + 1). \tag{3.20}$$

The MLE of θ and λ, say $\hat{\theta}_{\text{MLETII}}$ and $\hat{\lambda}_{\text{MLETII}}$, can be obtained by solving numerically the two nonlinear equations (3.18) and (3.19).

The MLE $\hat{\theta}_{\text{MLETII}}$ and $\hat{\lambda}_{\text{MLETII}}$, based on Type II censored data are strongly consistent and asymptotically normal (see, Zheng [11]), that is,

$$\left[\sqrt{n}\left(\hat{\theta}_{\text{MLETII}} - \theta\right), \ \sqrt{n}\left(\hat{\lambda}_{\text{MLETII}} - \lambda\right)\right] \longrightarrow N\left(\mathbf{0}, I_{[0, p]}^{-1}(\mathbf{\Omega})\right), \tag{3.21}$$

where $\mathbf{\Omega} = (\theta, \lambda)$ and $I_{[0, p]}(\mathbf{\Omega})$ is the Fisher information matrix (2.12) whose elements are given by (2.15), (2.16), and (2.17).

Now, consider the MLE of θ, based on the Type II censored data when the scale parameter λ is known. Without loss of generality, we can take $\lambda = 1$. If λ is known, the MLE of θ, say $\hat{\theta}_{\text{MLETIISCK}}$, could be obtained by solving numerically the nonlinear (3.18). It follows, by the asymptotic properties of the MLE, that

$$\hat{\theta}_{\text{MLETIISCK}} \longrightarrow N\left(\theta, \ \frac{1}{nI_{[0, p]}(\theta)}\right), \tag{3.22}$$

where $I_{[0, p]}(\theta)$ is defined in (2.15).

Let us consider the MLE of λ when the shape parameter θ is known. For known θ the MLE of λ, say $\hat{\lambda}_{\text{MLETIISHK}}$, can be obtained by solving numerically the nonlinear equation (3.19). It follows, by the asymptotic properties of the MLE, that

$$\hat{\lambda}_{\text{MLETIISHK}} \longrightarrow N\left(\lambda, \ \frac{1}{nI_{[0, p]}(\lambda)}\right), \tag{3.23}$$

where $I_{[0, p]}(\lambda)$ is defined in (2.16).

4. Estimators Based on Percentiles

If the data come from a distribution function which has a closed form, then it is quite natural to estimate the unknown parameters by fitting straight line to the theoretical points obtained by the distribution function and the sample percentile points. Murthy et al. [21] discussed this method for Weibull distribution while Gupta and Kundu [22] studied the generalized exponential distribution.

First, let us consider the case when both parameters are unknown. Since

$$F(x; \theta, \lambda) = \left[1 - e^{-\lambda x}(\lambda x + 1)\right]^{\theta}, \tag{4.1}$$

therefore,

$$\ln\left[1 - (F(x; \theta, \lambda))^{1/\theta}\right] = -\lambda x + \ln(\lambda x + 1). \tag{4.2}$$

Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ be the order statistics obtained by $EG(\theta, \lambda)$. If p_i denotes some estimate of $F(x_{i:n}; \theta, \lambda)$, then the estimate of θ and λ can be obtained by minimizing

$$\sum_{i=1}^{n} \left\{ \lambda x_{i:n} - \ln(\lambda x_{i:n} + 1) + \ln\left[1 - p_i^{1/\theta}\right] \right\}^2, \tag{4.3}$$

with respect to θ and λ. We call these estimators as percentile estimators (PCEs) and could be obtained by solving numerically the following two nonlinear equations:

$$\sum_{i=1}^{n} \left\{ \lambda x_{i:n} - \ln(\lambda x_{i:n} + 1) + \ln\left[1 - p_i^{1/\theta}\right] \right\} \left[1 - p_i^{1/\theta}\right]^{-1} p_i^{1/\theta} \frac{\ln p_i}{\theta^2} = 0, \tag{4.4}$$

$$\sum_{i=1}^{n} \left\{ \lambda x_{i:n} - \ln(\lambda x_{i:n} + 1) + \ln\left[1 - p_i^{1/\theta}\right] \right\} \left[x_{i:n} - (\lambda x_{i:n} + 1)^{-1} x_{i:n}\right] = 0. \tag{4.5}$$

Several estimators of p_i can be used here (see, Murthy et al. [21]). In this section, we mainly consider $p_i = i/(n+1)$, which is the expected value of $F(x_{i:n})$.

Now, let us consider the case when one parameter is known. If the shape parameter θ is known, then the PCE of λ, say $\hat{\lambda}_{\text{PCESHK}}$, can be obtained from (4.5).

