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Research Article

A Best Possible Double Inequality for Power Mean

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We answer the question: for any $p, q \in \mathbb{R}$ with $p \neq q$ and $p \neq -q$, what are the greatest value $\lambda = \lambda(p, q)$ and the least value $\mu = \mu(p, q)$, such that the double inequality $M_\lambda(a, b) < \sqrt{M_p(a, b)M_q(a, b)} < M_\mu(a, b)$ holds for all $a, b > 0$ with $a \neq b$? Where $M_p(a, b)$ is the pth power mean of two positive numbers a and b.

1. Introduction

For $p \in \mathbb{R}$, the pth power mean $M_p(a, b)$ of two positive numbers a and b is defined by

$$M_p(a,b) = \begin{cases} \left(\dfrac{a^p + b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases} \tag{1.1}$$

It is well known that $M_p(a, b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed a, $b > 0$ with $a \neq b$. Many classical means are special case of the power mean, for example, $M_{-1}(a, b) = H(a, b) = 2ab/(a + b)$, $M_0(a, b) = G(a, b) = \sqrt{ab}$, and $M_1(a, b) = A(a, b) = (a + b)/2$ are the harmonic, geometric, and arithmetic means of a and b, respectively. Recently, the power mean has been the subject of intensive research. In particular, many remarkable inequalities and properties for the power mean can be found in literature [1–15].

Let $L(a,b)=(a-b)/(\log a-\log b)$ and $I(a,b)=1/e(a^a/b^b)^{1/(a-b)}$ be the logarithmic and identric means of two positive numbers a and b with $a\neq b$, respectively. Then it is well known that

$$H(a,b)=M_{-1}(a,b)<G(a,b)=M_0(a,b)<L(a,b)<I(a,b)<A(a,b)=M_1(a,b) \tag{1.2}$$

for all $a,b>0$ with $a\neq b$.

In [16–22], the authors presented the sharp power mean bounds for L, I, $(IL)^{1/2}$, and $(L+I)/2$ as follows:

$$\begin{gathered} M_0(a,b)<L(a,b)<M_{1/3}(a,b), \qquad M_{2/3}(a,b)<I(a,b)<M_{\log 2}(a,b), \\ M_0(a,b)<\sqrt{L(a,b)I(a,b)}<M_{1/2}(a,b), \qquad \frac{1}{2}(L(a,b)+I(a,b))<M_{1/2}(a,b) \end{gathered} \tag{1.3}$$

for all $a,b>0$ with $a\neq b$.

Alzer and Qiu [12] proved that the inequality

$$\frac{1}{2}(L(a,b)+I(a,b))>M_p(a,b) \tag{1.4}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $p\leq \log 2/(1+\log 2)=0.40938\ldots$.

The following sharp bounds for the sum $\alpha A(a,b)+(1-\alpha)L(a,b)$, and the products $A^{\alpha}(a,b)L^{1-\alpha}(a,b)$ and $G^{\alpha}(a,b)L^{1-\alpha}(a,b)$ in terms of power means were proved in [5, 8] as follows:

$$\begin{gathered} M_{\log 2/(\log 2-\log\alpha)}(a,b)<\alpha A(a,b)+(1-\alpha)L(a,b)<M_{(1+2\alpha)/3}(a,b), \\ M_0(a,b)<A^{\alpha}(a,b)L^{1-\alpha}(a,b)<M_{(1+2\alpha)/3}(a,b), \\ M_0(a,b)<G^{\alpha}(a,b)L^{1-\alpha}(a,b)<M_{(1-\alpha)/3}(a,b) \end{gathered} \tag{1.5}$$

for any $\alpha\in(0,1)$ and all $a,b>0$ with $a\neq b$.

