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Research Article

Soft Rough Approximation Operators and Related Results

Zhaowen Li,[1] Bin Qin,[2] and Zhangyong Cai[3]

[1] *School of Science, Guangxi University for Nationalities, Nanning, Guangxi 530006, China*
[2] *School of Information and Statistics, Guangxi College of Finance and Economics, Nanning, Guangxi 530003, China*
[3] *Department of Mathematics, Guangxi Teachers College, Nanning, Guangxi 530023, China*

Correspondence should be addressed to Zhaowen Li; lizhaowen8846@126.com

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Soft set theory is a newly emerging tool to deal with uncertain problems. Based on soft sets, soft rough approximation operators are introduced, and soft rough sets are defined by using soft rough approximation operators. Soft rough sets, which could provide a better approximation than rough sets do, can be seen as a generalized rough set model. This paper is devoted to investigating soft rough approximation operations and relationships among soft sets, soft rough sets, and topologies. We consider four pairs of soft rough approximation operators and give their properties. Four sorts of soft rough sets are investigated, and their related properties are given. We show that Pawlak's rough set model can be viewed as a special case of soft rough sets, obtain the structure of soft rough sets, give the structure of topologies induced by a soft set, and reveal that every topological space on the initial universe is a soft approximating space.

1. Introduction

Most of traditional methods for formal modeling, reasoning, and computing are crisp, deterministic, and precise in character. However, many practical problems within fields such as economics, engineering, environmental science, medical science, and social sciences involve data that contain uncertainties. We cannot use traditional methods because of various types of uncertainties present in these problems.

There are several theories: probability theory, fuzzy set theory, theory of interval mathematics, and rough set theory [1], which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties (see [2]). For example, theory of probabilities can deal only with stochastically stable phenomena. To overcome these kinds of difficulties, Molodtsov [2] proposed a completely new approach, which is called soft set theory, for modeling uncertainty.

Presently, works on soft set theory are progressing rapidly. Maji et al. [3–5] further studied soft set theory, used this theory to solve some decision making problems, and devoted fuzzy soft sets combining soft sets with fuzzy sets. Roy et al. [6] presented a fuzzy soft set theoretic approach towards decision making problems. Jiang et al. [7] extended soft sets with description logics. Aktaş et al. [8] defined soft groups. Feng et al. [9, 10] investigated relationships among soft sets, rough sets, and fuzzy sets. Shabir et al. [11] investigated soft topological spaces. Ge et al. [12] discussed relationships between soft sets and topological spaces.

The purpose of this paper is to investigate soft rough approximation operators and relationships among soft sets, soft rough sets, and topologies.

The remaining part of this paper is organized as follows. In Section 2, we recall some basic concepts of rough sets and soft sets. In Section 3, we consider four pairs of soft rough approximation operators and give their properties. Four sorts of soft rough sets are introduced or investigated, and the fact that Pawlak's rough set model can be viewed as a special case of soft rough sets is proved. In Section 4, we investigate the relationships between soft sets and topologies, obtain the structure of topologies induced by a soft set, and reveal that every topological space on the initial universe is a soft approximating space. In Section 5, we give the related properties of soft rough sets and obtain the structure of soft rough sets. In Section 6, we prove that there exists a one-to-one correspondence between the set of all soft sets and the set of all formal contexts. Conclusion is in Section 7.

2. Overview of Rough Sets and Soft Sets

In this section, we recall some basic concepts about rough sets and soft sets.

Throughout this paper, U denotes initial universe, E denotes the set of all possible parameters, and 2^U denotes the family of all subsets of U. We only consider the case where both U and E are nonempty finite sets.

2.1. Rough Sets. Rough set theory was initiated by [1] for dealing with vagueness and granularity in information systems. This theory handles the approximation of an arbitrary subset of a universe by two definable or observable subsets called lower and upper approximations. It has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, expert systems, and many other fields (see [1, 13]).

Let R be an equivalence relation on U. The pair (U, R) is called a Pawlak approximation space. The equivalence relation R is often called an indiscernibility relation. Using the indiscernibility relation R, one can define the following two rough approximations:

$$R_*(X) = \{x \in U : [x]_R \subseteq X\},$$
$$R^*(X) = \{x \in U : [x]_R \cap X \neq \emptyset\}. \quad (1)$$

$R_*(X)$ and $R^*(X)$ are called the Pawlak lower approximation and the Pawlak upper approximation of X, respectively. In general, we refer to R_* and R^* as Pawlak rough approximation operators and $R_*(X)$ and $R^*(X)$ as Pawlak rough approximations of X.

The Pawlak boundary region of X is defined by the difference between these Pawlak rough approximations; that is, $\mathrm{Bnd}_R(X) = R^*(X) - R_*(X)$. It can easily be seen that $R_*(X) \subseteq X \subseteq R^*(X)$.

A set is Pawlak rough if its boundary region is not empty; otherwise, the set is crisp. Thus, X is Pawlak rough if $R_*(X) \neq R^*(X)$.

We may relax equivalence relations so that rough set theory is able to solve more complicated problems in practice. The classical rough set theory based on equivalence relations has been extended to binary relations [14].

Definition 1 (see [14]). Let R be a binary relation on U. The pair (U, R) is called an approximation space. Based on the approximation space (U, R), we define a pair of operations $\underline{R}$, $\overline{R} : 2^U \to 2^U$ as follows:

$$\underline{R}(X) = \{x \in U : R(x) \subseteq X\},$$
$$\overline{R}(X) = \{x \in U : R(x) \cap X \neq \emptyset\}, \quad (2)$$

where $X \in 2^U$ and $R(x) = \{y \in U : xRy\}$.

$\underline{R}(X)$ and $\overline{R}(X)$ are called the lower approximation and the upper approximation of X, respectively. In general, we refer to $\underline{R}$ and $\overline{R}$ as rough approximation operators and $\underline{R}(X)$ and $\overline{R}(X)$ as rough approximations of X.

X is called a definable set if $\underline{R}(X) = \overline{R}(X)$; X is called a rough set if $\underline{R}(X) \neq \overline{R}(X)$.