Now let us consider the case when the scale parameter λ is known. Without loss of generality, we can assume that $\lambda = 1$. If we denote $F(x; \theta) = F(x; \theta, 1)$, then

$$\ln F(x; \theta) = \theta \ln\left[1 - e^{-x}(x + 1)\right]. \tag{4.6}$$

Therefore, the PCE of θ, say $\hat{\theta}_{\text{PCESCK}}$, can be obtained by minimizing

$$\sum_{i=1}^{n} \left\{ \ln p_i - \theta \ln\left[1 - e^{-x_{i:n}}(x_{i:n} + 1)\right] \right\}^2, \tag{4.7}$$

with respect to θ and hence

$$\widehat{\theta}_{\text{PCESCK}} = \frac{\sum_{i=1}^{n}\left\{\ln p_i \ln[1 - e^{-x_{i:n}}(x_{i:n} + 1)]\right\}}{\sum_{i=1}^{n}\left\{\ln[1 - e^{-x_{i:n}}(x_{i:n} + 1)]\right\}^2}. \tag{4.8}$$

Interestingly, $\widehat{\theta}_{\text{PCESCK}}$ is also in a closed form like $\widehat{\theta}_{\text{MLESCK}}$ when λ is known.

5. Least-Squares and Weighted Least-Squares Estimators

The least-squares estimators and weighted least-squares estimators were originally proposed by Swain et al. [23] to estimate the parameters of Beta distributions. It can be described as follows. Suppose $Y_1, Y_2, \ldots, Y_n$ is a random sample of size n from a distribution function $G(\cdot)$ and $Y_{1:n} < Y_{2:n} < \cdots < Y_{n:n}$ denotes the order statistics of the observed sample. It is well known that

$$E(G(Y_{j:n})) = \frac{j}{n+1}, \qquad \text{Var}(G(Y_{j:n})) = \frac{j(n-j+1)}{(n+1)^2(n+2)}. \tag{5.1}$$

Using the expectations and the variances, two variants of the least-squares methods can be used.

Method 1. The least-squares estimators of the unknown parameters can be obtained by minimizing

$$\sum_{j=1}^{n}\left(G(Y_{j:n}) - \frac{j}{n+1}\right)^2, \tag{5.2}$$

with respect to the unknown parameters. Therefore, in case of EG distribution, the least-squares estimators of θ and λ, say $\widehat{\theta}_{\text{LSE}}$ and $\widehat{\lambda}_{\text{LSE}}$, respectively, can be obtained by minimizing

$$\sum_{j=1}^{n}\left\{\left[1 - e^{-\lambda x_{j:n}}(\lambda x_{j:n} + 1)\right]^{\theta} - \frac{j}{n+1}\right\}^2, \tag{5.3}$$

with respect to θ and λ.

It could be obtained by solving the following nonlinear equations:

$$\sum_{j=1}^{n}\left\{\left[1 - e^{-\lambda x_{j:n}}(\lambda x_{j:n} + 1)\right]^{\theta} - \frac{j}{n+1}\right\}\left[1 - e^{-\lambda x_{j:n}}(\lambda x_{j:n} + 1)\right]^{\theta}\ln\left[1 - e^{-\lambda x_{j:n}}(\lambda\ x_{j:n} + 1)\right] = 0, \tag{5.4}$$

$$\sum_{j=1}^{n}\left\{\left[1 - e^{-\lambda x_{j:n}}(\lambda x_{j:n} + 1)\right]^{\theta} - \frac{j}{n+1}\right\}\theta\lambda x_{j:n}^2 e^{-\lambda x_{j:n}}\left[1 - e^{-\lambda x_{j:n}}(\lambda\ x_{j:n} + 1)\right]^{\theta-1} = 0. \tag{5.5}$$

Method 2. The weighted least-squares estimators of the unknown parameters can be obtained by minimizing

$$\sum_{j=1}^{n} \frac{1}{\mathrm{Var}(G(Y_{j:n}))} \left(G(Y_{j:n}) - \frac{j}{n+1} \right)^2, \tag{5.6}$$

with respect to the unknown parameters. Therefore, in case of an EG distribution, the weighted least-squares estimators of θ and λ, say $\widehat{\theta}_{\mathrm{WLSE}}$ and $\widehat{\lambda}_{\mathrm{WLSE}}$, respectively, can be obtained by minimizing

$$\sum_{j=1}^{n} \frac{(n+1)^2(n+2)}{j(n-j+1)} \left\{ \left[1 - e^{-\lambda x_{j:n}}(\lambda x_{j:n} + 1) \right]^{\theta} - \frac{j}{n+1} \right\}^2, \tag{5.7}$$

with respect to θ and λ.