In [2, 7], the authors answered the question: for any $\alpha\in(0,1)$, what are the greatest values $p_1=p_1(\alpha)$, $p_2=p_2(\alpha)$, $p_3=p_3(\alpha)$, and $p_4=p_4(\alpha)$, and the least values $q_1=q_1(\alpha)$, $q_2=q_2(\alpha)$, $q_3=q_3(\alpha)$, and $q_4=q_4(\alpha)$, such that the inequalities

$$\begin{gathered} M_{p_1}(a,b)<P^{\alpha}(a,b)L^{1-\alpha}(a,b)<M_{q_1}(a,b), \\ M_{p_2}(a,b)<A^{\alpha}(a,b)G^{1-\alpha}(a,b)<M_{q_2}(a,b), \\ M_{p_3}(a,b)<G^{\alpha}(a,b)H^{1-\alpha}(a,b)<M_{q_3}(a,b), \\ M_{p_4}(a,b)<A^{\alpha}(a,b)H^{1-\alpha}(a,b)<M_{q_4}(a,b) \end{gathered} \tag{1.6}$$

hold for all $a,b>0$ with $a\neq b$?

In [4], the authors presented the greatest value $p = p(\alpha, \beta)$ and the least value $q = q(\alpha, \beta)$ such that the double inequality

$$M_p(a,b) < A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) < M_q(a,b) \tag{1.7}$$

holds for all $a, b > 0$ with $a \neq b$ and $\alpha, \beta > 0$ with $\alpha + \beta < 1$.

It is the aim of this paper to answer the question: for any $p, q \in \mathbb{R}$ with $p \neq q$ and $p \neq -q$, what are the greatest value $\lambda = \lambda(p, q)$ and the least value $\mu = \mu(p, q)$, such that the double inequality

$$M_{\lambda}(a,b) < \sqrt{M_p(a,b)M_q(a,b)} < M_{\mu}(a,b) \tag{1.8}$$

holds for all $a, b > 0$ with $a \neq b$?

2. Main Result

In order to establish our main result, we need a lemma which we present in this section.

Lemma 2.1. *Let $p, q \neq 0$, $p \neq q$ and $x > 1$. Then*

$$M_p(x,1)M_q(x,1) < M^2_{(p+q)/2}(x,1) \tag{2.1}$$

for $p + q > 0$, and

$$M_p(x,1)M_q(x,1) > M^2_{(p+q)/2}(x,1) \tag{2.2}$$

for $p + q < 0$.

Proof. From (1.1), we have

$$\begin{aligned} &\log\left[M_p(x,1)M_q(x,1)\right] - \log M^2_{(p+q)/2}(x,1) \\ &\quad = \frac{1}{p}\log\frac{1+x^p}{2} + \frac{1}{q}\log\frac{1+x^q}{2} - \frac{4}{p+q}\log\frac{1+x^{(p+q)/2}}{2}. \end{aligned} \tag{2.3}$$

Let

$$f(x) = \frac{1}{p}\log\frac{1+x^p}{2} + \frac{1}{q}\log\frac{1+x^q}{2} - \frac{4}{p+q}\log\frac{1+x^{(p+q)/2}}{2}, \tag{2.4}$$

then simple computations lead to

$$f(1) = 0, \tag{2.5}$$

$$f'(x) = \frac{\left(1 - x^{(p+q)/2}\right)\left(x^{p/2} - x^{q/2}\right)^2}{x(1+x^p)(1+x^q)\left(1+x^{(p+q)/2}\right)}. \tag{2.6}$$

Equation (2.6) implies that

$$f'(x) < 0 \tag{2.7}$$

for $p + q > 0$, and

$$f'(x) > 0 \tag{2.8}$$

for $p + q < 0$.

Therefore, inequality (2.1) follows from (2.3)–(2.5) and inequality (2.7), and inequality (2.2) follows from (2.3)–(2.5) and inequality (2.8).

