

Research Article

Higher-Order Weakly Generalized Epiderivatives and Applications to Optimality Conditions

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The notions of higher-order weakly generalized contingent epiderivative and higher-order weakly generalized adjacent epiderivative for set-valued maps are proposed. By virtue of the higher-order weakly generalized contingent (adjacent) epiderivatives, both necessary and sufficient optimality conditions are obtained for Henig efficient solutions to a set-valued optimization problem whose constraint set is determined by a set-valued map. The imposed assumptions are relaxed in comparison with those of recent results in the literature. Examples are provided to show some advantages of our notions and results.

1. Introduction

In the last several decades, several notions of derivatives (epiderivatives) for set-valued maps have been proposed and used for the formulation of optimality conditions in set-valued optimization problems. By virtue of contingent derivative (see [1]), Corley [2] investigated first-order Fritz John type necessary and sufficient optimality conditions for set-valued optimization problems. Jahn and Rauh [3] proposed the contingent epiderivative of a set-valued map and then obtained an unified necessary and sufficient optimality condition by employing the epiderivative. The essential differences between the definitions of the contingent derivative and the contingent epiderivative are that the graph is replaced by the epigraph and the derivative is single-valued. Chen and Jahn [4] introduced a notion of a generalized contingent epiderivative of a set-valued map and then established an unified necessary and sufficient conditions for a set-valued optimization problem. Lalitha and Arora [5] introduced a notion of a weak Clarke epiderivative and used it to establish optimality

criteria for a constrained set-valued optimization problem. As to other concepts of derivatives (epiderivatives) of set-valued maps and their applications, one can refer to [6–15]. Recently, second-order derivatives have also been proposed, for example, see [16, 17].

Until now, there are only a few papers to deal with higher-order optimality conditions and duality of set-valued optimization problems by virtue of the higher-order derivatives or epiderivatives introduced by the higher-order tangent sets. Since higher-order tangent sets introduced in [1], in general, are not cones and convex sets, there are some difficulties in studying higher-order optimality conditions for general set-valued optimization problems. Li et al. [18] studied some properties of higher-order tangent sets and higher-order derivatives introduced in [1], and then obtained higher-order necessary and sufficient optimality conditions for set-valued optimization problems under cone-concavity assumptions. By using these higher-order derivatives, they [19] also discussed higher-order Mond-Weir duality for constrained set-valued optimization problems based on weak efficiency. Li and Chen [20] proposed higher-order generalized contingent (adjacent) epiderivatives of set-valued maps and, then obtained higher-order Fritz John type necessary and sufficient conditions for Henig efficient solutions to a constrained set-valued optimization problem. Wang and Li [21] introduced generalized higher-order contingent (adjacent) epiderivatives of set-valued maps, and then investigated both necessary and sufficient conditions for Henig efficient solutions to set-valued optimization problems by employing the generalized higher-order contingent (adjacent) epiderivatives. Chen et al. [22] introduced higher-order weak contingent epiderivative and higher-order weak adjacent epiderivative for set-valued maps, and then investigated higher-order Mond-Weir type dual, higher-order Wolfe type dual, and higher-order optimality conditions to a constrained set-valued optimization problem by employing the higher-order weak adjacent (contingent) epiderivatives and Henig efficiency.

Motivated by the work reported in [5, 18–22], we first introduce the notions of higher-order weakly generalized contingent epiderivative, higher-order weakly generalized adjacent epiderivative for set-valued maps and generalized cone-convex set-valued maps. Second, we discuss some properties used in this paper and the existence of higher-order weakly generalized contingent epiderivative and higher-order weakly generalized adjacent epiderivative. Finally, based on higher-order weakly generalized contingent (adjacent) epiderivatives and Henig efficiency, we discuss higher-order optimality conditions to a constrained set-valued optimization problem.

The rest of the paper is organized as follows. In Section 2, we collect some of the concepts and some of their properties required for the paper. In Section 3, we define higher-order weakly generalized contingent epiderivative and higher-order weakly generalized adjacent epiderivative of set-valued maps and study existence and some properties of them. In Section 4, we establish higher-order necessary and sufficient optimality conditions to a constrained set-valued optimization problem.

2. Preliminaries and Notations

Throughout this paper, let X , Y , and Z be three real normed spaces, where the spaces Y and Z are partially ordered by nontrivial pointed closed convex cones $C \subset Y$ and $D \subset Z$ with $\text{int } C \neq \emptyset$ and $\text{int } D \neq \emptyset$, respectively. We assume that $0_X, 0_Y, 0_Z$ denote the origins of X, Y, Z , respectively, Y^* denotes the topological dual space of Y , and C^* denotes the dual cone of C , defined by $C^* = \{\varphi \in Y^* \mid \varphi(y) \geq 0, \forall y \in C\}$. Let M be a nonempty set in Y . The cone hull of M is defined by $\text{cone}(M) = \{ty \mid t \geq 0, y \in M\}$. Let E be a nonempty subset of X ,

and $F : E \rightarrow 2^Y$ and $G : E \rightarrow 2^Z$ be two given nonempty set-valued maps. The effective domain, the graph, and the epigraph of F are defined, respectively, by $\text{dom}(F) = \{x \in E \mid F(x) \neq \emptyset\}$, $\text{gph}(F) = \{(x, y) \in X \times Y \mid x \in E, y \in F(x)\}$ and $\text{epi}(F) = \{(x, y) \in X \times Y \mid x \in E, y \in F(x) + C\}$. The profile map $F_+ : E \rightarrow 2^Y$ is defined by $F_+(x) = F(x) + C$, for every $x \in \text{dom}(F)$. Let $y_0 \in Y$, $F(E) = \bigcup_{x \in E} F(x)$ and $(F - y_0)(x) = F(x) - \{y_0\} = \{y - y_0 \mid y \in F(x)\}$.

A nonempty convex subset B of the convex cone C is called a base of C , if

$$C = \text{cone}(B), \quad 0_Y \notin \text{cl}(B). \tag{2.1}$$

Suppose that C has a base B . Denote

$$\begin{aligned} C^\Delta(B) &= \{f \in C^* : \inf\{f(b) : b \in B\} > 0\}, \\ C_\epsilon(B) &= \text{cone}(B + \epsilon U) \quad \forall 0 < \epsilon < \delta, \end{aligned} \tag{2.2}$$

where $\delta = \inf\{\|b\| : b \in B\}$ and U is the closed unit ball of Y . It follows from [23] that, for $\delta > 0$, $\text{cl}(\text{int} C_\epsilon(B))$ is a closed convex pointed cone and $C \setminus \{0_Y\} \subset \text{int}(C_\epsilon(B))$ for all $0 < \epsilon < \delta$.