It could be found by solving the following nonlinear equations

$$\sum_{j=1}^{n} \frac{(n+1)^2(n+2)}{j(n-j+1)} \left\{ \left[1 - e^{-\lambda x_{j:n}}(\lambda x_{j:n} + 1) \right]^{\theta} - \frac{j}{n+1} \right\}$$
$$\times \left[1 - e^{-\lambda x_{j:n}}(\lambda x_{j:n} + 1) \right]^{\theta} \ln\left[1 - e^{-\lambda x_{j:n}}(\lambda x_{j:n} + 1) \right] = 0, \tag{5.8}$$

$$\sum_{j=1}^{n} \frac{(n+1)^2(n+2)}{j(n-j+1)} \left\{ \left[1 - e^{-\lambda x_{j:n}}(\lambda x_{j:n} + 1) \right]^{\theta} - \frac{j}{n+1} \right\}$$
$$\times \theta \lambda x_{j:n}^2 e^{-\lambda x_{j:n}} \left[1 - e^{-\lambda x_{j:n}}(\lambda x_{j:n} + 1) \right]^{\theta-1} = 0. \tag{5.9}$$

6. Method of Moment Estimators

In this section, we provide the method of moment estimators (MMEs) of the parameters of an EG distribution. If X follows $EG(\theta, \lambda)$, then

$$\mu = E(X) = \frac{\theta}{\lambda}[2 + A_1(\theta)], \tag{6.1}$$

$$\sigma^2 = V(X) = \frac{\theta}{\lambda^2} \left\{ 6 + A_2(\theta) - \theta[2 + A_1(\theta)]^2 \right\}, \tag{6.2}$$

where $A_1(\theta)$ and $A_2(\theta)$ are defined in (2.9).

It is well known that the principle of the moment's method is to equate the sample moments with the corresponding population.

From (6.1) and (6.2), we obtain the coefficient of variation (C.V.) as

$$\mathrm{C.V.} = \frac{\sigma}{\mu} = \frac{\sqrt{\theta[6 + A_2(\theta)] - \theta^2[2 + A_1(\theta)]^2}}{\theta[2 + A_1(\theta)]}. \tag{6.3}$$

Table 1: Bias estimates and MSEs of θ are presented, when λ is known.

n	Method	$\theta = 0.25$	$\theta = 0.5$	$\theta = 1$	$\theta = 2$	$\theta = 2.5$
10	MLE	0.13269	0.31581	0.12298	0.19146	0.24863
		0.04193	0.15449	0.15997	0.59425	0.95929
	MME	0.14114	0.24936	0.76652	-0.76236	1.21176
		0.01992	0.06218	0.58756	0.58119	1.46835
	PCE	0.05443	0.22400	-0.05708	-0.18646	-0.19735
		0.02452	0.08653	0.11220	0.47012	0.75097
	LSE	1.25613	0.92901	1.18688	0.20133	0.18193
		1.57786	0.86305	1.58971	1.02132	1.43291
	WLSE	1.21698	0.90029	1.13884	0.17699	0.16053
		1.48105	0.81052	1.47831	0.82830	1.32242
	UBE	0.09442	0.23423	0.01069	-0.02769	-0.02624
		0.02862	0.09922	0.11744	0.45242	0.72764
20	MLE	0.01488	0.02848	0.06164	0.09579	0.14010
		0.00356	0.01578	0.06943	0.23892	0.40589
	MME	-0.01385	-0.00755	0.09627	0.07783	0.06881
		0.01186	0.03202	0.12147	0.47954	0.72776
	PCE	-0.05946	-0.05620	-0.06172	-0.15758	-0.22235
		0.00671	0.02036	0.06388	0.23730	0.39565
	LSE	1.62596	1.36431	0.02848	0.07049	0.10051
		2.64656	1.86404	0.09260	0.32610	0.55121
	WLSE	2.21133	1.87305	0.02495	0.06392	0.29974
		4.89475	3.51203	0.08449	0.29810	0.90352
	UBE	0.00164	0.00206	0.00856	-0.00900	0.00809
		0.00302	0.01351	0.05930	0.20743	0.34867
50	MLE	0.01029	0.01194	0.02556	0.03783	0.05525
		0.00135	0.00541	0.02171	0.09390	0.12927
	MME	-0.02304	-0.01614	0.02354	0.03965	-0.01924
		0.00476	0.01191	0.03931	0.16665	0.23736
	PCE	-0.06374	-0.05750	-0.04838	-0.10306	-0.19602
		0.00523	0.01070	0.02443	0.10966	0.18408
	LSE	1.58813	1.04030	0.02102	0.01700	0.03799
		2.52287	1.08322	0.03064	0.11586	0.17602
	WLSE	2.20514	1.85472	0.01968	0.01582	-0.49704
		4.87453	3.44050	0.02725	0.10712	0.39058
	UBE	0.00508	0.00170	0.00505	-0.00292	0.00414
		0.00122	0.00506	0.02025	0.08881	0.12124
100	MLE	0.03297	0.14041	0.01201	0.02151	0.00560
		0.00109	0.00197	0.01105	0.04172	0.06166
	MME	0.07016	0.04276	0.03428	0.04947	-0.04117
		0.00492	0.00183	0.00117	0.00245	0.00170
	PCE	0.03915	0.16972	-0.03748	-0.07757	-0.10999
		0.00308	0.02880	0.01317	0.05372	0.08057
	LSE	0.19718	0.17144	0.00963	0.01773	-0.00567
		0.03888	0.03338	0.01525	0.05612	0.08245