Let

$$\begin{aligned}
E_0 &= \left\{(p,q) \in \mathbb{R}^2 : p = q\right\}, & E_0' &= \left\{(p,q) \in \mathbb{R}^2 : p = -q\right\},\\
E_1 &= \left\{(p,q) \in \mathbb{R}^2 : p,q > 0,\ p > q\right\}, & E_1' &= \left\{(p,q) \in \mathbb{R}^2 : p,q > 0,\ p < q\right\},\\
E_2 &= \left\{(p,q) \in \mathbb{R}^2 : p,q < 0,\ p > q\right\}, & E_2' &= \left\{(p,q) \in \mathbb{R}^2 : p,q < 0,\ p < q\right\},\\
E_3 &= \left\{(p,q) \in \mathbb{R}^2 : p > 0,\ q = 0\right\}, & E_3' &= \left\{(p,q) \in \mathbb{R}^2 :\ p = 0,\ q > 0\right\},\\
E_4 &= \left\{(p,q) \in \mathbb{R}^2 : p > 0,\ q < 0,\ p + q > 0\right\}, & E_4' &= \left\{(p,q) \in \mathbb{R}^2 : p < 0,\ q > 0,\ p + q > 0\right\},\\
E_5 &= \left\{(p,q) \in \mathbb{R}^2 : p = 0,\ q < 0\right\}, & E_5' &= \left\{(p,q) \in \mathbb{R}^2 : p < 0,\ q = 0\right\},\\
E_6 &= \left\{(p,q) \in \mathbb{R}^2 : p > 0,\ q < 0,\ p + q < 0\right\}, & E_6' &= \left\{(p,q) \in \mathbb{R}^2 : p < 0,\ q > 0,\ p + q < 0\right\}.
\end{aligned} \tag{2.9}$$

Then we clearly see that $\mathbb{R}^2 = \bigcup_{i=0}^{6} E_i \bigcup_{i=0}^{6} E_i'$, and it is not difficult to verify that the identity $\sqrt{M_p(a,b)M_q(a,b)} = M_{(p+q)/2}(a,b)$ holds for all $a, b > 0$ if $(p,q) \in E_0 \bigcup E_0'$. Let

$$\begin{gathered}
\lambda = \begin{cases} \dfrac{2pq}{(p+q)}, & (p,q) \in E_1 \bigcup E_1', \\ \dfrac{(p+q)}{2}, & (p,q) \in E_2 \bigcup E_2' \bigcup E_5 \bigcup E_5' \bigcup E_6 \bigcup E_6', \\ 0, & (p,q) \in E_3 \bigcup E_3' \bigcup E_4 \bigcup E_4', \end{cases} \\
\mu = \begin{cases} \dfrac{2pq}{(p+q)}, & (p,q) \in E_2 \bigcup E_2', \\ \dfrac{(p+q)}{2}, & (p,q) \in E_1 \bigcup E_1' \bigcup E_3 \bigcup E_3' \bigcup E_4 \bigcup E_4', \\ 0, & (p,q) \in E_5 \bigcup E_5' \bigcup E_6 \bigcup E_6'. \end{cases}
\end{gathered} \tag{2.10}$$

□

Then we have Theorem 2.2 as follows.

Theorem 2.2. *The double inequality*

$$M_\lambda(a,b) < \sqrt{M_p(a,b)M_q(a,b)} < M_\mu(a,b) \tag{2.11}$$

holds for all a, $b > 0$ with $a \neq b$, and $M_\lambda(a,b)$ and $M_\mu(a,b)$ are the best possible lower and upper power mean bounds for the geometric mean of $M_p(a,b)$ and $M_q(a,b)$.

Proof. From (1.1), we clearly see that $M_p(a,b)$ is symmetric and homogenous of degree 1. Without loss of generality, we assume that $b = 1$, $a = x > 1$ and $p > q$. We divide the proof of inequality (2.11) into three cases.