2.2. Soft Sets

Definition 2 (see [2]). Let A be a nonempty subset of E. A pair (f, A) is called a soft set over U, if f is a mapping given by $f : A \to 2^U$. We denote (f, A) by f_A.

In other words, a soft set over U is a parametrized family of subsets of the universe U. For $e \in A$, $f(e)$ may be considered as the set of e-approximate elements of the soft set f_A.

Definition 3 (see [3]). Let f_A and g_B be two soft sets over U.

(1) f_A is called a soft subset of g_B, if $A \subseteq B$ and $f(e) = g(e)$ for each $e \in A$. We denote it by $f_A \tilde{\subseteq} g_B$.

(2) f_A is called a soft super set of g_B, if $g_B \tilde{\subseteq} f_A$. We denote it by $f_A \tilde{\supseteq} g_B$.

Definition 4 (see [3]). Let f_A and g_B be two soft sets over U. f_A and g_B are called soft equal, if $A = B$ and $f(e) = g(e)$ for each $e \in A$. We denote it by $f_A = g_B$.

Obviously, $f_A = g_B$ if and only if $f_A \tilde{\subseteq} g_B$ and $f_A \tilde{\supseteq} g_B$.

Definition 5 (see [10, 12]). Let f_A be a soft set over U.

(1) f_A is called full, if $\bigcup_{a \in A} f(a) = U$.

(2) f_A is called partition, if $\{f(a) : a \in A\}$ forms a partition of U.

Obviously, every partition soft set is full.

Definition 6. Let f_A be a soft set over U.

(1) f_A is called keeping intersection, if for any $a, b \in A$, there exists $c \in A$ such that $f(a) \cap f(b) = f(c)$.

(2) f_A is called keeping union, if for any $a, b \in A$, there exists $c \in A$ such that $f(a) \cup f(b) = f(c)$.

(3) f_A is called topological, if $\{f(a) : a \in A\}$ is a topology on U.

Obviously, every topological soft set is full, keeping intersection and keeping union, and f_A is keeping intersection (resp., keeping union) if and only if for any $A' \subseteq A$, there exists $a' \in A$ such that $\bigcap_{a \in A'} f(a) = f(a')$ (resp., $\bigcup_{a \in A'} f(a) = f(a')$).

Example 7. Let $U = \{h_1, h_2, h_3, h_4, h_5\}$, $A = \{a_1, a_2, a_3, a_4\}$, and let f_A be a soft set over U, defined as follows:

$$\begin{aligned} f(a_1) &= \{h_1, h_2, h_5\}, \\ f(a_2) &= \emptyset, \\ f(a_3) &= \{h_3\}, \\ f(a_4) &= \{h_3, h_4\}. \end{aligned} \tag{3}$$

Obviously, f_A is not partition. We have

$$\begin{aligned} & f(a_3) \cap f(a_4) = \{h_3\} = f(a_3), \\ & f(a_1) \cap f(a_2) = f(a_1) \cap f(a_3) = f(a_1) \cap f(a_4) \\ & \qquad = f(a_2) \cap f(a_3) = f(a_2) \cap f(a_4) \\ & \qquad = \emptyset = f(a_2). \end{aligned} \tag{4}$$

Then, f_A is full and keeping intersection. But

$$f(a_1) \cup f(a_3) = \{h_1, h_2, h_3, h_5\} \neq f(a) \quad (\forall a \in A). \tag{5}$$

Thus, f_A is not keeping union.

Example 8. Let $U = \{h_1, h_2, h_3, h_4, h_5\}$, $A = \{a_1, a_2, a_3, a_4\}$, and let f_A be a soft set over U, defined as follows:

$$\begin{aligned} f(a_1) &= \{h_1\}, \\ f(a_2) &= \{h_1, h_2\}, \\ f(a_3) &= \{h_1, h_2, h_3\}, \\ f(a_4) &= \{h_1, h_2, h_3, h_4\}. \end{aligned} \tag{6}$$

Then, f_A is keeping intersection and keeping union. But f_A is not full.

Example 9. Let $U = \{h_1, h_2, h_3, h_4, h_5\}$, $A = \{a_1, a_2, a_3, a_4\}$, and let f_A be a soft set over U, defined as follows:

$$\begin{aligned} f(a_1) &= \{h_1\}, \\ f(a_2) &= \{h_2\}, \\ f(a_3) &= \{h_1, h_2\}, \\ f(a_4) &= U. \end{aligned} \tag{7}$$

Then, f_A is full and keeping union. But f_A is neither keeping intersection nor partition.

Example 10. Let $U = \{h_1, h_2, h_3, h_4, h_5\}$, $A = \{a_1, a_2, a_3, a_4\}$, and let f_A be a soft set over U, defined as follows:

$$\begin{aligned} f(a_1) &= \{h_1, h_2\}, \\ f(a_2) &= \{h_5\}, \\ f(a_3) &= \{h_3\}, \\ f(a_4) &= \{h_4\}. \end{aligned} \tag{8}$$

Obviously, f_A is partition. But

$$\begin{aligned} & f(a_2) \cap f(a_3) = \emptyset \neq f(a) \quad (\forall a \in A), \\ & f(a_1) \cup f(a_2) = \{h_1, h_2, h_5\} \neq f(a) \quad (\forall a \in A). \end{aligned} \tag{9}$$

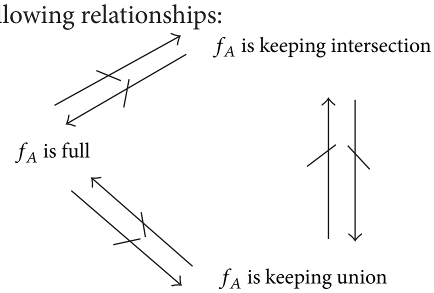
Thus, f_A is neither keeping intersection nor keeping union.