Table 1: Continued.

n	Method	$\theta = 0.25$	$\theta = 0.5$	$\theta = 1$	$\theta = 2$	$\theta = 2.5$
	WLSE	0.16700	0.22510	0.00934	0.01664	-0.00374
		0.02789	0.05388	0.01365	0.05063	0.07397
	UBE	0.03014	0.13401	0.00189	0.00129	-0.01945
		0.00091	0.00180	0.01069	0.04044	0.06078

The first entry is the simulated bias.
The second entry is simulated MSE.

The C.V. is independent of the scale parameter λ. Therefore, equating the sample C.V. with the population C.V., we obtain

$$\frac{S}{\overline{X}} = \frac{\sqrt{\theta[6 + A_2(\theta)] - \theta^2[2 + A_1(\theta)]^2}}{\theta[2 + A_1(\theta)]}, \tag{6.4}$$

where $S^2 = (\sum_{i=1}^{n}(X_i - \overline{X})^2)/(n-1)$ and $\overline{X} = (1/n)\sum_{i=1}^{n}X_i$. We need to solve (6.4) to obtain the MME of θ, say $\widehat{\theta}_{\text{MME}}$. Once we estimate θ, we can use (6.1) to obtain the MME of λ.

If the scale parameter is known (without loss of generality, we assume $\lambda = 1$), then the MME of θ, say $\widehat{\theta}_{\text{MMESCK}}$, can be obtained by solving the nonlinear equation:

$$\overline{X} = \mu, \tag{6.5}$$

that is,

$$\overline{X} = \theta[2 + A_1(\theta)]. \tag{6.6}$$

Now consider the case when the shape parameter θ is known, then the MME of λ, say $\widehat{\lambda}_{\text{MMESHK}}$, is

$$\widehat{\lambda}_{\text{MMESHK}} = \frac{\theta[2 + A_1(\theta)]}{\overline{X}}. \tag{6.7}$$

Note that (6.5) follows easily from (6.1). Although $\widehat{\lambda}_{\text{MMESHK}}$ is not an unbiased estimator of λ, $(1/\widehat{\lambda}_{\text{MMESHK}})$ is unbiased estimator of $(1/\lambda)$ and, therefore,

$$\text{Var}\left(\frac{\lambda}{\widehat{\lambda}_{\text{MMESHK}}}\right) = \frac{1}{n}\left[\frac{6 + A_2(\theta)}{\theta[2 + A_1(\theta)]^2} - 1\right]. \tag{6.8}$$

7. L-Moment Estimator

In this section, we propose a method of estimating the unknown parameter of an EG distribution based on the linear combination of order statistics (see, [24, 25]). The estimators obtained by this method are popularly known as L-moment estimators (LMEs). It is observed

Table 2: Bias estimates and MSEs of λ are presented, when θ is known.

