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Research Article

Type-K Exponential Ordering with Application to Delayed Hopfield-Type Neural Networks

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Order-preserving and convergent results of delay functional differential equations without quasimonotone condition are established under type-K exponential ordering. As an application, the model of delayed Hopfield-type neural networks with a type-K monotone interconnection matrix is considered, and the attractor result is obtained.

1. Introduction

Since monotone methods have been initiated by Kamke [1] and Müler [2], and developed further by Krasnoselskii [3, 4], Matano [5], and Smith [6], the theory and application of monotone dynamics have become increasingly important (see [7–18]).

It is well known that the quasimonotone condition is very important in studying the asymptotic behaviors of dynamical systems. If this condition is satisfied, the solution semiflows will admit order-preserving property. There are many interesting results, for example, [6, 8–12, 14–17] for competitive (cooperative) or type-K competitive (cooperative) systems and [6, 7, 13] for delayed systems. In particular, for the scalar delay differential equations of the form

$$x'(t) = g(x(t), x(t-r)), \tag{1.1}$$

if the quasimonotone condition $(\partial g(x, y))/\partial y > 0$ holds, then (1.1) generates an eventually strongly monotone semiflow on the space $C([-r, 0], \mathbb{R})$, which is one of sufficient conditions for obtaining convergent results. In other words, the right hand side of (1.1) must be strictly increasing in the delayed argument. This is a severe restriction, and so the quasimonotone conditions are not always satisfied in applications. Recently, many researchers have tried to relax the quasimonotone condition by introducing a new cone or partial ordering, for example, the exponential ordering [6, 18, 19]. In particular, Smith [6] and Wu and Zhao [18] considered a new cone parameterized by a nonnegative constant, which is applicable to a single equation. Replacing the previous constant by a quasipositive matrix, the exponential ordering is generalized to some delay differential systems by Smith [6] and Y. Wang and Y. Wang [19]. However, the above results are not suitable to the type-K systems (see [6] for its definition). A typical example is a Hopfield-type neural network model with a type-K monotone interconnection matrix, which implies that the interaction among neurons is not only excitatory but also inhibitory. For this purpose, we introduce a type-K exponential ordering and establish order-preserving and convergent results under the weak quasimonotone condition (WQM) (see Section 2) and then apply the result to a network model with a type-K monotone interconnection matrix.

This paper is arranged as follows. In next section, the type-K exponential ordering parameterized by a type-K monotone matrix is introduced, and convergent result is established. In Section 3, we apply our results to a delayed Hopfield-type neural network.

2. Type-K Exponential Ordering

In this section, we establish a new cone and introduce some order-preserving and convergent results.

Let (X_i, X_i^+), $i \in N = \{1, 2, \ldots, n\}$, be ordered Banach spaces with $\mathrm{Int}X_i^+ \neq \emptyset$. For $x_i, y_i \in X_i$, we write $x_i \leq_{X_i} y_i$ if $y_i - x_i \in X_i^+$; $x_i <_{X_i} y_i$ if $y_i - x_i \in X_i^+ \setminus \{0\}$; $x_i \ll_{X_i} y_i$ if $y_i - x_i \in \mathrm{Int}X_i^+$. For $k \in N$, we denote $I = \{1, 2, \ldots, \kappa\}$ and $J = N \setminus I = \{\kappa + 1, \ldots, n\}$. Thus, we can define the product space $X = \prod_{i=1}^{i=n} X_i$ which generates two cones $X^+ = \prod_{i=1}^{i=n} X_i^+$ and $K = \prod_{i=1}^{i=\kappa} X_i^+ \times \prod_{i=\kappa+1}^{i=n} (-X_i^+)$ with nonempty interiors $\mathrm{Int}X^+ = \prod_{i=1}^{i=n} \mathrm{Int}X_i^+$ and $\mathrm{Int}K = \prod_{i=1}^{i=\kappa} \mathrm{Int}X_i^+ \times \prod_{i=\kappa+1}^{i=n} (-\mathrm{Int}X_i^+)$. The ordering relation on X^+ and K is defined in the following way:

$$
\begin{gathered}
x \leq_X y \Longleftrightarrow x_i \leq_{X_i} y_i, \quad \forall i \in N, \\
x <_X y \Longleftrightarrow x \leq y, \quad x_i <_{X_i} y_i, \quad \text{for some } i \in N, \text{ that is, } x \leq_X y,\ x \neq y, \\
x \ll_X y \Longleftrightarrow x_i \ll_{X_i} y_i, \quad \forall i \in N, \\
x \leq_K y \Longleftrightarrow x_i \leq_{X_i} y_i, \quad \forall i \in I, \qquad x_i \geq_{X_i} y_i, \quad \forall i \in J, \\
x <_K y \Longleftrightarrow x \leq_K y, \quad x_i <_{X_i} y_i, \quad \text{for some } i \in I \quad \text{or} \quad x_i >_{X_i} y_i, \quad \text{for some } i \in J, \\
x \ll_K y \Longleftrightarrow x_i \ll_{X_i} y_i, \quad \forall i \in I, \qquad x_i \gg_{X_i} y_i, \quad \forall i \in J.
\end{gathered}
\tag{2.1}
$$

A semiflow on X is a continuous mapping Φ: $X \times \mathbb{R}_+ \to X$, $(x, t) \to \Phi(x, t)$, which satisfies (i) $\Phi_0 = id$ and (ii) $\Phi_t \cdot \Phi_s = \Phi_{t+s}$ for $t, s \in \mathbb{R}_+$. Here, $\Phi_t(x) \equiv \Phi(x, t)$ for $x \in X$ and $t \geq 0$. The *orbit* of x is denoted by $O(x)$:

$$
O(x) = \{\Phi_t(x) : t \geq 0\}. \tag{2.2}
$$

An *equilibrium point* is a point x for which $\Phi_t(x) = x$ for all $t \geq 0$. Let $\mathbf{E}$ be the set of all equilibrium points for Φ. The omega limit set $\omega(x)$ of x is defined in the usual way. A point $x \in X$ is called a *quasiconvergent point* if $\omega(x) \subset \mathbf{E}$. The set of all such points is denoted by $\mathbf{Q}$.

A point $x \in X$ is called a *convergent point* if $\omega(x)$ consists of a single point of $\mathbf{E}$. The set of all convergent points is denoted by $\mathbf{C}$.

The semiflow Φ is said to be *type-K monotone* provided

$$\Phi_t(x) \leqslant_K \Phi_t(y) \quad \text{whenever } x \leqslant_K y \ \forall t \geq 0. \tag{2.3}$$

Φ is called *type-K strongly order preserving* (for short type-K SOP), if it is type-K monotone, and whenever $x <_K y$, there exist open subsets U, V of X with $x \in U$, $y \in V$ and $t_0 > 0$, such that

$$\Phi_t(U) \leqslant_K \Phi_t(V) \quad \forall t \geq t_0. \tag{2.4}$$

The semiflow Φ is said to be *strongly type-K monotone* on X if Φ is type-K monotone, and whenever $x <_K y$ and $t > 0$, then $\Phi_t(x) \ll_K \Phi_t(y)$. We say that Φ is *eventually strongly type-K monotone* if it is type-K monotone, and whenever $x <_K y$, there exists $t_0 > 0$ such that $\Phi_{t_0}(x) \ll_K \Phi_{t_0}(y)$. Clearly, strongly type-K monotonicity implies eventually strongly type-K monotonicity.

An $n \times n$ matrix M is said to be *type-K monotone* if it has the following manner:

$$M = \begin{pmatrix} \overline{A} & -\overline{B} \\ -\overline{C} & \overline{D} \end{pmatrix}, \tag{2.5}$$

where $\overline{A} = (\mathrm{a}_{ij})_{k\times k}$ satisfies $(a_{ij}) \geqslant 0$ if $i \neq j$, similarly for the $(n-k) \times (n-k)$ matrix $\overline{D}$ and $\overline{B} \geqslant 0$, $\overline{C} \geqslant 0$.

In this paper, the following lemma is necessary.

Lemma 2.1. *If M is a type-K monotone matrix, then e^{Mt} remains type-K monotone with diagonal entries being strictly positive for all $t > 0$.*

Proof. The product of two type-K monotone matrices remains type-K monotone; the rest is obvious and we omit it here. □

Let $r > 0$ be fixed and let $C := C([-r, 0], X)$. The ordering relations on C are understood to hold pointwise. Consider the family of sets parameterized by type-K monotone matrix M given by

$$\widetilde{K}_M = \left\{ \phi = (\phi_1, \phi_2, \dots, \phi_n) \in C : \phi(s) \geq_K 0, \ s \in [-r, 0] \phi(t) \geq_K e^{M(t-s)} \phi(s), \ 0 \geq t \geq s \geq -r \right\}. \tag{2.6}$$

It is easy to see that $\widetilde{K}_M$ is a closed cone in C and generates a partial ordering on C which is written by $\geq_M$. Assume that $\phi \in C$ is differentiable on $(-r, 0)$, a similar argument to [18, lemma 2.1] implies that $\phi \geq_M 0$ if and only if $\phi(-r) \geq_K 0$ and $d\phi(s)/ds - M\phi(s) \geq_K 0$ for all $s \in (-r, 0)$.