Definition 2.1. Let $F : E \rightarrow 2^Y$ be a set-valued map, $x_0 \in E, y_0 \in F(x_0)$.

- (i) F is said to be C -convex on a convex set E , if $\text{epi} F$ is a convex set.
- (ii) F is said to be generalized C -convex at (x_0, y_0) on E , if $\text{cone}(\text{epi} F - \{(x_0, y_0)\})$ is convex.

Obviously, if F is C -convex on convex set E , then F is a generalized C -convex at (x_0, y_0) on E . But the converse does not hold. For Example, let $E = [-1, 1]$, $F(x) = \{y \in \mathbb{R}y \geq x^{2/3}\}$, for all $x \in E, (x_0, y_0) = (0, 0) \in \text{gph}(F)$. F is generalized R_+ -convex at (x_0, y_0) on E , but F is not R_+ -convex.

Definition 2.2. An element $y \in M$ is said to be a minimal point (resp., weakly minimal point) of M if $M \cap (\{y\} - C) = \{y\}$ (resp., $M \cap (\{y\} - \text{int} C) = \emptyset$). The set of all minimal points (resp., weakly minimal point) of M is denoted by $\text{Min}_c M$ (resp., $W\text{Min}_c M$).

Suppose that m is a positive integer, X is a normed space supplied with a distance d and K is a subset of X . We denote by $d(x, K) = \inf_{y \in K} d(x, y)$ the distance from x to K , where we set $d(x, \emptyset) = +\infty$.

Definition 2.3 (see [1]). Let x belong to a subset K of a normed space X and u_1, \dots, u_{m-1} be elements of X . One says the following

- (i) The subset

$$\begin{aligned} T_K^{(m)}(x, u_1, \dots, u_{m-1}) &= \lim_{h \rightarrow 0^+} \inf \frac{K - x - hu_1 - \dots - h^{m-1}u_{m-1}}{h^m} \\ &= \left\{ y \in X \mid \lim_{h \rightarrow 0^+} d\left(y, \frac{K - x - hu_1 - \dots - h^{m-1}u_{m-1}}{h^m}\right) = 0 \right\} \end{aligned} \tag{2.3}$$

is the m th-order contingent set of K at (x, u_1, \dots, u_{m-1}) .

(ii) The subset

$$\begin{aligned} T_K^{(m)}(x, u_1, \dots, u_{m-1}) &= \liminf_{h \rightarrow 0^+} \frac{K - x - hu_1 - \dots - h^{m-1}u_{m-1}}{h^m} \\ &= \left\{ y \in X \mid \lim_{h \rightarrow 0^+} d\left(y, \frac{K - x - hu_1 - \dots - h^{m-1}u_{m-1}}{h^m}\right) = 0 \right\} \end{aligned} \quad (2.4)$$

is the m th-order adjacent set of K at (x, u_1, \dots, u_{m-1}) .

From Proposition 3.2 in [18], we have the following result.

Proposition 2.4. *If K is convex, $x \in K$, and $u_i \in X, i = 1, \dots, m - 1$, then $T_K^{b(m)}(x, u_1, \dots, u_{m-1})$ is convex.*

3. Higher-Order Weakly Generalized Epiderivatives

Definition 3.1 (see [21]). Let x belong to a subset K of X and u_1, \dots, u_{m-1} be elements of X .

(i) The subset

$$\begin{aligned} G-T_K^{(m)}(x, u_1, \dots, u_{m-1}) &= \limsup_{h \rightarrow 0^+} \frac{\text{cone}(K - x) - hu_1 - \dots - h^{m-1}u_{m-1}}{h^m} \\ &= \left\{ y \in X \mid \liminf_{h \rightarrow 0^+} d\left(y, \frac{\text{cone}(K - x) - hu_1 - \dots - h^{m-1}u_{m-1}}{h^m}\right) = 0 \right\} \end{aligned} \quad (3.1)$$

is said to be the m th-order generalized contingent set of K at (x, u_1, \dots, u_{m-1}) .

(ii) The subset

$$\begin{aligned} G-T_K^{b(m)}(x, u_1, \dots, u_{m-1}) &= \liminf_{h \rightarrow 0^+} \frac{\text{cone}(K - x) - hu_1 - \dots - h^{m-1}u_{m-1}}{h^m} \\ &= \left\{ y \in X \mid \lim_{h \rightarrow 0^+} d\left(y, \frac{\text{cone}(K - x) - hu_1 - \dots - h^{m-1}u_{m-1}}{h^m}\right) = 0 \right\} \end{aligned} \quad (3.2)$$

is said to be the m th-order generalized adjacent set of K at (x, u_1, \dots, u_{m-1}) .

Remark 3.2.

(i) The following inclusion holds:

$$G-T_K^{b(m)}(x, u_1, \dots, u_{m-1}) \subseteq G-T_K^{(m)}(x, u_1, \dots, u_{m-1}). \quad (3.3)$$

- (ii) Both $G-T_K^{(m)}(x, u_1, \dots, u_{m-1})$ and $G-T_K^{b(m)}(x, u_1, \dots, u_{m-1})$ are closed.
- (iii) If $\text{cone}(K - \{x\})$ is convex, then $G-T_K^{b(m)}(x, u_1, \dots, u_{m-1})$ is convex.
- (iv) If $\text{cone}(K - \{x\})$ is convex, and $u_1, \dots, u_{m-1} \in K$, then

$$G-T_K^{(m)}(x, u_1 - x, \dots, u_{m-1} - x) = G-T_K^{b(m)}(x, u_1 - x, \dots, u_{m-1} - x). \quad (3.4)$$

Definition 3.3. Let $(x_0, y_0) \in \text{graph}(F)$, $(u_i, v_i) \in X \times Y, i = 1, 2, \dots, m - 1$.

- (i) The m th-order weakly generalized contingent epiderivative $D_{w-g}^{(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})$ of F at (x_0, y_0) with respect to (in short, w.r.t.) vectors $(u_1, v_1), \dots, (u_{m-1}, v_{m-1})$ is the set-valued map from X to Y defined by

$$\begin{aligned} & D_{w-g}^{(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x) \\ &= \text{WMin}_c \left\{ y \in Y : (x, y) \in G-T_{\text{epi}(F)}^{(m)}(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1}) \right\}. \end{aligned} \quad (3.5)$$

- (ii) The m th-order weakly generalized adjacent epiderivative $D_{w-g}^{b(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})$ of F at (x_0, y_0) w.r.t. vectors $(u_1, v_1), \dots, (u_{m-1}, v_{m-1})$ is the set-valued map from X to Y defined by

$$\begin{aligned} & D_{w-g}^{b(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x) \\ &= \text{WMin}_c \left\{ y \in Y : (x, y) \in G-T_{\text{epi}(F)}^{b(m)}(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1}) \right\}. \end{aligned} \quad (3.6)$$

To compare our derivatives with well-known derivatives, we recall some notions.