n	Method	$\theta = 0.25$	$\theta = 0.5$	$\theta = 1$	$\theta = 2$	$\theta = 2.5$
10	MLE	0.31811	0.18392	0.11145	0.09787	0.11787
		0.58672	0.21690	0.07703	0.04421	0.04936
	MME	0.32033	0.17616	0.11145	0.09702	0.10992
		0.78535	0.22157	0.07703	0.04474	0.04646
	PCE	− 0.05303	− 0.02370	− 0.02479	− 0.01796	− 0.01759
		0.16991	0.12909	0.06761	0.03890	0.03475
	LSE	− 0.49779	0.27450	0.31333	0.43391	− 0.13358
		0.24796	0.23511	0.22940	0.18828	0.02373
	WLSE	1.25407	0.24347	0.30201	0.04305	− 0.12468
		5.71291	0.31666	0.21657	0.03298	0.02343
	MLE	0.31811	0.18392	0.11145	0.09787	0.11787
		0.58672	0.21690	0.07703	0.04421	0.04936
20	MLE	0.93920	0.08715	0.03108	0.01178	0.02822
		0.22575	0.08646	0.02862	0.01488	0.01642
	MME	0.95002	0.08647	0.03110	0.01352	0.02613
		0.26947	0.08936	0.02861	0.01506	0.01567
	PCE	0.65305	− 0.07187	− 0.06136	− 0.04752	− 0.04746
		0.10416	0.06648	0.03426	0.01919	0.01706
	LSE	1.08709	0.71135	0.02553	0.02546	− 0.00643
		0.21023	0.58339	0.03482	0.36011	0.01492
	WLSE	0.42226	0.42004	0.01817	0.00580	− 0.01538
		0.26128	0.21520	0.03193	0.01579	0.01510
	MLE	0.93920	0.08715	0.03108	0.01178	0.02822
		0.22575	0.08646	0.02861	0.01488	0.01642
50	MLE	0.86277	0.04295	0.00711	0.01066	0.01775
		0.07361	0.02587	0.01054	0.00593	0.00615
	MME	0.87038	0.04343	0.00711	0.01123	0.01500
		0.08332	0.02685	0.01054	0.00598	0.00589
	PCE	0.67724	− 0.05961	− 0.04379	−0.02532	− 0.03890
		0.04971	0.02590	0.01420	0.00802	0.00690
	LSE	0.60209	0.00401	0.00374	0.00952	− 0.01785
		0.04424	0.02550	0.01336	0.00675	0.00543
	WLSE	0.33053	− 0.04580	0.00322	0.00918	− 0.04761
		0.19229	0.01685	0.01214	0.00638	0.01093
	MLE	0.86277	0.04295	0.00711	0.01066	0.01775
		0.07361	0.02587	0.01054	0.00593	0.00615
100	MLE	− 0.21280	− 0.12827	0.00632	0.00470	0.00250
		0.05254	0.02181	0.00480	0.00292	0.00222
	MME	− 0.25157	− 0.14261	0.00632	0.00517	0.00245
		0.06987	0.02544	0.00480	0.00293	0.00225
	PCE	− 0.17997	− 0.10697	− 0.02831	− 0.01738	− 0.01960
		0.04306	0.01985	0.00681	0.00405	0.00344
	LSE	− 0.16872	− 0.07718	0.00509	0.00372	0.00268
		0.02847	0.00596	0.00585	0.00329	0.00250

Table 2: Continued.

n	Method	$\theta = 0.25$	$\theta = 0.5$	$\theta = 1$	$\theta = 2$	$\theta = 2.5$
	WLSE	-0.31292	-0.14028	0.00484	0.00359	0.00234
		0.10323	0.01968	0.00545	0.00310	0.00234
	MLE	-0.21279	-0.12827	0.00633	0.00470	0.00250
		0.05254	0.02181	0.00480	0.00292	0.00222

The first entry is the simulated bias.
The second entry is simulated MSE.

(see, Gupta and Kundu [15]) that the LMEs have certain advantages over the conventional moment estimators.

The standard method to compute the L-moment estimators is to equate the sample L-moments with the population L-moments.

First, we discuss the case of obtaining the LMEs when both the parameters of an EG distribution are unknown. If $x_{1:n} \leq x_{2:n} \leq \cdots \leq x_{n:n}$ denote the ordered sample, then using the same notation as in [15, 25], we obtain the first and second sample L-moments as

$$l_1 = \frac{1}{n} \sum_{i=1}^{n} x_{i:n}, \qquad l_2 = \frac{2}{n(n-1)} \sum_{i=1}^{n} (i-1)x_{i:n} - l_1. \tag{7.1}$$

Similarly, the first two population L-moments (see David and Nagaraja [24]) are

$$\lambda_1 = \mu = E(X), \qquad \lambda_2 = \left(\frac{1}{2}\right)(\mu_{2:2} - \mu_{1:2}) = \int_{-\infty}^{\infty} x[2F(x) - 1]f(x)dx, \tag{7.2}$$

respectively, where $\mu_{i:n} = E(X_{i:n})$.

Then, for EG(θ, λ), we obtain

$$\lambda_1 = \frac{\theta}{\lambda}[2 + A_1(\theta)], \qquad \lambda_2 = \frac{2\theta}{\lambda}[2 + A_1(2\theta)] - \frac{\theta}{\lambda}[2 + A_1(\theta)], \tag{7.3}$$

where $A_1(\theta)$ is defined by (2.9) and

$$A_r(2\theta) = \sum_{j=1}^{\infty} \sum_{k=0}^{j} (-1)^j \binom{2\theta - 1}{j}\binom{j}{k} \frac{\Gamma(r+k+2)}{(1+j)^{r+k+2}}, \qquad r = 1, 2, \ldots. \tag{7.4}$$

Therefore, LMEs can be obtained by solving the following two equations:

$$l_1 = \frac{\theta}{\lambda}[2 + A_1(\theta)], \tag{7.5}$$

$$l_2 = \frac{2\theta}{\lambda}[2 + A_1(2\theta)] - \frac{\theta}{\lambda}[2 + A_1(\theta)]. \tag{7.6}$$