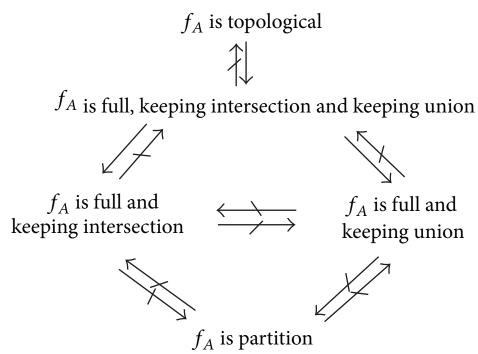
Example 11. Let $U = \{h_1, h_2, h_3, h_4, h_5\}$, $A = \{a_1, a_2, a_3, a_4\}$ and let f_A be a soft set over U, defined as follows

$$\begin{aligned} f(a_1) &= \{h_1\}, \\ f(a_2) &= \{h_1, h_4\}, \\ f(a_3) &= \{h_1, h_3, h_4\}, \\ f(a_4) &= X. \end{aligned} \tag{10}$$

Obviously, f_A is full, keeping intersection and keeping union. But f_A is not topological.

From Examples 7, 8, 9, and 10 and 11, we have the following relationships:





3. Soft Rough Approximation Operators and Soft Rough Sets

Soft rough sets, which could provide a better approximation than rough sets do, can be seen as a generalized rough set model (see Example 4.6 in [10]), and defining soft rough sets and some related concepts needs using soft rough approximation operators based on soft sets. Thus, soft rough approximation operators deserve further research.

In this section, we consider a pair of soft rough approximation operators which are presented by Feng et al. in [9, 10], proposing three pairs of soft rough approximation operators and giving their properties. Four sorts of soft rough sets are defined by using four pairs of soft rough approximation operators.

3.1. Soft Rough Approximation Operators $\underline{apr}_P$ *and* $\overline{apr}_P$

Definition 12 (see [9, 10]). Let f_A be a soft set over U. Then, the pair $P = (U, f_A)$ is called a soft approximation space. We define a pair of operators $\underline{\text{apr}}_P, \overline{\text{apr}}_P : 2^U \to 2^U$ as follows:

$$\begin{aligned} \underline{\text{apr}}_P(X) &= \{u \in U : \exists a \in A, \text{ s.t. } u \in f(a) \subseteq X\}, \\ \overline{\text{apr}}_P(X) &= \{u \in U : \exists a \in A, \\ &\qquad \text{s.t. } u \in f(a),\ f(a) \cap X \neq \emptyset\}. \end{aligned} \tag{11}$$

$\underline{\text{apr}}_P$ and $\overline{\text{apr}}_P$ are called the soft P-lower approximation operator on U and the soft P-upper approximation operator on U, respectively. In general, we refer to $\underline{\text{apr}}_P$ and $\overline{\text{apr}}_P$ as soft P-rough approximations operator on U.

$\underline{\text{apr}}_P(X)$ and $\overline{\text{apr}}_P(X)$ are called the soft P-lower approximation and the soft P-upper approximation of X, respectively. In general, we refer to $\underline{\text{apr}}_P(X)$ and $\overline{\text{apr}}_P(X)$ as soft rough approximations of X with respect to P.

X is called a soft P-definable set if $\underline{\text{apr}}_P(X) = \overline{\text{apr}}_P(X)$; X is called a soft P-rough set if $\underline{\text{apr}}_P(X) \neq \overline{\text{apr}}_P(X)$.

Moreover, the sets

$$\begin{aligned} \text{Pos}_P(X) &= \underline{\text{apr}}_P(X), \\ \text{Neg}_P(X) &= U - \overline{\text{apr}}_P(X), \\ \text{Bnd}_P(X) &= \overline{\text{apr}}_P(X) - \underline{\text{apr}}_P(X) \end{aligned} \tag{12}$$

are called the soft P-positive region, the soft P-negative region, and the soft P-boundary region of A, respectively.

Proposition 13 (see [9, 10]). *Let f_A be a soft set over U, and let $P = (U, f_A)$ be a soft approximation space. Then, the following properties hold for any $X, Y \in 2^U$.*

(1) $\underline{apr}_P(X) = \bigcup\{f(a) : a \in A \text{ and } f(a) \subseteq X\} \subseteq X$; $\overline{apr}_P(X) = \bigcup\{f(a) : a \in A \text{ and } f(a) \cap X \neq \emptyset\}$.

(2) $\underline{apr}_P(\emptyset) = \overline{apr}_P(\emptyset) = \emptyset$; $\underline{apr}_P(U) = \overline{apr}_P(U) = \bigcup_{a \in A} f(a)$.

(3) $X \subseteq Y \Rightarrow \underline{apr}_P(X) \subseteq \underline{apr}_P(Y)$; $X \subseteq Y \Rightarrow \overline{apr}_P(X) \subseteq \overline{apr}_P(Y)$.

(4) $\overline{apr}_P(X \cup Y) = \overline{apr}_P(X) \cup \overline{apr}_P(Y)$.

(5) $\underline{apr}_P(\underline{apr}_P(X)) = \underline{apr}_P(X)$; $\underline{apr}_P(\overline{apr}_P(X)) = \overline{apr}_P(X)$.

Proposition 14. *Let f_A be a soft set over U, and let $P = (U, f_A)$ be a soft approximation space. Then, the following properties hold.*

(1) *If f_A is full, then*

(a) $\underline{apr}_P(X) \subseteq X \subseteq \overline{apr}_P(X)$ *for any* $X \in 2^U$;

(b) $\underline{apr}_P(U) = \overline{apr}_P(U) = U$.