Case 1. $(p,q) \in E_1 \bigcup E_2$. Then from Lemma 2.1, we clearly see that

$$\sqrt{M_p(x,1)M_q(x,1)} < M_{(p+q)/2}(x,1) \tag{2.12}$$

for $(p,q) \in E_1$, and

$$\sqrt{M_p(x,1)M_q(x,1)} > M_{(p+q)/2}(x,1) \tag{2.13}$$

for $(p,q) \in E_2$.

From (1.1), we get

$$\begin{aligned} &\log\left[M_p(x,1)M_q(x,1)\right] - \log M^2_{2pq/(p+q)}(x,1) \\ &\quad = \frac{1}{p}\log\frac{1+x^p}{2} + \frac{1}{q}\log\frac{1+x^q}{2} - \frac{p+q}{pq}\log\frac{1+x^{2pq/(p+q)}}{2}. \end{aligned} \tag{2.14}$$

Let

$$F(x) = \frac{1}{p}\log\frac{1+x^p}{2} + \frac{1}{q}\log\frac{1+x^q}{2} - \frac{p+q}{pq}\log\frac{1+x^{2pq/(p+q)}}{2}, \tag{2.15}$$

then simple computations lead to

$$F(1) = 0, \tag{2.16}$$

$$F'(x) = \frac{x^q G(x)}{x(1+x^p)(1+x^q)\left(1+x^{2pq/(p+q)}\right)}, \tag{2.17}$$

where

$$G(x) = x^{p-q} - x^{(2pq+p^2-q^2)/(p+q)} + 2x^p - x^{2pq/(p+q)} - 2x^{q(p-q)/(p+q)} + 1, \tag{2.18}$$

$$G(1) = 0, \tag{2.19}$$

$$G'(x) = x^{(pq-q^2-p-q)/(p+q)} H(x), \tag{2.20}$$

where

$$H(x) = (p-q)x^{p(p-q)/(p+q)} - \frac{2pq+p^2-q^2}{p+q}x^p + 2px^{(p^2+q^2)/(p+q)} - \frac{2pq}{p+q}x^q - \frac{2q(p-q)}{p+q}, \tag{2.21}$$

$$H(1) = \frac{2(p-q)^2}{p+q}, \tag{2.22}$$

$$H'(x) = \frac{p}{p+q}x^{p-1}I(x), \tag{2.23}$$

where

$$I(x) = (p-q)^2x^{-2pq/(p+q)} + 2\left(p^2+q^2\right)x^{-q(p-q)/(p+q)} - 2q^2x^{-(p-q)} - 2pq - p^2 + q^2, \tag{2.24}$$

$$I(1) = 2(p-q)^2, \tag{2.25}$$

$$I'(x) = \frac{2q(p-q)}{p+q}x^{(q^2-pq-p-q)/(p+q)}J(x), \tag{2.26}$$

where

$$J(x) = -p(p-q)x^{-q} + q(p+q)x^{-p(p-q)/(p+q)} - p^2 - q^2, \tag{2.27}$$

$$J(1) = -2p(p-q), \tag{2.28}$$

$$J'(x) = pq(p-q)x^{-q-1}\left(1 - x^{(q^2-p^2+2pq)/(p+q)}\right). \tag{2.29}$$

If $(p,q) \in E_1$, then (2.15), (2.18), (2.21), (2.22), (2.24), (2.25), (2.27), and (2.28) lead to

$$\lim_{x\to+\infty} F(x) = 0, \tag{2.30}$$

$$\lim_{x\to+\infty} G(x) = -\infty, \tag{2.31}$$

$$\lim_{x\to+\infty} H(x) = -\infty, \tag{2.32}$$

$$H(1) > 0, \tag{2.33}$$

$$\lim_{x\to+\infty} I(x) = -2pq - p^2 + q^2 < 0, \tag{2.34}$$

$$I(1) > 0, \tag{2.35}$$

$$\lim_{x\to+\infty} J(x) = -\left(p^2+q^2\right) < 0, \tag{2.36}$$

$$J(1) < 0. \tag{2.37}$$

We divide the discussion into two subcases.