Definition 3.4 (see [22]). Let $(x_0, y_0) \in \text{graph}(F)$, $(u_i, v_i) \in X \times Y, i = 1, 2, \dots, m - 1$.

- (i) The m th-order weakly contingent epiderivative $D_w^{(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})$ of F at (x_0, y_0) w.r.t. vectors $(u_1, v_1), \dots, (u_{m-1}, v_{m-1})$ is the set-valued map from X to Y defined by

$$\begin{aligned} & D_w^{(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x) \\ &= \text{WMin}_c \left\{ y \in Y : (x, y) \in T_{\text{epi}(F)}^{(m)}(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1}) \right\}. \end{aligned} \quad (3.7)$$

- (ii) The m th-order weakly adjacent epiderivative $D_w^{b(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})$ of F at (x_0, y_0) w.r.t. vectors $(u_1, v_1), \dots, (u_{m-1}, v_{m-1})$ is the set-valued map from X to Y defined by

$$\begin{aligned} D_w^{b(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x) \\ = \text{WMin}_c \left\{ y \in Y : (x, y) \in T_{\text{epi}(F)}^{b(m)}(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1}) \right\}. \end{aligned} \quad (3.8)$$

Definition 3.5 (see [20]). Let $(x_0, y_0) \in \text{graph}(F)$, $(u_i, v_i) \in X \times Y, i = 1, 2, \dots, m - 1$.

- (i) The m th-order generalized contingent epiderivative $D_g^{(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})$ of F at $(x_0, y_0) \in \text{gph}(F)$ w.r.t. vectors $(u_1, v_1), \dots, (u_{m-1}, v_{m-1})$ is the set-valued map from X to Y defined by

$$\begin{aligned} D_g^{(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x) \\ = \text{Min}_c \left\{ y \in Y : (x, y) \in T_{\text{epi}(F)}^{(m)}(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1}) \right\}, \\ x \in \text{dom} \left[D^{(m)}F_+(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1}) \right]. \end{aligned} \quad (3.9)$$

- (ii) The m th-order generalized adjacent epiderivative $D_g^{b(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})$ of F at $(x_0, y_0) \in \text{gph}(F)$ w.r.t. vectors $(u_1, v_1), \dots, (u_{m-1}, v_{m-1})$ is the set-valued map from X to Y defined by

$$\begin{aligned} D_g^{b(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x) \\ = \text{Min}_c \left\{ y \in Y : (x, y) \in T_{\text{epi}(F)}^{b(m)}(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1}) \right\}, \\ x \in \text{dom} \left[D^{b(m)}F_+(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1}) \right]. \end{aligned} \quad (3.10)$$

We now discuss the properties of the m th-order weakly generalized contingent epiderivative and adjacent epiderivative, for which we recall the following definitions.

Definition 3.6 (See [5, 24]).

- (i) The cone C is called Daniell, if any decreasing sequence in Y having a lower bound converges to its infimum.
- (ii) A subset M of Y is said to be minorized, if there exists a $y \in Y$ such that $M \subseteq \{y\} + C$.
- (iii) The weak domination property is said to hold for a subset H of Y if $H \subset \text{WMin}_c H + \text{int} C \cup \{0_Y\}$.

Using properties of higher-order tangent sets [1], we have the following result.

Proposition 3.7. *Let $(x_0, y_0) \in \text{gph}(F)$, $u_i \in X, v_1 \in Y$. If the sets $\{y \in Y \mid (x - x_0, y) \in G-T_{\text{epi}(F)}^{(m)}(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})\}$ and $\{y \in Y \mid (x - x_0, y) \in G-T_{\text{epi}(F)}^{b(m)}(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})\}$ fulfill the weak domination property for all $x \in E$, then for any $x \in E$,*

- (i) $D_g^{(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x - x_0) \subseteq D_{w-g}^{(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x - x_0) + C,$
- (ii) $D_w^{(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x - x_0) \subseteq D_{w-g}^{(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x - x_0) + C,$
- (iii) $D_g^{b(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x - x_0) \subseteq D_{w-g}^{b(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x - x_0) + C,$
- (iv) $D_w^{b(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x - x_0) \subseteq D_{w-g}^{b(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x - x_0) + C.$

Remark 3.8. The reverse inclusions in Proposition 3.7 may not hold. The following examples explain the case, where we only take $m = 1, 2$.

Example 3.9. Let $X = R, Y = R^2, E = R_+, C = R_+^2, F(x) = \{(y_1, y_2) \in R^2 : y_1 \geq x^{2/3}, y_2 \geq 0\}$, for all $x \in E, (x_0, y_0) = (0, (0, 0)) \in \text{gph}(F)$. Then,

$$\begin{aligned} T_{\text{epi}(F)}(x_0, y_0) &= T_{\text{epi}(F)}^b(x_0, y_0) = \{(x, y) \mid x = 0, y_1 \in R_+, y_2 \in R_+\}, \\ G-T_{\text{epi}(F)}(x_0, y_0) &= G-T_{\text{epi}(F)}^b(x_0, y_0) = R \times (R_+ \times R_+). \end{aligned} \quad (3.11)$$

Hence, for any $x \in R_+ \setminus \{0\}$,

$$\begin{aligned} D_g F(x_0, y_0)(x - x_0) &= D_g^b F(x_0, y_0)(x - x_0) = \emptyset, \\ D_w F(x_0, y_0)(x - x_0) &= D_w^b F(x_0, y_0)(x - x_0) = \emptyset, \\ D_{w-g} F(x_0, y_0)(x - x_0) &= D_{w-g}^b F(x_0, y_0)(x - x_0) \\ &= \{(y_1, y_2) \in R^2 \mid y_1 = 0, y_2 \in R_+\} \\ &\quad \cup \{(y_1, y_2) \in R^2 \mid y_1 \in R_+, y_2 = 0\}. \end{aligned} \quad (3.12)$$