Table 3: Bias estimates and MSEs of θ are presented, when λ is unknown.

n	Method	$\theta = 0.25$	$\theta = 0.5$	$\theta = 1$	$\theta = 2$	$\theta = 2.5$
10	MLE	0.21113	0.47216	1.52019	1.72396	1.14780
		0.22611	0.22293	2.31097	2.97203	4.56281
	MME	0.76727	0.49996	1.14769	1.11707	1.34613
		0.58875	0.39533	2.71090	3.59468	12.94580
	PCE	0.59915	0.52863	-0.81238	-1.80674	2.06424
		0.35898	0.27945	0.65997	3.26431	4.26110
	LSE	1.42344	4.79954	3.30292	1.81180	2.00336
		2.02618	23.03550	10.9093	3.28262	4.01345
	WLSE	1.37249	4.13538	2.33451	2.88664	1.94695
		1.88374	17.10140	5.44994	8.33268	3.79060
	LME	0.25377	0.33619	-0.83525	0.42838	-0.30403
		0.06440	0.11302	0.69764	0.18351	0.09243
20	MLE	0.05642	-0.26499	0.20639	0.51890	-0.63219
		0.06938	0.11276	0.28192	1.76134	3.30157
	MME	-0.02744	0.02945	0.24985	0.70164	0.68147
		0.11071	0.07633	0.45191	3.40144	5.12740
	PCE	-0.05994	0.16020	-0.30593	-0.59678	-0.49013
		0.83344	0.20841	0.90643	2.82742	2.21411
	LSE	4.17385	1.64517	0.13219	1.99406	-0.11114
		17.52740	2.71021	0.94279	5.38358	3.77045
	WLSE	4.53046	2.49667	0.06400	0.71230	-0.14265
		20.61810	6.23523	0.60372	4.52342	3.73948
	LME	-0.12299	-0.05096	0.14087	0.34621	0.28202
		0.00526	0.05250	0.32047	1.91264	4.40559
50	MLE	0.02917	0.04433	0.06944	0.17997	-0.35615
		0.00322	0.01287	0.05652	0.36878	2.45495
	MME	-0.02159	-0.03043	0.07974	0.20995	0.01896
		0.05408	0.02100	0.09118	0.58875	0.81280
	PCE	0.09865	0.30284	-0.23200	-0.37726	-0.72448
		0.04254	0.13493	0.57164	0.62198	1.36530
	LSE	4.04285	1.41485	0.02900	0.91823	-0.67959
		16.38400	2.00264	0.21018	1.64307	1.06572
	WLSE	4.46262	2.30001	0.02562	0.62267	-0.87230
		19.94850	5.29033	0.16917	1.25187	1.41209
	LME	-0.00570	-0.07578	0.02864	0.12927	-0.07695
		0.00259	0.01867	0.05908	0.38234	0.64602
100	MLE	0.04925	0.33618	0.04200	0.07415	0.08994
		0.00243	0.11302	0.00176	0.12379	0.23124
	MME	0.19042	0.14178	-0.15128	0.19460	-0.09501
		0.048974	0.02010	0.04628	0.23351	0.17784
	PCE	-0.12753	0.03137	-0.15234	-0.27866	-0.33585
		0.01626	0.00098	0.11004	0.36332	0.55157
	LSE	7.15894	0.36587	0.08370	0.38356	0.15933
		23.58748	0.13386	0.00701	0.14712	0.02539

Table 3: Continued.

n	Method	$\theta = 0.25$	$\theta = 0.5$	$\theta = 1$	$\theta = 2$	$\theta = 2.5$
	WLSE	6.48572	0.65182	0.06534	0.30304	0.11907
		25.85176	0.42487	0.00427	0.09183	0.01418
	LME	0.20830	0.28730	-0.03486	0.47106	0.04308
		0.04339	0.08254	0.00121	0.22190	0.00186

The first entry is the simulated bias.
The second entry is simulated MSE.

First, we obtain the LME of θ, say $\widehat{\theta}_{\text{LME}}$, as the solution of the following nonlinear equation:

$$\frac{2 + 2A_1(2\theta) - A_1(\theta)}{2 + A_1(\theta)} = \frac{l_2}{l_1}. \tag{7.7}$$

Once $\widehat{\theta}_{\text{LME}}$ is obtained, the LME of λ, say $\widehat{\lambda}_{\text{LME}}$, can be obtained from (7.5) as

$$\widehat{\lambda}_{\text{LME}} = \frac{\widehat{\theta}_{\text{LME}}\left[2 + A_1\left(\widehat{\theta}_{\text{LME}}\right)\right]}{l_1}. \tag{7.8}$$

Note that if θ or λ is known, then the LME of λ or θ is the same as the corresponding MME obtained in Section 6.