Subcase 1.1. $(p,q) \in E_1$. Then (2.26) and (2.29) together with inequalities (2.36) and (2.37) imply that $I(x)$ is strictly decreasing in $[1,+\infty)$. In fact, if $(q^2-p^2+2pq)/(p+q) \geq 0$, then (2.29) and inequality (2.37) imply that $J(x) < 0$ for $x \in [1,+\infty)$. If $(q^2-p^2+2pq)/(p+q) < 0$, then (2.29) and inequality (2.36) lead to the conclusion that $J(x) < 0$ for $x \in [1,+\infty)$.

From inequalities (2.34) and (2.35) together with the monotonicity of $I(x)$, we know that there exists $\lambda_1 > 1$ such that $I(x) > 0$ for $x \in [1,\lambda_1)$ and $I(x) < 0$ for $x \in (\lambda_1,+\infty)$. Then (2.23) leads to the conclusion that $H(x)$ is strictly increasing in $[1,\lambda_1]$ and strictly decreasing in $[\lambda_1,+\infty)$.

It follows from (2.32) and (2.33) together with the piecewise monotonicity of $H(x)$ that there exists $\lambda_2 > \lambda_1 > 1$ such that $H(x) > 0$ for $[1,\lambda_2]$ and $H(x) < 0$ for $(\lambda_2,+\infty)$. Then (2.20) leads to the conclusion that $G(x)$ is strictly increasing in $[1,\lambda_2]$ and strictly decreasing in $[\lambda_2,+\infty)$.

From (2.17), (2.19) and (2.31) together with the piecewise monotonicity of $G(x)$, we clearly see that there exists $\lambda_3 > \lambda_2 > 1$ such that $F(x)$ is strictly increasing in $[1,\lambda_3]$ and strictly decreasing in $[\lambda_3,+\infty)$.

Therefore, $\sqrt{M_p(x,1)M_q(x,1)} > M_{2pq/(p+q)}(x,1)$ follows from (2.14)–(2.16) and (2.30) together with the piecewise monotonicity of $F(x)$.

Subcase 1.2. $(p,q) \in E_2$. Then (2.30) and (2.35) again hold, and (2.18), (2.21), (2.22), and (2.28) lead to

$$\lim_{x\to+\infty} G(x) = +\infty, \tag{2.38}$$

$$\lim_{x\to+\infty} H(x) = +\infty, \tag{2.39}$$

$$H(1) < 0, \tag{2.40}$$

$$J(1) > 0. \tag{2.41}$$

It follows from (2.29) and inequalities $(q^2-p^2+2pq)/(p+q) < 0$ and (2.41) that $J(x) > 0$ for $x \in [1,+\infty)$. Then (2.26) and inequality (2.35) lead to the conclusion that $I(x) > 0$ for $x \in [1,+\infty)$. Therefore, $H(x)$ is strictly increasing in $[1,+\infty)$ follows from (2.23).

It follows from (2.20) and (2.39) together with inequality (2.40) and the monotonicity of $H(x)$ that there exists $\mu_1 > 1$ such that $G(x)$ is strictly decreasing in $[1,\mu_1]$ and strictly increasing in $[\mu_1,+\infty)$.

From (2.17), (2.19) and (2.38) together with the piecewise monotonicity of $G(x)$, we clearly see that there exists $\mu_2 > \mu_1 > 1$ such that $F(x)$ is strictly decreasing in $[1,\mu_2]$ and strictly increasing in $[\mu_2,+\infty)$.

Therefore, $\sqrt{M_p(x,1)M_q(x,1)} < M_{2pq/(p+q)}(x,1)$ follows from (2.14)–(2.16) and (2.30) together with the piecewise monotonicity of $F(x)$.