(2) *If f_A is keeping union, then*

(a) *for any $X \in 2^U$, there exists $a \in A$ such that $\underline{apr}_P(X) = f(a)$;*

(b) *for any $X \in 2^U$, there exists $a \in A$ such that $\overline{apr}_P(X) = f(a)$.*

(3) *If f_A is keeping intersection, then*

$$\underline{apr}_P(X \cap Y) = \underline{apr}_P(X) \cap \underline{apr}_P(Y) \quad \text{for any } X, Y \in 2^U. \tag{13}$$

(4) *If f_A is partition, then*

$$\underline{apr}_P(X \cap Y) = \underline{apr}_P(X) \cap \underline{apr}_P(Y) \quad \text{for any } X, Y \in 2^U. \tag{14}$$

(5) *If f_A is full and keeping union, then*

$$\overline{apr}_P(X) = U \quad \text{for any } X \in 2^U \setminus \{\emptyset\}. \tag{15}$$

Proof. (1)(a) By Proposition 13, $\underline{\text{apr}}_P(X) \subseteq X$. Suppose that $X - \overline{\text{apr}}_P(X) \neq \emptyset$. Pick

$$x \in X - \overline{\text{apr}}_P(X) \neq \emptyset. \tag{16}$$

Since f_A is full, $U = \bigcup_{a \in A} f(a)$. So, $x \in f(a)$ for some $a \in A$. $x \in X$ implies $f(a) \cap X \neq \emptyset$. Thus, $x \in \overline{\text{apr}}_P(X) \neq \emptyset$, contradiction. Hence,

$$X \subseteq \overline{\text{apr}}_P(X). \tag{17}$$

(1)(b) This holds by (1) and Proposition 13.
(2) This holds by Proposition 13.
(3) By Proposition 13,

$$\underline{\text{apr}}_P(X \cap Y) \subseteq \underline{\text{apr}}_P(X) \cap \underline{\text{apr}}_P(Y). \tag{18}$$

Suppose that $\underline{\text{apr}}_P(X) \cap \underline{\text{apr}}_P(Y) - \underline{\text{apr}}_P(X \cap Y) \neq \emptyset$. Pick

$$x \in \underline{\text{apr}}_P(X) \cap \underline{\text{apr}}_P(Y) - \underline{\text{apr}}_P(X \cap Y). \tag{19}$$

Then, there exist $a, b \in A$ such that $x \in f(a) \subseteq X$ and $x \in f(b) \subseteq Y$. Since f_A is keeping intersection, then $f(a) \cap f(b) = f(c)$ for some $c \in A$. This implies $x \in f(c) \subseteq X \cap Y$. Thus, $x \in \underline{\text{apr}}_P(X \cap Y)$, contradiction. Hence,

$$\underline{\text{apr}}_P(X \cap Y) \supseteq \underline{\text{apr}}_P(X) \cap \underline{\text{apr}}_P(Y). \tag{20}$$

Therefore,

$$\underline{\text{apr}}_P(X \cap Y) = \underline{\text{apr}}_P(X) \cap \underline{\text{apr}}_P(Y). \tag{21}$$

(4) Suppose that $\underline{\text{apr}}_P(X)\cap\underline{\text{apr}}_P(Y)-\underline{\text{apr}}_P(X\cap Y)\neq\emptyset$. Pick

$$x\in\underline{\text{apr}}_P(X)\cap\underline{\text{apr}}_P(Y)-\underline{\text{apr}}_P(X\cap Y). \tag{22}$$

Then, there exist $a,b\in A$ such that $x\in f(a)\subseteq X$ and $x\in f(b)\subseteq Y$. Since f_A is partition, then $f(a)=f(b)$. This implies $x\in f(a)\subseteq X\cap Y$. So, $x\in\underline{\text{apr}}_P(X\cap Y)$, contradiction. Thus,

$$\underline{\text{apr}}_P(X\cap Y)\supseteq\underline{\text{apr}}_P(X)\cap\underline{\text{apr}}_P(Y). \tag{23}$$

Therefore,

$$\underline{\text{apr}}_P(X\cap Y)=\underline{\text{apr}}_P(X)\cap\underline{\text{apr}}_P(Y). \tag{24}$$

(5) Since f_A is full and keeping union, then $U=\bigcup_{a\in A}f(a)=f(a^*)$ for some $a^*\in A$. For each $X\in 2^U\setminus\{\emptyset\}$ and each $u\in U$, $u\in f(a^*)$ and $f(a^*)\cap X=X\neq\emptyset$, and then $\overline{\text{apr}}_P(X)=U$. □

3.2. Soft Rough Approximation Operators $\underline{apr}'_P$ and $\overline{apr}'_P$, $\underline{apr}''_P$ and $\overline{apr}''_P$, and $\underline{apr}'''_P$ and $\overline{apr}'''_P$

Definition 15. Let f_A be a soft set over U.

(1) Define a binary relation R_f on U by

$$xR_fy\Longleftrightarrow\exists a\in A,\quad\{x,y\}\subseteq f(a) \tag{25}$$

for each $x,y\in U$. Then, R_f is called the binary relation induced by f_A on U.

(2) For each $x\in U$, define a successor neighborhood $R_{f_s}(x)$ of x in U by

$$\left(R_f\right)_s(x)=\left\{y\in U:xR_fy\right\}. \tag{26}$$

Since the following Proposition 16 is clear, we omit its proof.

Proposition 16. *Let f_A be a soft set over U, and let R_f be the binary relation induced by f_A on U. Then, the following properties hold.*

(1) *R_f is a symmetric relation.*

(2) *If f_A is full, then R_f is a reflexive relation.*

(3) *If f_A is partition, then R_f is an equivalence relation.*

Proposition 17. *Let f_A be a soft set over U, and let R_f be the binary relation induced by f_A on U. Then, the following properties hold.*

(1) *If $x\in f(a)$ with $a\in A$, then $f(a)\subseteq(R_f)_s(x)$.*

(2) *If f_A is partition and $x\in f(a)$ with $a\in A$, then $f(a)=(R_f)_s(x)$.*

(3) *If f_A is keeping union, then for each $x\in U$, there exists $a\in A$ such that $(R_f)_s(x)=f(a)$.*

Proof. (1) This is obvious.