Consider the abstract functional differential equation

$$x'(t) = f(x_t), \tag{2.7}$$

where $f : D \to X$ is continuous and satisfies a local Lipschitz condition on each compact subset of D and D is an open subset of C. By the standard equation theory, the solution $x(t, \phi)$ of (2.7) can be continued to the maximal interval of existence $[0, \sigma_\phi)$. Moreover, if $\sigma_\phi > r$, then $x(t, \phi)$ is a classical solution of (2.7) for $t \in (r, \sigma_\phi)$. In this section, for simplicity, we assume that, for each $\phi \in D$, (2.7) admits a solution $x(t, \phi)$ defined on $[0, \infty)$. Therefore, (2.7) generates a semiflow on C by $\Phi_t(\phi) \equiv x_t(\phi)$, where $x_t(\phi)(s) = x(t + s, \phi)$ for $t \geq 0$ and $-r \leq s \leq 0$.

In the following, we will seek a sufficient condition for the solution of (2.7) to preserve the ordering $\geq_M$.

(WQM) Whenever $\phi, \psi \in D$, $\psi \geq_M \phi$, then

$$f(\psi) - f(\phi) \geq_K M(\psi(0) - \phi(0)). \tag{2.8}$$

Theorem 2.2. *Suppose that (WQM) holds. If $\psi \geq_M \phi$, then $x_t(\psi) \geq_M x_t(\phi)$ for all $t \geq 0$.*

Proof. Let $\eta \in \mathrm{Int}K$. For any $\varepsilon > 0$, define $f_\varepsilon(\phi) = f(\phi) + \varepsilon\eta$ for $\phi \in D$, and let $x_t^\varepsilon(\psi)$ be a unique solution of the following equation:

$$\begin{aligned} x'(t) &= f_\varepsilon(x_t), \quad t \geq 0, \\ x(s) &= \psi(s), \quad -r \leq s \leq 0. \end{aligned} \tag{2.9}$$

Let $y^\epsilon(t) = x^\epsilon(t, \psi) - x(t, \phi)$ and define

$$S = \{t \in [0, \infty) : y_t^\epsilon \geq_M 0\}. \tag{2.10}$$

Since $\psi \geq_M \phi$, S is closed and nonempty. We first prove the following two claims.

Claim 1. If $t_0 \in S$, there exists $\delta_0 > 0$ such that $[t_0, t_0 + \delta_0] \subset S$.

According to the integral expression of (2.9) we have

$$y^\epsilon(t) = e^{M(t-s)} y^\epsilon(s) + \int_s^t e^{M(\tau - s)} \left[f(x_\tau^\epsilon(\psi)) - f(x_\tau(\phi)) - M(x^\epsilon(\tau, \psi) - x(\tau, \phi)) + \epsilon\eta \right] d\tau. \tag{2.11}$$

Since $t_0 \in S$ and (WQM) hold, we have

$$f(x_t^\epsilon(\psi)) - f(x_t(\phi)) - M(x^\epsilon(t, \psi) - x(t, \phi)) + \epsilon\eta|_{t=t_0} \geq_K \epsilon\eta \gg_K 0. \tag{2.12}$$

By the characteristic of a cone, there is $\delta_0 > 0$ such that

$$f(x_t^\epsilon(\psi)) - f(x_t(\phi)) - M(x^\epsilon(t, \psi) - x(t, \phi)) + \epsilon\eta \geq_K 0, \quad \forall t \in [t_0, t_0 + \delta_0]. \tag{2.13}$$

By Lemma 2.1, we have

$$y^{\epsilon}(t) \geq_K e^{M(t-s)} y^{\epsilon}(s), \quad \forall t_0 \leq s \leq t \leq t_0 + \delta_0, \tag{2.14}$$

which, together with the definition of $\widetilde{K}_M$, implies that

$$x_t^{\epsilon}(\psi) \geq_M x_t(\phi), \quad \forall t \in [t_0, t_0 + \delta_0]. \tag{2.15}$$

Claim 2. Let $S_1 = \{t : [0, t] \subset S\}$. Then $\sup S_1 = \infty$.