Case 2. $(p,q) \in E_3 \bigcup E_5$. Clearly, we have $M_0(x,1) < \sqrt{M_p(x,1)M_q(x,1)}$ for $(p,q) \in E_3$ and $M_0(x,1) > \sqrt{M_p(x,1)M_q(x,1)}$ for $(p,q) \in E_5$. Therefore, we need only to prove that

$$\sqrt{M_0(x,1)M_r(x,1)} < M_{r/2}(x,1) \tag{2.42}$$

for $r > 0$, and

$$\sqrt{M_0(x,1)M_r(x,1)} > M_{r/2}(x,1) \tag{2.43}$$

for $r < 0$.

From (1.1), one has

$$\log[M_0(x,1)M_r(x,1)] - \log M_{r/2}^2(x,1) = \frac{1}{2}\log x + \frac{1}{r}\log\frac{1+x^r}{2} - \frac{4}{r}\log\frac{1+x^{r/2}}{2}. \tag{2.44}$$

Let

$$f(x) = \frac{1}{2}\log x + \frac{1}{r}\log\frac{1+x^r}{2} - \frac{4}{r}\log\frac{1+x^{r/2}}{2}, \tag{2.45}$$

then simple computations lead to

$$f(1) = 0, \tag{2.46}$$

$$f'(x) = -\frac{\left(x^{r/2}-1\right)^3}{2x(1+x^r)\left(1+x^{r/2}\right)}. \tag{2.47}$$

If $r > 0$ (or $r < 0$, resp.), then (2.47) leads to the conclusion that $f(x)$ is strictly decreasing (or increasing, resp.) in $[1,+\infty)$. Therefore, inequalities (2.42) and (2.43) follow from (2.44)–(2.46) and the monotonicity of $f(x)$.

Case 3. $(p,q) \in E_4 \bigcup E_6$. Then from Lemma 2.1, we clearly see that $M_{(p+q)/2}(x,1) > \sqrt{M_p(x,1)M_q(x,1)}$ for $(p,q) \in E_4$ and $\sqrt{M_p(x,1)M_q(x,1)} > M_{(p+q)/2}(x,1)$ for $(p,q) \in E_6$. Therefore, we need only to prove that

$$\sqrt{M_p(x,1)M_q(x,1)} > M_0(x,1) \tag{2.48}$$

for $(p,q) \in E_4$, and

$$\sqrt{M_p(x,1)M_q(x,1)} < M_0(x,1) \tag{2.49}$$

for $(p,q) \in E_6$.

From (1.1), we get

$$\log\left[M_p(x,1)M_q(x,1)\right] - \log M_0^2(x,1) = \frac{1}{p}\log\frac{1+x^p}{2} + \frac{1}{q}\log\frac{1+x^q}{2} - \log x. \tag{2.50}$$

Let

$$f(x) = \frac{1}{p}\log\frac{1+x^p}{2} + \frac{1}{q}\log\frac{1+x^q}{2} - \log x, \tag{2.51}$$

then simple computations lead to

$$f(1) = 0, \tag{2.52}$$

$$f'(x) = \frac{x^{p+q} - 1}{x(1 + x^p)(1 + x^q)}. \tag{2.53}$$

If $(p,q) \in E_4$ (or E_6, resp.), then (2.53) implies that $f(x)$ is strictly increasing (or decreasing, resp.) in $[1, +\infty)$. Therefore, inequalities (2.48) and (2.49) follow from (2.50)–(2.52) and the monotonicity of $f(x)$.

Next, we prove that $M_\lambda(a,b)$ and $M_\mu(a,b)$ are the best possible lower and upper power mean bounds for the geometric mean of $M_p(a,b)$ and $M_q(a,b)$. We divide the proof into six cases.