(2) Suppose that $y\in(R_f)_s(x)$. Then, xR_fy, and so $\{x,y\}\subseteq f(b)$ for some $b\in A$. Since f_A is partition and $f(a)\cap f(b)\neq\emptyset$, then $f(a)=f(b)$. Thus, $y\in f(a)$. This implies $f(a)\supseteq(R_f)_s(x)$. By (1),

$$f(a)=\left(R_f\right)_s(x). \tag{27}$$

(3) Suppose that $y\in(R_f)_s(x)$. Then, xR_fy, and so $\{x,y\}\subseteq f(a_y)$ for some $a_y\in A$. By (1), $f(a_y)\subseteq(R_f)_s(x)$. Thus, $\{y\}\subseteq f(a_y)\subseteq(R_f)_s(x)$. This implies

$$\left(R_f\right)_s(x)=\bigcup\left\{f\left(a_y\right):y\in\left(R_f\right)_s(x)\right\}. \tag{28}$$

Since f_A is keeping union, then $\bigcup\{f(a_y):y\in(R_f)_s(x)\}=f(a)$ for some $a\in A$. Thus,

$$\left(R_f\right)_s(x)=f(a). \tag{29}$$

□

Definition 18. Let f_A be a soft set over U, and let $P=(U,f_A)$ be a soft approximation space. We define three pairs of soft rough approximation operations: $2^U\to 2^U$ as follows:

(1)

$$\underline{\text{apr}}'_P(X)=\left\{x\in U:\left(R_f\right)_s(x)\subseteq X\right\},$$
$$\overline{\text{apr}}'_P(X)=\left\{x\in U:\left(R_f\right)_s(x)\cap X\neq\emptyset\right\}. \tag{30}$$

X is called a soft P'-definable set if $\underline{\text{apr}}_P(X)=\overline{\text{apr}}'_P(X)$. X is called a soft P'-rough set if $\underline{\text{apr}}'_P(X)\neq\overline{\text{apr}}'_P(X)$. The sets

$$\text{Pos}'_P(X)=\underline{\text{apr}}'_P(X),$$
$$\text{Neg}'_P(X)=U-\overline{\text{apr}}'_P(X), \tag{31}$$
$$\text{Bnd}'_P(X)=\overline{\text{apr}}'_P(X)-\underline{\text{apr}}'_P(X)$$

are called the soft P'-positive region, the soft P'-negative region, and the soft P'-boundary region of X, respectively. Consider,

(2)

$$\underline{\text{apr}}''_P(X)=\bigcup\left\{\left(R_f\right)_s(x):x\in U,\ \left(R_f\right)_s(x)\subseteq X\right\},$$
$$\overline{\text{apr}}''_P(X)=U-\underline{\text{apr}}''_P(U-X). \tag{32}$$

X is called a soft P''-definable set if $\underline{\text{apr}}''_P(X)=\overline{\text{apr}}''_P(X)$. X is called a soft P''-rough set if $\underline{\text{apr}}''_P(X)\neq\overline{\text{apr}}''_P(X)$. The sets

$$\text{Pos}''_P(X)=\underline{\text{apr}}''_P(X),$$
$$\text{Neg}''_P(X)=U-\overline{\text{apr}}''_P(X), \tag{33}$$
$$\text{Bnd}''_P(X)=\overline{\text{apr}}''_P(X)-\underline{\text{apr}}''_P(X)$$

are called the soft P''-positive region, the soft P''-negative region, and the soft P''-boundary region of X, respectively. Consider,

(3)

$$\overline{\mathrm{apr}}_P'''(X) = \cup\left\{(R_f)_s(x) : x \in U,\ (R_f)_s(x) \cap X \neq \emptyset\right\},$$
$$\underline{\mathrm{apr}}_P'''(X) = U - \overline{\mathrm{apr}}_P'''(U - X). \tag{34}$$

X is called a soft P'''-definable set if $\underline{\mathrm{apr}}_P'''(X) = \overline{\mathrm{apr}}_P'''(X)$. X is called a soft P'''-rough set if $\underline{\mathrm{apr}}_P'''(X) \neq \overline{\mathrm{apr}}_P'''(X)$. The sets

$$\mathrm{Pos}_P'''(X) = \underline{\mathrm{apr}}_P'''(X),$$
$$\mathrm{Neg}_P'''(X) = U - \overline{\mathrm{apr}}_P'''(X), \tag{35}$$
$$\mathrm{Bnd}_P'''(X) = \overline{\mathrm{apr}}_P'''(X) - \underline{\mathrm{apr}}_P'''(X)$$

are called the soft P'''-positive region, the soft P'''-negative region, and the soft P'''-boundary region of X, respectively.

In general, we also refer to $\underline{\mathrm{apr}}_P'(X)$ and $\overline{\mathrm{apr}}_P'(X)$, $\underline{\mathrm{apr}}_P''(X)$ and $\overline{\mathrm{apr}}_P''(X)$, and $\underline{\mathrm{apr}}_P'''(X)$ and $\overline{\mathrm{apr}}_P'''(X)$ as soft rough approximations of X with respect to P', P'', P''', respectively.

It is not very difficult to prove Propositions 19, 20, and 21 (see [15]).

Proposition 19. *Let f_A be a soft set over U, and let $P = (U, f_A)$ be a soft approximation space. Then, the following properties hold for any $X, Y \in 2^U$.*

(1) $\underline{apr}_P'(X) \subseteq X$. *If f_A is full, then*

$$\underline{apr}_P'(X) \subseteq X \subseteq \overline{apr}_P'(X). \tag{36}$$

(2) $\overline{apr}_P'(\emptyset) = \emptyset$; $\underline{apr}_P'(U) = U$. *If f_A is full, then*

$$\underline{apr}_P'(\emptyset) = \overline{apr}_P'(\emptyset) = \emptyset;$$
$$\underline{apr}_P'(U) = \overline{apr}_P'(U) = U. \tag{37}$$

(3) $X \subseteq Y \Rightarrow \underline{apr}_P'(X) \subseteq \underline{apr}_P'(Y)$; $X \subseteq Y \Rightarrow \overline{apr}_P'(X) \subseteq \overline{apr}_P'(Y)$.

(4) $\underline{apr}_P'(X \cap Y) = \underline{apr}_P'(X) \cap \underline{apr}_P'(Y)$.

(5) $\overline{apr}_P'(X \cup Y) = \overline{apr}_P'(X) \cup \overline{apr}_P'(Y)$.

(6) $\underline{apr}_P'(U - X) = U - \overline{apr}_P'(X)$; $\overline{apr}_P'(U - X) = U - \underline{apr}_P'(X)$.