If $t^* = \sup S_1 < \infty$, then there is a sequence $\{t_n\} \subset S_1 \subset S$ such that $t_n \to t^*$ as $n \to \infty$. From the closeness of S we have $t^* \in S$. By Claim 1, $[t^*, t^* + \delta^*] \subset S$ for some $\delta^* > 0$, which contradicts the definition of t^*. Therefore, $\sup S_1 = \infty$, which implies $S = [0, \infty)$.

Since $f_\epsilon \to f$ uniformly on bounded subset of D as $\epsilon \to 0^+$, then

$$\lim_{\epsilon \to 0^+} x_t^{\epsilon}(\psi) = x_t(\psi), \quad \forall t \geq 0. \tag{2.16}$$

Letting $\epsilon \to 0^+$ in $y_t^{\epsilon} = x_t^{\epsilon}(\psi) - x_t(\phi) \geq_M 0$, we have $x_t(\psi) - x_t(\phi) \geq_M 0$, which implies that $x_t(\psi) \geq_M x_t(\phi)$. □

By the definition of the semiflow Φ_t, it is easy to see from (WQM) that Φ_t is monotone with respect to $\geq_M$ in the sense that $\Phi_t(\psi) \geq_M \Phi_t(\phi)$ whenever $\psi \geq_M \phi$ for all $t \geq 0$.

As we all know the strongly order-preserving property is necessary for obtaining some convergent results. However, it is easy to check that the cone $\widetilde{K}_M$ has empty interior on C; we cannot, therefore, expect to show that the semiflow generated by (2.7) is eventually strongly type-K monotone in C. Let $\varphi(\cdot) \in \operatorname{Int} K$ and define

$$\begin{gathered} C_\varphi = \{\phi \in C : \text{there exist } \gamma \geq 0 \text{ such that } -\gamma\varphi \leq_M \phi \leq_M \gamma\varphi\}, \\ \|\phi\|_\varphi = \inf\{\gamma \geq 0 : -\gamma\varphi \leq_M \phi \leq_M \gamma\varphi\}. \end{gathered} \tag{2.17}$$

It is easy to check that $(C_\varphi, \|\phi\|_\varphi)$ is a Banach space, $K_M = C_\varphi \cap \widetilde{K}_M$ is a cone with nonempty interior $\operatorname{Int} K_M$ (see [20]), and $i : C_\varphi \to C$ is continuous. Using the smoothing property of the semiflow Φ on C^+ and fundamental theory of abstract functional differential equations, we deduce that for all $t > r$, $\Phi_t C \subset C \cap C_\varphi$, $\Phi_t : C \to C \cap C_\varphi$ is continuous, and $\Phi_t(\psi - \phi) \in \operatorname{Int} K_M$ for any $\psi, \phi \in C$ with $\psi >_M \phi$. Thus, from Theorem 2.2, type-K strongly order-preserving property can be obtained.

Theorem 2.3. *Assume that* (WQM) *holds. If $\psi >_M \phi$, then $x_t(\psi) \gg_M x_t(\phi)$ in K_M for all $t \geq r$.*

In order to obtain the main result of this paper, which says that the generic solution converges to equilibrium, the corresponding compactness assumption will be required.

(A1) f maps bounded subset of D to bounded subset of $\mathbb{R}^n$. Moreover, for each compact subset A of D, there exists a closed and bounded subset $B = B(A)$ of D such that $x_t(\phi) \in B$ for each $\phi \in A$ and all large t.