Example 3.10. Suppose that $X = R, Y = R^2, E = X, C = R_+^2$. Let $F : E \rightarrow 2^{R^2}$ be a set-valued map with $F(x) = \{(y_1, y_2) \in R^2 \mid y_1 \geq x^4, y_2 \geq x^2\}$, $(x_0, y_0) = (0, (0, 0)) \in \text{gph}(F)$

and $(u, v) = (1, (0, 0))$. Then $T_{\text{epi}(F)}^{(2)}(x_0, y_0, u, v) = T_{\text{epi}(F)}^{b(2)}(x_0, y_0, u, v) = R \times (R_+ \times [1, +\infty))$,
 $G-T_{\text{epi}(F)}^{(2)}(x_0, y_0, u, v) = G-T_{\text{epi}(F)}^{b(2)}(x_0, y_0, u, v) = R \times (R_+ \times R_+)$. Therefore, for any $x \in E$,

$$\begin{aligned} D_g^{(2)}F(x_0, y_0, u, v)(x - x_0) &= D_g^{b(2)}F(x_0, y_0, u, v)(x - x_0) = \{(0, 1)\}, \\ D_w^{(2)}F(x_0, y_0, u, v)(x - x_0) &= D_w^{b(2)}F(x_0, y_0, u, v)(x - x_0) \\ &= \{(0, y_2) \mid y_2 \geq 1\} \cup \{(y_1, 1) \mid y_1 \geq 0\}, \\ D_{w-g}^{(2)}F(x_0, y_0, u, v)(x - x_0) &= D_{w-g}^{b(2)}F(x_0, y_0, u, v)(x - x_0) \\ &= \{(y_1, 0) \mid y_1 \geq 0\} \cup \{(0, y_2) \mid y_2 \geq 0\}. \end{aligned} \quad (3.13)$$

Example 3.11. Let $X = R, Y = R^2, E = X, C = R_+^2, F(x) = \{(y_1, y_2) \in R^2 : y_1 \geq x^{4/3}, y_2 \in R\}$, for all $x \in E, (x_0, y_0) = (0, (0, 0)) \in \text{gph}(F)$, and $(u, v) = (1, (0, 0))$. Then,

$$\begin{aligned} T_{\text{epi}(F)}^{(2)}(x_0, y_0, u, v) &= T_{\text{epi}(F)}^{b(2)}(x_0, y_0, u, v) = \emptyset, \\ G-T_{\text{epi}(F)}^{(2)}(x_0, y_0, u, v) &= G-T_{\text{epi}(F)}^{b(2)}(x_0, y_0, u, v) = R \times (R_+ \times R). \end{aligned} \quad (3.14)$$

Hence, for any $x \in E$,

$$\begin{aligned} D_g^{(2)}F(x_0, y_0, u, v)(x - x_0) &= D_g^{b(2)}F(x_0, y_0, u, v)(x - x_0) = \emptyset, \\ D_w^{(2)}F(x_0, y_0, u, v)(x - x_0) &= D_w^{b(2)}F(x_0, y_0, u, v)(x - x_0) = \emptyset, \\ D_{w-g}^{(2)}F(x_0, y_0, u, v)(x - x_0) &= D_{w-g}^{b(2)}F(x_0, y_0, u, v)(x - x_0) \\ &= \{(y_1, y_2) \in R^2 \mid y_1 = 0, y_2 \in R\}. \end{aligned} \quad (3.15)$$

We now discuss the existence of the m th-order weakly generalized contingent epiderivative and adjacent epiderivative.

Theorem 3.12. *Let C be a closed convex-pointed cone and let C be Daniell.*

- (i) *If the set $P(x) := \{y \in Y : (x, y) \in G-T_{\text{epi}(F)}^{(m)}(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})\}$ is minorized for every $x \in \text{dom}(P)$, then $D_{w-g}^{(m)}(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x)$ exists for all $x \in \text{dom}(P)$.*
- (ii) *If the set $Q(x) := \{y \in Y : (x, y) \in G-T_{\text{epi}(F)}^{b(m)}(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})\}$ is minorized for every $x \in \text{dom}(Q)$, then $D_{w-g}^{b(m)}(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x)$ exists for all $x \in \text{dom}(Q)$.*

Proof. From Remark 3.2 (ii), we know that m th-order generalized contingent set and m th-order generalized adjacent set are closed. Then we can prove them as the proof of Theorem 3.1 in [22]. \square

Now we discuss some crucial propositions of the m th-order weakly generalized contingent epiderivative and adjacent epiderivative.

Proposition 3.13. *Let $x_0 \in E$, $y_0 \in F(x_0)$, $(u_i, v_i) \in \{0_X\} \times C$. If the set $P(x - x_0) := \{y \in Y \mid (x - x_0, y) \in G-T_{\text{epi}(F)}^{(m)}(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})\}$ fulfills the weak domination property for all $x \in E$, then for all $x \in E$,*

$$F(x) - \{y_0\} \subset D_{w-g}^{(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x - x_0) + C. \quad (3.16)$$

Proof. It follows from Proposition 3.9 in [21] and the weak domination property of $P(x - x_0)$ that the result holds. \square

From the proof process of Proposition 3.13, we have the following result.

Corollary 3.14. *Let $x_0 \in E$, $y_0 \in F(x_0)$, $(u_i, v_i) \in \{0_X\} \times C$. If the set $P(x - x_0) := \{y \in Y \mid (x - x_0, y) \in G-T_{\text{epi}(F)}^{b(m)}(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})\}$ fulfills the weak domination property for all $x \in E$, then for all $x \in E$,*

$$F(x) - \{y_0\} \subset D_{w-g}^{b(m)}F(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})(x - x_0) + C. \quad (3.17)$$

Remark 3.15. Since the cone-convexity and cone-concavity assumptions are omitted, Proposition 3.13 improves [18, Theorem 4.1], [20, Proposition 3.1] and [22, Proposition 3.1].

Proposition 3.16. *Let E be a nonempty subset of X , $x_0 \in E$, $y_0 \in F(x_0)$, and let $u_i \in E, v_i \in F(u_i) + C, i = 1, 2, \dots, m - 1$. If F is generalized C -convex at (x_0, y_0) on E , and the set $Q(x - x_0) := \{y \in Y \mid (x - x_0, y) \in G-T_{\text{epi}(F)}^{(m)}(x_0, y_0, u_1 - x_0, v_1 - y_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0)\}$ fulfills the weak domination property for all $x \in E$, then for any $x \in E$,*

$$F(x) - \{y_0\} \subset D_{w-g}^{(m)}F(x_0, y_0, u_1 - x_0, v_1 - y_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0)(x - x_0) + C. \quad (3.18)$$

Proof. Take any $x \in E$, $y \in F(x)$ and a sequence $\{h_n\}$ with $h_n \rightarrow 0^+$. Since F is generalized C -convex at (x_0, y_0) on E , $\text{cone}(\text{epi}(F) - \{(x_0, y_0)\})$ is convex, and then

$$h_n(u_1 - x_0, v_1 - y_0) + \dots + h_n^{m-1}(u_{m-1} - x_0, v_{m-1} - y_0) \in \text{cone}(\text{epi} F - \{(x_0, y_0)\}). \quad (3.19)$$