(7) $\overline{apr}_P'(\underline{apr}_P'(X)) \subseteq X \subseteq \underline{apr}_P'(\overline{apr}_P'(X))$.

Proposition 20. *Let f_A be a soft set over U, and let $P = (U, f_A)$ be a soft approximation space. Then, the following properties hold for any $X \in 2^U$.*

(1)

$$\underline{apr}_P''(X) \subseteq X \subseteq \overline{apr}_P''(X). \tag{38}$$

(2) $\underline{apr}_P''(\emptyset) = \emptyset$; $\overline{apr}_P''(U) = U$. *If f_A is full, then*

$$\underline{apr}_P'(\emptyset) = \overline{apr}_P'(\emptyset) = \emptyset;$$
$$\underline{apr}_P'(U) = \overline{apr}_P'(U) = U. \tag{39}$$

(3) $\underline{apr}_P''(U - X) = U - \overline{apr}_P''(X)$; $\overline{apr}_P''(U - X) = U - \underline{apr}_P''(X)$.

(4) $\underline{apr}_P''(\underline{apr}_P''(X)) = \underline{apr}_P''(X)$; $\overline{apr}_P''(\overline{apr}_P''(X)) = \overline{apr}_P''(X)$.

Proposition 21. *Let f_A be a soft set over U, and let $P = (U, f_A)$ be a soft approximation space. Then, the following properties hold for any $X, Y \in 2^U$.*

(1) *If f_A is full, then*

$$\underline{apr}_P'''(X) \subseteq X \subseteq \overline{apr}_P'''(X). \tag{40}$$

(2) $\overline{apr}_P'''(\emptyset) = \emptyset$; $\underline{apr}_P'''(U) = U$. *If f_A is full, then*

$$\underline{apr}_P'''(\emptyset) = \overline{apr}_P'''(\emptyset) = \emptyset;$$
$$\underline{apr}_P'''(U) = \overline{apr}_P'(U) = U. \tag{41}$$

(3) $X \subseteq Y \Rightarrow \underline{apr}_P'''(X) \subseteq \underline{apr}_P'''(Y)$; $X \subseteq Y \Rightarrow \overline{apr}_P'''(X) \subseteq \overline{apr}_P'''(Y)$.

(4) $\underline{apr}_P'''(X \cap Y) = \underline{apr}_P'''(X) \cap \underline{apr}_P'''(Y)$.

(5) $\overline{apr}_P'''(X \cup Y) = \overline{apr}_P'''(X) \cup \overline{apr}_P'''(Y)$.

(6) $\underline{apr}_P'''(U - X) = U - \overline{apr}_P'''(X)$; $\overline{apr}_P'''(U - X) = U - \underline{apr}_P'''(X)$.

(7) $\underline{apr}_P'''(\underline{apr}_P'''(X)) \subseteq \underline{apr}_P'''(X)$; $\overline{apr}_P'''(\overline{apr}_P'''(X)) \supseteq \overline{apr}_P'''(X)$.

(8) $\overline{apr}_P'''(\underline{apr}_P'''(X)) \subseteq X \subseteq \underline{apr}_P'''(\overline{apr}_P'''(X))$.

Example 22. Let $U = \{h_1, h_2, h_3, h_4, h_5\}$, $A = \{a_1, a_2, a_3, a_4\}$, and let f_A be a soft set over U, defined as follows:

$$f(a_1) = \{h_2\},$$
$$f(a_2) = \{h_1, h_4\},$$
$$f(a_3) = \{h_3\}, \tag{42}$$
$$f(a_4) = \{h_1, h_3\}.$$

Obviously, f_A is not full. We have

$$\begin{aligned}(R_f)_s(h_1) &= \{h_1, h_3, h_4\},\\ (R_f)_s(h_2) &= \{h_2\},\\ (R_f)_s(h_3) &= \{h_1, h_3\},\\ (R_f)_s(h_4) &= \{h_1, h_4\},\\ (R_f)_s(h_5) &= \emptyset.\end{aligned} \tag{43}$$

Let $X = \{h_3, h_5\}$, $Y = \{h_1, h_2, h_3\}$, and $Z = \{h_2, h_4, h_5\}$.

(1) We have

$$\overline{\mathrm{apr}}'_P(X) = \{h_1, h_3\}. \tag{44}$$

Thus,

$$X \not\subseteq \overline{\mathrm{apr}}'_P(X). \tag{45}$$

(2) We have

$$\begin{aligned}\underline{\mathrm{apr}}'_P(X) &= \{h_5\},\\ \underline{\mathrm{apr}}'_P(Y) &= \{h_2, h_3, h_5\}.\end{aligned} \tag{46}$$

Thus,

$$\underline{\mathrm{apr}}'_P(X) \subseteq \underline{\mathrm{apr}}'_P(Y) \nRightarrow X \subseteq Y. \tag{47}$$

(3) We have

$$\begin{aligned}\overline{\mathrm{apr}}'_P(X) &= \{h_1, h_3\},\\ \overline{\mathrm{apr}}'_P(Y) &= \{h_1, h_2, h_3, h_4\}.\end{aligned} \tag{48}$$

Thus,

$$\overline{\mathrm{apr}}'_P(X) \subseteq \overline{\mathrm{apr}}'_P(Y) \nRightarrow X \subseteq Y. \tag{49}$$

(4) We have

$$\begin{aligned}\underline{\mathrm{apr}}'_P(Y) &= \{h_2, h_3, h_5\},\\ \underline{\mathrm{apr}}'_P(Z) &= \{h_2, h_5\},\\ \underline{\mathrm{apr}}'_P(Y \cup Z) &= U.\end{aligned} \tag{50}$$

Thus,

$$\underline{\mathrm{apr}}'_P(Y \cup Z) \neq \underline{\mathrm{apr}}'_P(y) \cup \underline{\mathrm{apr}}'_P(Z). \tag{51}$$