8. Numerical Experiments and Discussions

In this section, we present the results of some numerical experiments to compare the performance of the different estimators proposed in the previous sections. We apply Monte Carlo simulations to compare the performance of different estimators, mainly with respect to their biases and mean squared errors (MSEs) for different sample sizes and for different parameter values. Since λ is the scale parameter and all the estimators are scale invariant, we take $\lambda = 1$ in all our computations. We set $\theta = 0.25, 0.5, 1, 2, 2.5$ and $n = 10, 20, 50, 100$. We compute the bias estimates and MSEs over 1000 replications for different cases.

First of all, we consider the estimation of θ when λ is known. In this case, the MLE, unbiased estimator (UBE), and PCE of θ can be obtained from (3.4), (3.11), and (4.8) respectively. The least-squares and weighted least-squares estimators of θ can be obtained by solving numerically the nonlinear equations (5.4) and (5.8), respectively. The MME (similarly LME) of θ can be obtained by solving numerically the nonlinear equation (6.5) as well. The results are reported in Table 1.

It is observed in Table 1 that for each method the MSEs decrease as sample size increases at $\theta \geq 1$. It indicates that all the methods deduce asymptotically unbiased and consistent estimators of the shape parameter θ for known λ. Moreover, all methods (except PCE and MME) usually overestimate θ whereas PCE underestimate θ in all cases that are considered and for some values of θ in case UBE. Therefore, the estimates of all methods are underestimate for most values of θ except MLE that forms overestimate for all θ. Also, all estimators are unbiased except LSE and WLSE that are the worst in biasness. The estimates of all methods are consistent except for some values at $\theta \leq 0.5$ because of the shape of the curve (reversed J shaped).

Table 4: Bias estimates and MSEs of λ are presented, when θ is unknown.

n	Method	$\theta = 0.25$	$\theta = 0.5$	$\theta = 1$	$\theta = 2$	$\theta = 2.5$
10	MLE	0.23192	-0.37506	1.51322	0.30659	0.17913
		0.41451	0.14067	2.28984	0.09400	0.12062
	MME	1.49108	0.22006	0.22579	0.18151	0.15722
		3.41454	0.21805	0.14019	0.16686	0.38909
	PCE	0.48838	-0.37506	-0.63758	-0.77034	-0.58675
		0.23851	0.14067	0.40651	0.59343	0.34428
	LSE	1.96123	1.93765	1.19535	0.67881	0.38439
		3.84642	3.75448	1.42885	0.46078	0.14776
	WLSE	1.93382	1.77460	0.95789	0.86839	0.36372
		3.73965	3.14921	0.91755	0.75410	0.13230
	LME	1.24369	0.58045	-0.86891	0.25729	-0.27276
		1.54676	0.33692	0.75500	0.06620	0.07440
20	MLE	1.12958	-0.78577	0.09648	0.06996	-0.32552
		0.98810	0.83260	0.07399	0.05256	0.31230
	MME	-0.21298	-0.02978	0.09269	0.08362	0.06518
		15.79690	1.26516	0.09956	0.07370	0.06565
	PCE	0.96517	0.41578	-0.25434	-0.20545	-0.13402
		2.64459	0.51577	0.27145	0.30745	0.19009
	LSE	1.08501	0.13210	0.12885	0.23836	-0.05068
		0.12089	0.01752	0.27372	0.44210	0.14692
	WLSE	0.95772	0.05630	0.07882	0.16095	-0.09438
		0.05037	0.00333	0.21817	0.58381	0.20759
	LME	0.26872	-0.06348	0.05770	0.04002	0.00806
		0.09852	0.09493	0.08266	0.05849	0.05390
50	MLE	0.94981	0.08882	0.03655	0.02594	-0.20388
		0.13613	0.05372	0.02425	0.01849	0.26018
	MME	0.17219	-0.03711	0.03235	0.02871	-0.00462
		9.21814	0.31441	0.03334	0.02669	0.02245
	PCE	1.59704	0.64277	-0.16087	-0.11791	-0.15752
		1.49896	0.44799	0.25011	0.05786	0.09803
	LSE	1.01034	0.12268	0.06734	0.26566	-0.15567
		0.07085	0.01507	0.05965	0.09425	0.04834
	WLSE	0.94417	0.04855	0.03741	0.19030	-0.24839
		0.04104	0.00241	0.08950	0.18141	0.10215
	LME	0.75545	-0.08930	0.01673	0.01864	-0.00924
		0.05648	0.04131	0.02619	0.01999	0.01877
100	MLE	-0.24206	0.04165	-0.10807	0.012484	0.00873
		0.05859	0.00173	0.01168	0.00768	0.00833
	MME	0.16552	0.02524	-0.07416	0.04744	0.06259
		0.05546	0.00064	0.02102	0.01807	0.01740
	PCE	-0.51381	-0.27116	-0.10042	-0.07722	-0.07335
		0.26401	0.07353	0.04582	0.02565	0.02310
	LSE	0.15872	0.14976	-0.09193	0.08286	0.06580
		0.06445	0.02243	0.00845	0.00687	0.00433