It follows from $h_n > 0$ and $\text{cone}(\text{epi} F - \{(x_0, y_0)\})$ is a convex cone that

$$\begin{aligned} (x_n, y_n) &:= h_n(u_1 - x_0, v_1 - y_0) + \dots + h_n^{m-1}(u_{m-1} - x_0, v_{m-1} - y_0) \\ &+ h_n^m(x - x_0, y - y_0) \in \text{cone}(\text{epi} F - \{(x_0, y_0)\}). \end{aligned} \quad (3.20)$$

We obtain that

$$(x - x_0, y - y_0) = \frac{(x_n, y_n) - h_n(u_1 - x_0, v_1 - y_0) - \dots - h_n^{m-1}(u_{m-1} - x_0, v_{m-1} - y_0)}{h_n^m}, \quad (3.21)$$

which implies that

$$(x - x_0, y - y_0) \in G-T_{\text{epi}(F)}^{(m)}(x_0, y_0, u_1 - x_0, v_1 - y_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0), \quad (3.22)$$

that is, $y - y_0 \in P(x - x_0)$. By the definition of m th-order weakly generalized contingent epiderivative and the weak domination property, we have

$$P(x - x_0) \subset D_{w-g}^{(m)}(x_0, y_0, u_1 - x_0, v_1 - y_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0)(x - x_0) + C. \quad (3.23)$$

Thus,

$$F(x) - \{y_0\} \subset D_{w-g}^{(m)}F(x_0, y_0, u_1 - x_0, v_1 - y_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0)(x - x_0) + C, \quad (3.24)$$

and the proof is complete. \square

Remark 3.17. Since the cone-convexity assumptions are replaced by generalized cone-convexity assumptions, Proposition 3.16 improves [18, Theorem 4.1], [20, Proposition 3.1] and [22, Proposition 3.1]. The following example explains the case, where we only take $m = 2$.

Example 3.18. Let $X = R, Y = R^2, E = [-1, 1], C = R_+^2, F(x) = \{(y_1, y_2) \in R^2 : y_1 \geq x^{2/3}, y_2 \geq x^{2/3}\}$, for all $x \in E$ and let $(x_0, y_0) = (0, (0, 0)) \in \text{graph}(F)$.

Naturally, F is generalized C -convex at (x_0, y_0) on E , and F is not C -convex on E . Let $u = 1, v = (1, 1) \in F(1) + C$. Then

$$\begin{aligned} T_{\text{epi}(F)}^{(2)}(x_0, y_0, u - x_0, v - y_0) &= \emptyset, \\ G-T_{\text{epi}(F)}^{(2)}(x_0, y_0, u - x_0, v - y_0) &= \{(x, y) \in R \times R^2 \mid y_1 \geq x, y_2 \geq x\}. \end{aligned} \quad (3.25)$$

Hence, the conditions of Proposition 3.16 are satisfied. For any $x \in X$,

$$\begin{aligned} D_g^{(2)}F(x_0, y_0, u - x_0, v - y_0)(x - x_0) &= D_w^{(2)}F(x_0, y_0, u - x_0, v - y_0)(x - x_0) = \emptyset, \\ D_{w-g}^{(2)}F(x_0, y_0, u - x_0, v - y_0)(x - x_0) &= \{(y_1, y_2) \in R^2 \mid y_1 = x, y_2 \geq x\} \\ &\cup \{(y_1, y_2) \in R^2 \mid y_1 \geq x, y_2 = x\}. \end{aligned} \quad (3.26)$$

Thus, for any $x \in E$,

$$F(x) - \{y_0\} \subset D_{w-g}^{(m)}F(x_0, y_0, u_1 - x_0, v_1 - y_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0)(x - x_0) + C. \quad (3.27)$$

Since F is not C -convex and C -concave on E and $T_{\text{epi}(F)}^{(2)}(x_0, y_0, u - x_0, v - y_0) = \emptyset$, the assumptions of [18, Theorem 4.1], [20, Proposition 3.1] and [22, Proposition 3.1] are not satisfied. Therefore [18, Theorem 4.1], [20, Proposition 3.1] and [22, Proposition 3.1] are unusable here.

Corollary 3.19. *Let E be a nonempty convex subset of X , $x, x_0 \in E$, $y_0 \in F(x_0)$. Let $u_i \in E, v_i \in F(u_i) + C, i = 1, 2, \dots, m - 1$. If F is generalized C -convex at (x_0, y_0) on E , and the set $\{y \in Y \mid (x - x_0, y) \in G-T_{\text{epi}(F)}^{b(m)}(x_0, y_0, u_1 - x_0, v_1 - y_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0)\}$ fulfills the weak domination property for all $x \in E$, then*

$$F(x) - \{y_0\} \subset D_{w-g}^{b(m)}F(x_0, y_0, u_1 - x_0, v_1 - y_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0)(x - x_0) + C. \quad (3.28)$$

4. Higher-Order Optimality Conditions

In this section, we discuss the higher-order optimality Conditions of Henig efficient solutions for constrained set-valued optimization problems. The notation $(F, G)(x)$ is used to denote $F(x) \times G(x)$. Firstly, we recall the definition of interior tangent cone of a set and state a result regarding it from [16].

The interior tangent cone of K at x_0 is defined as

$$IT_K(x_0) = \{u \in X \mid \exists \lambda > 0, \forall t \in (0, \lambda), \forall u' \in B_X(u, \lambda), x_0 + tu' \in K\}, \quad (4.1)$$

where $B_X(u, \lambda)$ stands for the closed ball centered at $u \in X$ and of radius λ .

Lemma 4.1 (see [13]). *If $K \subset X$ is convex, $x_0 \in K$, and $\text{int } K \neq \emptyset$, then*

$$IT_{\text{int } K}(x_0) = \text{intcone}(K - \{x_0\}). \quad (4.2)$$

Consider the following set-valued optimization problem:

$$(SP) \begin{cases} \min & F(x), \\ \text{s.t.} & G(x) \cap (-D) \neq \emptyset, x \in E, \end{cases} \quad (4.3)$$

let $K := \{x \in E \mid G(x) \cap (-D) \neq \emptyset\}$ and $F(K) := \bigcup_{x \in K} f(x)$. Let $x_0 \in K, y_0 \in F(x_0)$, (x_0, y_0) is said to be a Henig efficient solution of problem (SP), if for some $\epsilon \in (0, \delta)$,

$$(F(K) - \{y_0\}) \cap (-\text{int}(C_\epsilon(B))) = \emptyset. \quad (4.4)$$

Lemma 4.2 (see [13]). *Let $x_0 \in K, y_0 \in F(x_0)$. If there exists $\phi \in C^\Delta(B)$ such that*

$$\phi(y) \geq \phi(y_0), \quad \forall y \in F(K), \quad (4.5)$$

then (x_0, y_0) is a Henig efficient solution of problem (SP).