(5) We have

$$\begin{aligned}\overline{\mathrm{apr}}'_P(Y) &= \{h_1, h_2, h_3, h_4\},\\ \overline{\mathrm{apr}}'_P(Z) &= \{h_1, h_2, h_4\},\\ \overline{\mathrm{apr}}'_P(Y \cap Z) &= \{h_2\}.\end{aligned} \tag{52}$$

Thus,

$$\overline{\mathrm{apr}}'_P(Y \cap Z) \neq \overline{\mathrm{apr}}'_P(Y) \cap \underline{\mathrm{apr}}'_P(Z). \tag{53}$$

3.3. The Relationships among Four Pairs of Soft Rough Approximation Operators

Lemma 23. *Let f_A be a soft set over U, and let $P = (U, f_A)$ be a soft approximation space. Then, the following properties hold for any $X \in 2^U$.*

(1) *If f_A is full, then*

$$\underline{apr}_P(X) \supseteq \underline{apr}'_P(X). \tag{54}$$

(2) *If f_A is full and keeping union, then*

$$\overline{apr}_P(X) \supseteq \overline{apr}'_P(X). \tag{55}$$

(3) *If f_A is partition, then*

(a) $\underline{apr}_P(X) = \underline{apr}'_P(X)$;

(b) $\overline{apr}_P(X) = \overline{apr}'_P(X)$.

Proof. (1) Suppose that $x \in \underline{\mathrm{apr}}'_P(X)$. Then, $(R_f)_s(x) \subseteq X$. Since f_A is full, then $x \in f(a)$ for some $a \in A$. By Proposition 17, $f(a) \subseteq (R_f)_s(x)$. Thus, $x \in f(a) \subseteq X$. This implies $x \in \underline{\mathrm{apr}}_P(X)$. Thus,

$$\underline{\mathrm{apr}}_P(X) \supseteq \underline{\mathrm{apr}}'_P(X). \tag{56}$$

(2) If $X = \emptyset$, then $\overline{\mathrm{apr}}_P(X) = \emptyset = \overline{\mathrm{apr}}'_P(X)$. If $X \neq \emptyset$, by Proposition 14, $\overline{\mathrm{apr}}_P(X) = U$.

Hence,

$$\overline{\mathrm{apr}}_P(X) \supseteq \overline{\mathrm{apr}}'_P(X). \tag{57}$$

(3)(a) Suppose that $x \in \underline{\mathrm{apr}}_P(X)$. Then, $x \in f(a) \subseteq X$ for some $a \in A$. Since f_A is partition and $x \in f(a)$, then $f(a) = (R_f)_s(x)$ by Proposition 17. This implies $x \in \underline{\mathrm{apr}}'_P(X)$. Thus,

$$\underline{\mathrm{apr}}_P(X) \subseteq \underline{\mathrm{apr}}'_P(X). \tag{58}$$

By (1),

$$\underline{\mathrm{apr}}_P(X) = \underline{\mathrm{apr}}'_P(X). \tag{59}$$

(3)(b) This is similar to the proof of (3) (a).

Suppose that $x \in \overline{\mathrm{apr}}'_P(X)$. Then, $(R_f)_s(x) \cap X \neq \emptyset$. Since f_A is full, then $x \in f(a)$ for some $a \in A$. Since f_A is partition and $x \in f(a)$, then $f(a) = (R_f)_s(x)$ by Proposition 17. This implies $x \in \overline{\mathrm{apr}}_P(X)$. Thus,

$$\overline{\mathrm{apr}}_P(X) \supseteq \overline{\mathrm{apr}}'_P(X). \tag{60}$$

Hence, $\overline{\mathrm{apr}}_P(X) = \overline{\mathrm{apr}}'_P(X)$. □

By Propositions 13 and 17, we have Lemma 24.

Lemma 24. *Let f_A be a soft set over U, and let $P = (U, f_A)$ be a soft approximation space. If f_A is keeping union, then for any $X \in 2^U$,*

$$\underline{apr}_P(X) \supseteq \underline{apr}''_P(X), \qquad \overline{apr}_P(X) \supseteq \overline{apr}'''_P(X). \tag{61}$$

Lemma 25. *Let f_A be a soft set over U, and let $P = (U, f_A)$ be a soft approximation space. Then, the following properties hold for any $X \in 2^U$.*

(1) *If f_A is full, then*

$$\underline{apr}'''_P(X) \subseteq \underline{apr}'_P(X) \subseteq \underline{apr}''_P(X) \subseteq X \subseteq \overline{apr}''_P(X) \subseteq \overline{apr}'_P(X) \subseteq \overline{apr}'''_P(X). \quad (62)$$

(2) *If f_A is partition, then*

(a) $\underline{apr}'_P(X) = \underline{apr}''_P(X) = \underline{apr}'''_P(X)$;

(b) $\overline{apr}'_P = \overline{apr}''_P(X) = \overline{apr}'''_P(X)$.

Proof. (1) Suppose that $x \in \underline{\mathrm{apr}}'_P(X)$. Then, $(R_f)_s(x) \subseteq X$. Since f_A is full, then $x \in (R_f)_s(x) \subseteq X$ by Proposition 17. This implies $x \in \underline{\mathrm{apr}}''_P(X)$. Thus,

$$\underline{\mathrm{apr}}'_P(X) \subseteq \underline{\mathrm{apr}}''_P(X). \quad (63)$$

Suppose that $\underline{\mathrm{apr}}'''_P(X) - \underline{\mathrm{apr}}'_P(X) \neq \emptyset$. Pick

$$x \in \underline{\mathrm{apr}}'''_P(X) - \underline{\mathrm{apr}}'_P(X). \quad (64)$$

$x \notin \underline{\mathrm{apr}}'_P(X)$ implies $(R_f)_s(x) \nsubseteq X$. So, $(R_f)_s(x) \cap (U - X) \neq \emptyset$. Since f_A is full, then $x \in (R_f)_s(x)$ by Proposition 17. This implies $x \in \overline{\mathrm{apr}}''_P(U - X)$. Thus, $x \notin U - \overline{\mathrm{apr}}''_P(U - X) = \underline{\mathrm{apr}}'''_P(X)$, contradiction.