Case A. $(p,q) \in E_1$. Then for any $\epsilon \in (0, (p+q)/2)$ and $x > 0$, from (1.1), one has

$$\begin{aligned} &M_p(1+x,1)M_q(1+x,1) - M^2_{(p+q)/2-\epsilon}(1+x,1) \\ &\quad = \left[\frac{1+(1+x)^p}{2}\right]^{1/p}\left[\frac{1+(1+x)^q}{2}\right]^{1/q} - \left[\frac{1+(1+x)^{(p+q)/2-\epsilon}}{2}\right]^{4/(p+q-2\epsilon)}, \end{aligned} \tag{2.54}$$

$$\lim_{x \to +\infty} \frac{M^2_{2pq/(p+q)+\epsilon}(x,1)}{M_p(x,1)M_q(x,1)} = 2^{\epsilon(p+q)^2/pq[2pq+\epsilon(p+q)]} > 1. \tag{2.55}$$

Letting $x \to 0$ and making use of Taylor expansion, we get

$$\begin{aligned} &\left[\frac{1+(1+x)^p}{2}\right]^{1/p}\left[\frac{1+(1+x)^q}{2}\right]^{1/q} - \left[\frac{1+(1+x)^{(p+q)/2-\epsilon}}{2}\right]^{4/(p+q-2\epsilon)} \\ &\quad = \frac{\epsilon}{4}x^2 + o\left(x^2\right). \end{aligned} \tag{2.56}$$

Equations (2.54) and (2.56) together with inequality (2.55) imply that for any $\epsilon \in (0, p+q/2)$, there exist $\delta_1 = \delta_1(\epsilon) > 0$ and $X_1 = X_1(p,q,\epsilon) > 1$ such that $\sqrt{M_p(1+x,1)M_q(1+x,1)} > M_{(p+q)/2-\epsilon}(1+x,1)$ for $x \in (0, \delta_1)$ and $\sqrt{M_p(x,1)M_q(x,1)} < M_{2pq/(p+q)+\epsilon}(x,1)$ for $x \in (X_1, +\infty)$.

Case B. $(p,q) \in E_2$. Then for $\epsilon \in (0, -(p+q)/2)$ and $x > 0$, making use of (1.1) and Taylor expansion, we have

$$M^2_{(p+q)/2+\epsilon}(1+x,1) - M_p(1+x,1)M_q(1+x,1) = \frac{\epsilon}{4}x^2 + o\left(x^2\right) \quad (x \longrightarrow 0), \tag{2.57}$$

$$\lim_{x \to +\infty} \frac{M_p(x,1)M_q(x,1)}{M^2_{2pq/(p+q)-\epsilon}(x,1)} = 2^{\epsilon(p+q)^2/pq[2pq-\epsilon(p+q)]} > 1. \tag{2.58}$$

Equation (2.57) and inequality (2.58) imply that for any $\epsilon \in (0, -(p+q)/2)$, there exist $\delta_2 = \delta_2(\epsilon) > 0$ and $X_2 = X_2(p,q,\epsilon) > 1$ such that $M_{(p+q)/2+\epsilon}(1+x,1) > \sqrt{M_p(1+x,1)M_q(1+x,1)}$ for $x \in (0,\delta_2)$ and $\sqrt{M_p(x,1)M_q(x,1)} > M_{2pq/(p+q)-\epsilon}(x,1)$ for $x \in (X_2,+\infty)$.

Case C. $(p,q) \in E_3$. Then for $\epsilon \in (0,p/2)$ and $x > 0$, making use of (1.1) and Taylor expansion, we have

$$M_p(1+x,1)M_0(1+x,1) - M^2_{p/2-\epsilon}(1+x,1) = \frac{\epsilon}{4}x^2 + o\left(x^2\right) \quad (x \longrightarrow 0),$$
$$\lim_{x\to+\infty}\frac{M^2_{\epsilon}(x,1)}{M_p(x,1)M_0(x,1)} = +\infty. \tag{2.59}$$

Equation (2.59) leads to the conclusion that for any $\epsilon \in (0,p/2)$, there exist $\delta_3 = \delta_3(\epsilon) > 0$ and $X_3 = X_3(p,\epsilon) > 1$ such that $\sqrt{M_p(1+x,1)M_0(1+x,1)} > M_{p/2-\epsilon}(1+x,1)$ for $x \in (0,\delta_3)$ and $M_\epsilon(x,1) > \sqrt{M_p(x,1)M_0(x,1)}$ for $x \in (X_3,+\infty)$.