Theorem 4.3. Suppose that C has a base B . Let $(u_i, v_i, w_i) \in X \times (-C) \times (-D)$, $i = 1, 2, \dots, m-1$, $y_0 \in F(x_0)$, $z_0 \in G(x_0) \cap (-D)$, $\delta = \inf\{\|b\| : b \in B\}$. If (x_0, y_0) is a Henig efficient solution of (SP) , then there exists $\epsilon \in (0, \delta)$ such that

$$\begin{aligned} & \left[D_{w-g}^{(m)}(F, G)(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \dots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)(x) \right. \\ & \left. + \{(0_Y, z_0)\} \right] \cap -\text{int}(C_\epsilon(B) \times D) = \emptyset. \end{aligned} \quad (4.6)$$

for all $x \in X$.

Proof. Since (x_0, y_0) is a Henig efficient solution of (SP) , there exists an $\epsilon \in (0, \delta)$ such that $(F(K) - \{y_0\}) \cap -\text{int}(C_\epsilon(B)) = \emptyset$. Then

$$\text{cone}(F(K) + C - \{y_0\}) \cap -\text{int}(C_\epsilon(B)) = \emptyset. \quad (4.7)$$

If $D_{w-g}^{(m)}(F, G)(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \dots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)(x) = \emptyset$ for some $x \in X$, then the result (4.6) holds trivially. So we next prove that for any $x \in \Omega := \text{dom}[D_{w-g}^{(m)}(F, G)(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \dots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)]$, the result (4.6) holds.

Assume that (4.6) does not hold. Then there exist $\bar{x} \in \Omega$ and $(\bar{y}, \bar{z}) \in X \times Y$ such that

$$(\bar{y}, \bar{z}) \in D_{w-g}^{(m)}(F, G)(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \dots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)(\bar{x}), \quad (4.8)$$

$$(\bar{y}, \bar{z}) + (0_Y, z_0) \in -(\text{int}(C_\epsilon(B)) \times \text{int} D). \quad (4.9)$$

It follows from (4.8) and the definition of m th-order weakly generalized contingent epiderivative that there exist sequences $\{h_n\}$ with $h_n \rightarrow 0^+$ and $\{(x_n, y_n, z_n)\} \subseteq \text{cone}(\text{epi}(F, G) - \{(x_0, y_0, z_0)\})$ such that

$$\frac{(x_n, y_n, z_n) - h_n(u_1, v_1, w_1 + z_0) - \dots - h_n^{m-1}(u_{m-1}, v_{m-1}, w_{m-1} + z_0)}{h_n^m} \longrightarrow (\bar{x}, \bar{y}, \bar{z}). \quad (4.10)$$

From (4.9) and (4.10), there exists a sufficiently large N_1 such that

$$y_n - h_n v_1 - \dots - h_n^{m-1} v_{m-1} \in -\text{int}(C_\epsilon(B)), \quad \text{for } n > N_1, \quad (4.11)$$

$$\begin{aligned} \bar{z}_n &:= \frac{z_n - h_n(w_1 + z_0) - \dots - h_n^{m-1}(w_{m-1} + z_0)}{h_n^m} \\ &= \frac{h_n + \dots + h_n^{m-1}}{h_n^m} \left(\frac{z_n - h_n w_1 - \dots - h_n^{m-1} w_{m-1}}{h_n + \dots + h_n^{m-1}} - z_0 \right) \longrightarrow \bar{z} \end{aligned} \quad (4.12)$$

$$\in -(\text{int} D + z_0) \subset -\text{intcone}(D + z_0).$$

Since $v_1, \dots, v_{m-1} \in -C$, $h_n > 0$ and C is a convex cone,

$$h_n v_1 + \dots + h_n^{m-1} v_{m-1} \in -C. \quad (4.13)$$

By (4.11) and (4.13), we get

$$y_n \in -\text{int}(C_\epsilon(B)), \quad \text{for } n > N_1. \quad (4.14)$$

According to (4.12) and Lemma 4.1, we obtain $-\bar{z} \in IT_{\text{int}D}(-z_0)$. Then, it follows from the definitions of $IT_{\text{int}D}(-z_0)$ that $\exists \lambda > 0$, for all $t \in (0, \lambda)$, for all $u' \in B_Z(-\bar{z}, \lambda)$, $-z_0 + tu' \in \text{int}D$. Since $h_n \rightarrow 0^+$ and (4.12), there exists a sufficiently large N_2 such that

$$\begin{aligned} \frac{h_n^m}{h_n + \dots + h_n^{m-1}} &\in (0, \lambda), \quad \text{for } n > N_2, \\ -z_0 + \frac{h_n^m}{h_n + \dots + h_n^{m-1}}(-\bar{z}_n) &\in \text{int}D, \quad \text{for } n > N_2, \end{aligned} \quad (4.15)$$

that is,

$$\frac{z_n - h_n w_1 - \dots - h_n^{m-1} w_{m-1}}{h_n + \dots + h_n^{m-1}} \in -\text{int}D, \quad \text{for } n > N_2. \quad (4.16)$$

It follows from $h_n > 0, w_1, \dots, w_{m-1} \in -D$ and D is a convex cone that

$$z_n \in -\text{int}D, \quad \text{for } n > N_2. \quad (4.17)$$

Since $z_n \in \text{cone}(G(x_n) + D - \{z_0\})$, there exist $\lambda_n \geq 0, \tilde{z}_n \in G(x_n)$ and $d_n \in D$ such that $z_n = \lambda_n(\tilde{z}_n + d_n - z_0)$. It follows from (4.17) that $\tilde{z}_n \in G(x_n) \cap (-D)$, for $n > N_2$, and then

$$x_n \in K, \quad \text{for } n > N_2. \quad (4.18)$$

Then it follows from (4.14) that

$$y_n \in \text{cone}(F(K) + C - \{y_0\}) \cap -\text{int}(C_\epsilon(B)), \quad \text{for } n > \max\{N_1, N_2\}, \quad (4.19)$$

which contradicts (4.7). Thus (4.6) holds and the proof is complete. \square

From $D + \text{int}D \subseteq \text{int}D, C + \text{int}C_\epsilon(B) \subseteq \text{int}C_\epsilon(B)$ and Theorem 4.3, we have the following corollary.