Theorem 2.4. *Assume that* (*WQM*) *and (A1) hold. Then the set of convergent points in D contains an open and dense subset. If* $\mathbf{E}$ *consists of a single point, it attracts all solutions of* (2.7)*. If the initial value* $x_0 \geq_K 0 (x_0 \leq_K 0)$ *and* $\mathbf{E}$ *consists of two points or more, we conclude that all solutions converge to one of these.*

Proof. By Theorem 2.3, the semiflow is eventually strongly monotone in K_M. Let $\hat{e} = (\hat{1}, \ldots, \hat{1}, -\hat{1}, \ldots, -\hat{1}) \in K$, where $\hat{1}$ denotes a constant mapping defined on C; that is, $\hat{1}(s) = 1$ for all $s \in [-r, 0]$. Obviously, $\hat{e} \geq_M \hat{0}$. For any $\psi \in D$, either the sequence of points $\psi + (1/n)\hat{e}$ or $\psi - (1/n)\hat{e}$ is eventually contained in D and approaches ψ as $n \to \infty$, and, hence, each point of D can be approximated either from above or from below in D with respect to $\geq_M$. The assumption (A1) implies the compactness; that is, $O(x)$ has compact closure in X for each $x \in X$ (see [6]). Therefore, from [6, Theorem 1.4.3], we deduce that the set of quasiconvergent points contains an open and dense subset of D. From the proof of [6, Theorem 6.3.1], we know that the set $\mathbf{E}$ is totally ordered by $\geq_M$. Reference [6, Remark 1.4.2] implies that the set of convergent points contains an open and dense subset of D. The last two assertions can be obtained from [6, Theorems 2.3.1 and 2.3.2]. □

Remark 2.5. The above theorem implies that there exists an equilibrium attracting all solutions with initial values in the cone K. If $\mathbf{E}$ consists of a single element, the equilibrium attracts all solutions with initial values in D.

3. Delayed Hopfield-Type Neural Networks

In this section, we will apply our main result to the following system of delayed differential equations:

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j\left(x_j\left(t - r_j\right)\right) + I_i, \quad i = 1, 2, \ldots, n, \tag{3.1}$$

where $a_i > 0$ and $r_j \geq 0$ are constant, $i, j = 1, \ldots, n$. The interconnection matrix $(a_{ij})_{n \times n}$ is type-K monotone with the elements in the diagonal being nonnegative. In this situation, the interaction among neurons is not only excitatory but also inhibitory. The external input functions I_i are constants or periodic. The activation functions $f = (f_1, \ldots, f_n) : D \to \mathbb{R}$, where D is an open subset of $X = C([-r, 0], \mathbb{R}^n)$ with $r = \max\{r_j \mid j \in N\}$, satisfy (A1) and following property.

(A2) There exist constants L_j such that $|f_j(x) - f_j(y)| \leq L_j |x - y|$ for $j = 1, \ldots, n$.

First, we consider the case that the external input functions I_i are constants.

Theorem 3.1. *Equation* (3.1) *has an equilibrium which attracts all its solutions coming from the initial value* $\phi \geq_K 0$ *with* $\phi(0)$ *being bounded.*

Proof. From [21, Theorem 1], we deduce that (3.1) admits at least an equilibrium; that is, the equilibrium points set $\mathbf{E}$ is nonempty.

For $\phi \in X$, we define

$$F_i(\phi) = -a_i\phi_i(0) + \sum_{j=1}^{n} a_{ij} f_j\left(\phi_j(-r_j)\right) + I_i. \tag{3.2}$$

Choosing $M = \operatorname{diag}\{-\mu, \dots, -\mu\}$ with $\mu > 0$, and denoting $L = \max_{1\le j\le n} L_j$, $\alpha = \max_{1\le i,j\le n}|a_{ij}|$ and $\beta = \max_{1\le j\le n} a_j$. Since $\phi(0)$ is bounded, for ψ, $\phi \in D$ with $\psi \ge_M \phi$, there exist $\overline{m} \ge 0$ and $\underline{m} \ge 0$ with $\overline{m} \ge \underline{m}$ such that

$$\begin{gathered} \underline{m} \le \psi_j(0) - \phi_j(0) \le \overline{m}, \quad \forall i \in I, \\ -\overline{m} \le \psi_j(0) - \phi_j(0) \le -\underline{m}, \quad \forall i \in J. \end{gathered} \tag{3.3}$$