Case D. $(p,q) \in E_4$. Then for $\epsilon \in (0,(p+q)/2)$ and $x > 0$, making use of (1.1) and Taylor expansion, we have

$$M_p(1+x,1)M_q(1+x,1) - M^2_{(p+q)/2-\epsilon}(1+x,1) = \frac{\epsilon}{4}x^2 + o\left(x^2\right) \quad (x \longrightarrow 0),$$
$$\lim_{x\to+\infty}\frac{M^2_{\epsilon}(x,1)}{M_p(x,1)M_q(x,1)} = +\infty. \tag{2.60}$$

Equation (2.60) implies that for any $\epsilon \in (0,(p+q)/2)$, there exist $\delta_4 = \delta_4(\epsilon) > 0$ and $X_4 = X_4(p,q,\epsilon) > 1$ such that $M_{(p+q)/2-\epsilon}(1+x,1) < \sqrt{M_p(1+x,1)M_q(1+x,1)}$ for $x \in (0,\delta_4)$ and $M_\epsilon(x,1) > \sqrt{M_p(x,1)M_q(x,1)}$ for $x \in (X_4,+\infty)$.

Case *E*. $(p,q) \in E_5$. Then for any $\epsilon \in (0,-q/2)$ and $x > 0$, making use of (1.1) and Taylor expansion, one has

$$M^2_{q/2+\epsilon}(1+x,1) - M_0(1+x,1)M_q(1+x,1) = \frac{\epsilon}{4}x^2 + o\left(x^2\right) \quad (x \longrightarrow 0),$$
$$\lim_{x\to+\infty}\frac{M_0(x,1)M_q(x,1)}{M^2_{-\epsilon}(x,1)} = +\infty. \tag{2.61}$$

Equation (2.61) leads to the conclusion that for any $\epsilon \in (0,-q/2)$, there exist $\delta_5 = \delta_5(\epsilon) > 0$ and $X_5 = X_5(q,\epsilon) > 1$ such that $M_{q/2+\epsilon}(1+x,1) > \sqrt{M_0(1+x,1)M_q(1+x,1)}$ for $x \in (0,\delta_5)$ and $M_{-\epsilon}(x,1) < \sqrt{M_0(x,1)M_q(x,1)}$ for $x \in (X_5,+\infty)$.

Case F. $(p,q)\in E_6$. Then for any $\epsilon\in(0,-(p+q)/2)$ and $x>0$, making use of (1.1) and Taylor expansion, one has

$$M^2_{(p+q)/2+\epsilon}(1+x,1)-M_p(1+x,1)M_q(1+x,1)=\frac{\epsilon}{4}x^2+o\left(x^2\right)\qquad(x\longrightarrow 0),$$
$$\lim_{x\to+\infty}\frac{M_p(x,1)M_q(x,1)}{M^2_{-\epsilon}(x,1)}=+\infty. \tag{2.62}$$

Equation (2.62) shows that for any $\epsilon\in(0,-(p+q)/2)$, there exist $\delta_6=\delta_6(\epsilon)>0$ and $X_6=X_6(p,q,\epsilon)>1$ such that $M_{(p+q)/2+\epsilon}(1+x,1)>\sqrt{M_p(1+x,1)M_q(1+x,1)}$ for $x\in(0,\delta_6)$ and $\sqrt{M_p(x,1)M_q(x,1)}>M^2_{-\epsilon}(x,1)$ for $x\in(X_6,+\infty)$. □

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