Corollary 4.4. *Suppose that C has a base B . Let $(u_i, v_i, w_i) \in X \times (-C) \times (-D), i = 1, 2, \dots, m-1, y_0 \in F(x_0), z_0 \in G(x_0) \cap (-D), \delta = \inf\{\|b\| : b \in B\}$. If (x_0, y_0) is a Henig efficient solution of (SP), then there exists $\epsilon \in (0, \delta)$ such that*

$$\begin{aligned} &\left[D_{w-g}^{(m)}(F, G)(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \dots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)(X) \right. \\ &\quad \left. + C \times D + \{(0_Y, z_0)\} \right] \cap (-\text{int}(C_\epsilon(B) \times D)) = \emptyset. \end{aligned} \quad (4.20)$$

Theorem 4.5. Suppose that C has a base B , $\delta = \inf\{\|b\| : b \in B\}$. Let $y_0 \in F(x_0)$, $(u_i, v_i - y_0, w_i) \in X \times (-C) \times (-D)$, $i = 1, 2, \dots, m-1$. If (x_0, y_0) is a Henig efficient solution of (SP), then for some $\epsilon \in (0, \delta)$ and for any $z_0 \in G(x_0) \cap (-D)$,

$$\begin{aligned} & \left[D_{w-g}^{(m)}(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(X) \right. \\ & \left. + C \times D + \{(0_Y, z_0)\} \right] \cap -\text{int}(C_\epsilon(B) \times D) = \emptyset. \end{aligned} \quad (4.21)$$

Proof. If $D_{w-g}^{(m)}(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(x) = \emptyset$ for some $x \in X$, then the result holds trivially. So we suppose that for any $x \in \Omega := \text{dom}[D_{w-g}^{(m)}(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)]$. Then the proof of the fact follows on the lines of [20, Theorem 4.1] by using m th-order weakly generalized contingent epiderivative instead of m th-order generalized adjacent epiderivative. \square

Theorem 4.6. Let $(u_i, v_i, w_i) \in X \times (-C) \times (-D)$, $i = 1, 2, \dots, m-1$, $y_0 \in F(x_0)$, $z_0 \in G(x_0) \cap (-D)$, let B be base of C and $\delta = \inf\{\|b\| : b \in B\}$. Suppose that the following conditions are satisfied:

- (i) (F, G) is generalized (C, D) -convex at (x_0, y_0, z_0) on nonempty set E ,
- (ii) the pair (x_0, y_0) is a Henig efficient solution of (SP),
- (iii) $P(x) := \{(y, z) \in (Y, Z) \mid (x - x_0, y, z) \in G-T_{\text{epi}(F, G)}^{(m)}(x_0, y_0, u_1, v_1, w_1 + z_0, \dots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)\}$ fulfills the weak domination property for all $x \in E$, $(0_Y, 0_Z) \in [D_{w-g}^{(m)}(F, G)(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \dots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)](0_X)$.

Then there exist $\phi \in C^\Delta(B)$ and $\psi \in D^*$ such that

$$\emptyset(y) + \psi(z) \geq 0, \quad (z_0) = 0, \quad (4.22)$$

for all $(y, z) \in \Delta(x) := D_{w-g}^{(m)}(F, G)(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \dots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)(x)$ and $x \in \Omega := \text{dom } \Delta$.

Proof. Let $z_0 \in G(x_0) \cap (-D)$. Define

$$M = \bigcup_{x \in \Omega} \Delta(x) + C \times D + (0_Y, z_0). \quad (4.23)$$

By the similar line of proof for convexity of M in Theorem 5.1 in [20], we obtain that M is a convex set. It follows from Corollary 4.4 that

$$M \cap (-(\text{int}(\text{cone}(\epsilon U + B)) \times \text{int } D)) = \emptyset. \quad (4.24)$$

By the separation theorem of convex sets, there exist $\phi \in Y^*$ and $\psi \in Z^*$, not both zero functionals such that for all $(\bar{y}, \bar{z}) \in M$, $(y, z) \in -(\text{int}(\text{cone}(\epsilon U + B)) \times \text{int } D)$, we have

$$\phi(\bar{y}) + \psi(\bar{z}) \geq \phi(y) + \psi(z). \quad (4.25)$$

It follows from (4.25) that

$$\phi(y) \leq \psi(z), \quad \forall (y, z) \in -\text{int}(\text{cone}(\epsilon U + B)) \times \text{int } D, \quad (4.26)$$

$$\phi(\bar{y}) + \psi(\bar{z}) \geq 0, \quad \forall (\bar{y}, \bar{z}) \in M. \quad (4.27)$$

Whence

$$\phi(y) + \psi(z) \geq 0, \quad (4.28)$$

for all $(y, z) \in \Delta(x)$ and $x \in \Omega$.

From (4.26), we obtain that ψ is bounded below on the $\text{int } D$. Then $\psi(z) \geq 0$, for all $z \in \text{int } D$. Naturally $\psi \in D^*$ and

$$\phi(b) \geq \phi(u), \quad \forall b \in B, u \in \epsilon U. \quad (4.29)$$

Since U is symmetry, there exists a $u_0 \in \epsilon U$ such that $\phi(u_0) = t > 0$. Then $\phi \in C^\Delta(B)$.

From (4.27) and condition (iii), we have $\psi(z_0) \geq 0$. Since $z_0 \in -D$ and $\psi \in D^*$, $\psi(z_0) \leq 0$. So

$$\psi(z_0) = 0, \quad (4.30)$$

and the proof is complete. \square

Remark 4.7. We notice that a Kuhn-Tucker type necessary optimality condition in Theorem 4.6 is obtained under weaker assumptions than those assumed in [20, Theorem 5.1] and [22, Theorem 6.1]. The following example explains the case, where we only take $m = 2$.

Example 4.8. Suppose that $X = Y = Z = R, E = X, C = D = R_+, B = \{2\}$. Let $F : E \rightarrow 2^Y$ be a set-valued map with

$$F(x) = \{y \in R : y \geq x^{2/3}\}, \quad x \in E, \quad (4.31)$$

and $G : E \rightarrow Z$ be a set-valued map with

$$G(x) = \{z \in R : z \geq x^{4/5}\}, \quad x \in E. \quad (4.32)$$

Let $x_0 = 0 \in E, y_0 = 0 \in F(x_0), z_0 = 0 \in G(x_0)$. Naturally, (F, G) is generalized $C \times D$ -convex at (x_0, y_0, z_0) on E . Consider the following constrained set-valued optimization problem (4.6):

$$\begin{aligned} \min, & \quad F(x), \\ \text{s.t.} & \quad x \in E, G(x) \cap (-D) \neq \emptyset. \end{aligned} \quad (4.33)$$