From (A2) and the definition of $\widetilde{K}_M$, if $\psi \ge_M \phi$, then

$$\begin{aligned} &F_i(\psi) - F_i(\phi) + \mu\left(\psi_i(0) - \phi_i(0)\right) \\ &\quad = \left(\mu - a_i\right)\left(\psi_i(0) - \phi_i(0)\right) + \sum_{j=1}^{n} a_{ij}\left(f_j\left(\psi_j(-r_j)\right) - f_j\left(\phi_j(-r_j)\right)\right) \\ &\quad \ge \left(\mu - a_i\right)\left(\psi_i(0) - \phi_i(0)\right) - \sum_{j=1}^{k} a_{ij} L_j\left(\psi_j(-r_j) - \phi_j(-r_j)\right) \\ &\qquad - \sum_{j=k+1}^{n} a_{ij} L_j\left(\psi_j(-r_j) - \phi_j(-r_j)\right) \\ &\quad \ge \left(\mu - a_i\right)\left(\psi_i(0) - \phi_i(0)\right) - \sum_{j=1}^{k} a_{ij} L_j e^{\mu r_j}\left(\psi_j(0) - \phi_j(0)\right) \\ &\qquad - \sum_{j=k+1}^{n} a_{ij} L_j e^{\mu r_j}\left(\psi_j(0) - \phi_j(0)\right) \\ &\quad \ge \left(\mu - \beta\frac{\overline{m}}{\underline{m}} - n\alpha L e^{\mu r}\frac{\overline{m}}{\underline{m}}\right)\underline{m}, \end{aligned} \tag{3.4}$$

for all $i \in I$. By a similar argument we have

$$F_i(\psi) - F_i(\phi) + \mu\left(\psi_i(0) - \phi_i(0)\right) \le \left(\mu - \beta\frac{\overline{m}}{\underline{m}} - n\alpha L e^{\mu r}\frac{\overline{m}}{\underline{m}}\right)\left(-\underline{m}\right) \tag{3.5}$$

for all $i \in J$. Let $H = \beta\overline{m}/\underline{m}$ and let $G = n\alpha L\overline{m}/\underline{m}$, and define $g(\mu) = \mu - H - Ge^{\mu r}$. If $r = 0$, we have $g(\mu) \ge 0$ for $\mu \ge H + G$. If $r > 0$ and $Ge^{Hr}r < 1/e$, we deduce that $g(\mu)$ reaches its positive maximum value at $\mu = H + (1/r)\ln(1/Ge^{Hr}r) > 0$. Thus, there exists a positive constant μ such that (WQM) holds; the conclusion can be obtained by Remark 2.5. $\square$

For the case of the external input functions I_i being periodic functions, we have following result.

Theorem 3.2. *For any periodic external input function* $I(t) = (I_1(t), \ldots, I_n(t))$*,* $I_i(t+\omega) = I_i(t)$*,* $i = 1, \ldots, n$*, (3.1) admits a unique periodic solution* $x^*(t)$ *and all other solutions which come from the initial value* $\phi \geq_K 0$ *with* $\phi(0)$ *being bounded converge to it as* $t \to \infty$*.*

Proof. Let $x(t) = x(t, \phi)$ be the solution of (3.1) for $t \geq 0$ with $x(s) = \phi(s)$ for $s \in [-r, 0]$. From the properties of the solution semiflow we have

$$x(t+\omega) = x(t+\omega, \phi) = x(t, x(\omega, \phi)). \tag{3.6}$$

From the proof of Theorem 3.1, we know that there exists a type-K monotone matrix such that (WQM) holds; Theorem 2.4 tells us that every orbit of (3.1) is convergent to a same equilibrium, denoted by ϕ^*, and then,

$$\lim_{n \to \infty} x(n\omega, \phi) = \phi^*. \tag{3.7}$$

We have, therefore,

$$x(\omega, \phi^*) = x\left(\omega, \lim_{n \to \infty} x(n\omega, \phi)\right) = \lim_{n \to \infty} x(\omega, x(n\omega, \phi)) = \lim_{n \to \infty} x((n+1)\omega, \phi) = \phi^*. \tag{3.8}$$

From (3.6) and (3.8) we deduce that

$$x(t+\omega, \phi^*) = x(t, x(\omega, \phi^*)) = x(t, \phi^*). \tag{3.9}$$

Therefore, $x(t, \phi^*) =: x^*(t)$ is a unique periodic solution of (3.1). Using the conclusion of Theorem 2.4 again, we have

$$\lim_{t \to \infty} x(t, \phi) = \lim_{t \to \infty} x(t, x(t, \phi)) = \lim_{t \to \infty} x(t, \phi^*). \tag{3.10}$$

Since $x^*(t)$ is a periodic solution, the proof is complete. □

Remark 3.3. Neural networks have important applications, such as to content-addressable memory [22], shortest path problem [23], and sorting problem [24]. Generally, the monotonicity is always assumed. Here, we relax the monotone condition, and hence neural networks have more extensive applications.

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