Table 4: Continued.

n	Method	$\theta = 0.25$	$\theta = 0.5$	$\theta = 1$	$\theta = 2$	$\theta = 2.5$
	WLSE	0.78422	0.34511	-0.10204	0.06591	0.06105
		0.03259	0.11910	0.01041	0.00434	0.00373
	LME	-0.01798	0.13656	0.01622	0.11112	0.02734
		0.00032	0.01865	0.00026	0.01235	0.00075

The first entry is the simulated bias.
The second entry is simulated MSE.

In the context of computational complexities, UBE, MLE, and PCE are the easiest to compute. They do not involve solvable nonlinear equation, whereas the LSE, WLSE, and MME involve solvable nonlinear equations and they need to be calculated by some iterative processes. Comparing all the methods, we conclude that for known scale parameter, the UBE should be used for estimating θ.

The negative sign in some results of the first entry only, in cells in all tables because of calculating of bias (see, Abouammoh and Alshingiti [26]).

Now consider the estimation of λ when θ is known. In this case, the MLE, PCE, LSE, and WLSE of λ can be obtained by solving the nonlinear equations (3.13), (4.5), (5.5), and (5.9), respectively, but the MME (LME is exactly the same) of λ can be obtained directly from (6.7). The results are reported in Table 2.

In this case, it is observed, at the same sample size, that the value of θ increases for all methods the MSEs decrease. Comparing the computational complexities of the different estimators, it is observed that when the shape parameter is known, PCE and MME can be computed directly, while some iterative techniques are needed to compute MLE, LSE, and WLSE. We apply Newton-Raphson method using Mathematica 6 to solve the nonlinear equations required. Comparing all methods, we conclude that all the estimates are consistent except WLSE and LSE for some θ.

Also, for most estimates, the MSEs decrease as the values of θ decrease. We recommend to use PCE for estimate at $\theta \leq 0.5$, $n = 10, 20$, and to use LSE at $\theta \leq 0.5$, $n = 100$, while use WLSE at $\theta \leq 0.5$, $n = 50$. All the estimates are consistent and unbiased at $\theta \geq 1$ for all values of n.

Finally, consider the estimation of θ and λ when both of them are unknown. The $\hat{\lambda}_{\text{MLE}}$ can be obtained by solving the nonlinear equation (3.6), once $\hat{\lambda}_{\text{MLE}}$ is obtained, $\hat{\theta}_{\text{MLE}}$ can be obtained from (3.4). The PCE of θ and λ can be obtained by solving the nonlinear equations (4.4) and (4.5). Similarly, LSE of θ and λ can be obtained by solving the nonlinear equations (5.4) and (5.5). Also, WLSE of θ and λ can be obtained by solving the nonlinear equations (5.8) and (5.9). The $\hat{\theta}_{\text{MME}}$ or $\hat{\theta}_{\text{LME}}$ can be obtained by solving the nonlinear equation (6.4) or (7.7), and then $\hat{\lambda}_{\text{MME}}$ or $\hat{\lambda}_{\text{LME}}$ can be obtained from (6.7) or (7.8). The results for θ and λ are presented in Tables 3 and 4, respectively.

It is observed in Tables 3 and 4 that for each method, the MSEs decrease as sample size increases. It indicates that all the methods deduce asymptotically unbiased and consistent estimators of θ and λ when both are unknown.

Comparing the performance of all the estimators, it is observed that as far as the minimum biases are concerned, the MLE performs. Considering the MSEs, the MLE and PCE perform better than the rest in most cases considered. The performances of the LSE's and WLSE's for $\theta \leq 1$ are the worst as far as the bias or MSE's are concerned. Moreover, it is observed from Table 4 for PCE method that the MSE's of $\hat{\lambda}$ depend on θ, that is, for $\theta < 2.5$,

and $n > 10$ as θ, increases the MSEs of $\widehat{\lambda}$ decrease. Most of the estimators are consistent and most of the estimators PCE are underestimate for all n.

Now if we consider the computational complexities, it is observed that MLEs, MMEs, and LMEs involve one-dimensional optimization, whereas PCEs, LSEs, and WLSEs involve two-dimensional optimization. Considering all the points above, we recommend to use MLE's for estimating θ and λ when both are unknown.

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