By definition, (x_0, y_0) is a Henig efficient solution of (4.6). Take $(u_1, v_1, w_1) = (-1, 0, 0) \in \text{gph}(D_{w-g}(F, G)(x_0, y_0, z_0))$. By directly calculating, we have

$$\begin{aligned} G-T_{\text{epi}(F,G)}^{(2)}(x_0, y_0, z_0, u_1, v_1, w_1 + z_0) &= \left\{ (x, y, z) \in E \times R^2 : y \geq 0, z \geq 0 \right\}, \\ (0_Y, 0_Z) \in D_{w-g}^{(2)}(F, G)(x_0, y_0, z_0, u_1, v_1, w_1 + z_0)(0_X). \end{aligned} \quad (4.34)$$

Then, the conditions of Theorem 4.6 are satisfied, and

$$\begin{aligned} D_{w-g}^{(2)}(F, G)(x_0, y_0, z_0, u_1, v_1, w_1 + z_0)(x) \\ = \left\{ (y, z) \in R^2 : y = 0, z \geq 0 \right\} \cup \left\{ (y, z) \in R^2 : y \geq 0, z = 0 \right\}. \end{aligned} \quad (4.35)$$

Simultaneously, take $\phi = 1 \in C^\Delta(B)$ and $\varphi = 1 \in D^*$. Obviously, the 2nd-order necessary optimality condition of Theorem 4.6 holds.

Since neither F nor G is R_+ -convex on nonempty convex set E and $T_{\text{epi}(F,G)}^{(2)}(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0) = \emptyset$, the assumptions of [20, Theorem 5.1] and [22, Theorem 6.1] are not satisfied. Therefore [20, Theorem 5.1] and [22, Theorem 6.1] are unusable here.

As a direct consequence of Theorem 4.6, we get the following corollary.

Corollary 4.9 (See [21]). *Let $(u_i, v_i, w_i) \in X \times (-C) \times (-D), i = 1, 2, \dots, m-1, y_0 \in F(x_0)$, let B be base of C and $\delta = \inf\{\|b\| : b \in B\}$. Suppose that the following conditions are satisfied:*

- (i) F and G are C -convex and D -convex on nonempty convex set E , respectively,
- (ii) the pair (x_0, y_0) is a Henig efficient solution of (SP),
- (iii) $(0_Y, 0_Z) \in [G - D^{(m)}(F, G)(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \dots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)](0_X)$, for any $z_0 \in G(x_0) \cap (-D)$.

Then for any $z_0 \in G(x_0) \cap (-D)$, there exist $\phi \in C^\Delta(B)$ and $\varphi \in D^*$ such that

$$\phi(y) + \varphi(z) \geq 0, \quad \varphi(z_0) = 0, \quad (4.36)$$

for all $(y, z) \in G-D^{(m)}(F, G)(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \dots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)(x)$ and $x \in \Omega : = \text{dom } G-D^{(m)}(F, G)(x_0, y_0, z_0, u_1, v_1, w_1 + z_0, \dots, u_{m-1}, v_{m-1}, w_{m-1} + z_0)$.

Theorem 4.10. *Suppose that C has a base B . Let $x, x_0 \in E, y_0 \in F(x_0), z_0 \in G(x_0) \cap (-D)$. Suppose that the following conditions are satisfied:*

- (i) $u_i \in E, v_i \in F(u_i) + C, w_i \in G(u_i) + D, i = 1, 2, \dots, m-1$,
- (ii) (F, G) is generalized $C \times D$ -convex at (x_0, y_0, z_0) on E ,
- (iii) if the set $\{(y, z) \in (Y, Z) \mid (x - x_0, y, z) \in G-T_{\text{epi}(F,G)}^{(m)}(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)\}$ fulfills the weak domination property for all $x \in E$,

(iv) there exist $\phi \in C^\Delta(B)$ and $\psi \in D^*$ such that

$$\phi(y) + \psi(z) \geq 0, \quad \psi(z_0) = 0, \quad (4.37)$$

for all $(y, z) \in D_{w-g}^{(m)}(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \dots, u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(x - x_0)$, for all $x \in K$.

Then the pair (x_0, y_0) is a Henig efficient solution of (SP).

Proof. It follows from Proposition 3.16 that

$$\begin{aligned} (y - y_0, z - z_0) \in D_{w-g}^{(m)}(F, G)(x_0, y_0, z_0, u_1 - x_0, v_1 - y_0, w_1 - z_0, \dots, \\ u_{m-1} - x_0, v_{m-1} - y_0, w_{m-1} - z_0)(x - x_0) + C \times D, \end{aligned} \quad (4.38)$$

for all $(y, z) \in (F, G)(x)$, $x \in K$. Then,

$$\phi(y - y_0) + \psi(z - z_0) \geq 0, \quad \forall (y, z) \in (F, G)(x). \quad (4.39)$$

Thus, for any $x \in K$, there exists a $\bar{z} \in G(x)$ with $\bar{z} \in -D$ such that $\psi(\bar{z}) \leq 0$. It follows from $\psi(z_0) = 0$ and (4.39) that

$$\phi(y) \geq \phi(y_0), \quad \forall y \in F(K). \quad (4.40)$$

Whence it follows from Lemma 4.2 that (x_0, y_0) is a Henig efficient solution of (SP). \square

Similarly as in the proof of Theorem 4.10, it follows from Proposition 3.13 that we have the following result.

Theorem 4.11. Suppose that C has a base B . Let $x, x_0 \in E$, $y_0 \in F(x_0)$, $z_0 \in G(x_0) \cap (-D)$. Suppose that the following conditions are satisfied:

- (i) $(u_i, v_i, w_i) \in \{0_X\} \times C \times D, i = 1, 2, \dots, m - 1$,
- (ii) If the set $\{(y, z) \in (Y, Z) \mid (x - x_0, y, z) \in G-T_{\text{epi}(F, G)}^{(m)}(x_0, y_0, u_1, v_1, \dots, u_{m-1}, v_{m-1})\}$ fulfills the weak domination property for all $x \in E$,
- (iii) there exist $\phi \in C^\Delta(B)$ and $\psi \in D^*$ such that

$$\phi(y) + \psi(z) \geq 0, \quad \psi(z_0) = 0, \quad (4.41)$$

for all $(y, z) \in D_{w-g}^{(m)}(F, G)(x_0, y_0, z_0, u_1, v_1, w_1, \dots, u_{m-1}, v_{m-1}, w_{m-1})(x - x_0)$, for all $x \in K$.

Then the pair (x_0, y_0) is a Henig efficient solution of (CP).

Remark 4.12.

- (i) Since Theorem 4.11 does not involve the convexity, it improves [20, Theorem 5.4] and [22, Theorem 6.2].

- (ii) If we use m th-order weakly generalized adjacent epiderivative instead of the m th-order weak generalized contingent epiderivative in Theorems 4.3, 4.5, 4.6, 4.10 and 4.11, then the corresponding results for m th-order weakly generalized adjacent epiderivative still hold.

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