Hence, $\underline{\mathrm{apr}}'''_P(X) \subseteq \underline{\mathrm{apr}}'_P(X)$.

By Proposition 20,

$$\underline{\mathrm{apr}}''_P(X) \subseteq X \subseteq \overline{\mathrm{apr}}''_P(X). \quad (65)$$

Since

$$\underline{\mathrm{apr}}'''_P(U - X) \subseteq \underline{\mathrm{apr}}'_P(U - X) \subseteq \underline{\mathrm{apr}}''_P(U - X), \quad (66)$$

then

$$U - \underline{\mathrm{apr}}'''_P(U - X) \supseteq U - \underline{\mathrm{apr}}'_P(U - X) \supseteq U - \underline{\mathrm{apr}}''_P(U - X). \quad (67)$$

By Propositions 19, 20, and 21,

$$\overline{\mathrm{apr}}''_P(X) \subseteq \overline{\mathrm{apr}}'_P(X) \subseteq \overline{\mathrm{apr}}'''_P(X). \quad (68)$$

(2)(a) Suppose that $x \in \underline{\mathrm{apr}}''_P(X)$. Then, there exists $y \in U$ such that $x \in (R_f)_s(y) \subseteq X$. Since f_A is partition, then R_f is an equivalence relation by Proposition 16. Thus, $x \in (R_f)_s(y)$ follows $(R_f)_s(x) = (R_f)_s(y)$. So, $(R_f)_s(x) \subseteq X$. This implies $x \in \underline{\mathrm{apr}}'_P(X)$. Hence,

$$\underline{\mathrm{apr}}'_P(X) \supseteq \underline{\mathrm{apr}}''_P(X). \quad (69)$$

By (1),

$$\underline{\mathrm{apr}}'_P(X) = \underline{\mathrm{apr}}''_P(X). \quad (70)$$

Suppose that $\underline{\mathrm{apr}}'_P(X) \cap \overline{\mathrm{apr}}'''_P(U - X) \neq \emptyset$. Pick

$$x \in \underline{\mathrm{apr}}'_P(X) \cap \overline{\mathrm{apr}}'''_P(U - X). \quad (71)$$

$x \in \underline{\mathrm{apr}}'_P(X)$ implies $(R_f)_s(x) \subseteq X$. $x \in \overline{\mathrm{apr}}'''_P(U - X)$ implies that there exists $y \in U$ such that $x \in (R_f)_s(y)$ and $(R_f)_s(y) \cap (U - X) \neq \emptyset$. So, $(R_f)_s(y) \nsubseteq X$. Note that R_f is an equivalence relation. Then, $(R_f)_s(x) = (R_f)_s(y)$. Thus, $(R_f)_s(x) \nsubseteq X$, contradiction.

Hence, $\underline{\mathrm{apr}}'_P(X) \cap \overline{\mathrm{apr}}'''_P(U - X) = \emptyset$.

This proves that

$$\underline{\mathrm{apr}}'_P(X) \subseteq U - \overline{\mathrm{apr}}'''_P(U - X) = \underline{\mathrm{apr}}'''_P(X). \quad (72)$$

By (1),

$$\underline{\mathrm{apr}}'_P(X) = \underline{\mathrm{apr}}'''_P(X). \quad (73)$$

(2)(b) By (2)(a),

$$\underline{\mathrm{apr}}'_P(U - X) = \underline{\mathrm{apr}}''_P(U - X) = \underline{\mathrm{apr}}'''_P(U - X). \quad (74)$$

Then,

$$U - \underline{\mathrm{apr}}'_P(U - X) = U - \underline{\mathrm{apr}}''_P(U - X) = U - \underline{\mathrm{apr}}'''_P(U - X). \quad (75)$$

By Propositions 19, 20, and 21,

$$\overline{\mathrm{apr}}'_P(X) = \overline{\mathrm{apr}}''_P(X) = \overline{\mathrm{apr}}'''_P(X). \quad (76)$$

□

Example 26. Let $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$, $A = \{a_1, a_2, a_3, a_4, a_5\}$, and let f_A be a soft set over U, defined as follows:

$$\begin{aligned} f(a_1) &= \{h_1\}, \\ f(a_2) &= \{h_6\}, \\ f(a_3) &= \{h_2, h_5\}, \\ f(a_4) &= \{h_2, h_4\}, \\ f(a_5) &= \{h_1, h_3, h_5\}. \end{aligned} \quad (77)$$

Obviously, f_A is full. We have

$$\begin{aligned} (R_f)_s(h_1) &= \{h_1, h_3, h_5\}, \\ (R_f)_s(h_2) &= \{h_2, h_4, h_5\}, \\ (R_f)_s(h_3) &= \{h_1, h_3, h_5\}, \\ (R_f)_s(h_4) &= \{h_2, h_4\}, \\ (R_f)_s(h_5) &= \{h_1, h_2, h_3, h_5\}. \end{aligned} \quad (78)$$

Let $X = \{h_2, h_4, h_6\}$. We have

$$\begin{aligned} \underline{\mathrm{apr}}_P(X) &= \{h_2, h_4, h_6\}, \\ \underline{\mathrm{apr}}'_P(X) &= \{h_4\}, \\ \underline{\mathrm{apr}}''_P(X) &= \{h_2, h_4\}, \\ \overline{\mathrm{apr}}_P(X) &= \{h_2, h_4, h_5, h_6\}, \\ \overline{\mathrm{apr}}'_P(X) &= \{h_2, h_4, h_5\}, \\ \overline{\mathrm{apr}}''_P(X) &= \{h_1, h_2, h_3, h_4, h_5\}. \end{aligned} \quad (79)$$

Thus,

$$\begin{aligned} \underline{\mathrm{apr}}_P(X) &\neq \underline{\mathrm{apr}}'_P(X), \\ \underline{\mathrm{apr}}'_P(X) &\neq \underline{\mathrm{apr}}''_P(X), \\ \overline{\mathrm{apr}}_P(X) &\neq \overline{\mathrm{apr}}'_P(X), \\ \overline{\mathrm{apr}}'_P(X) &\neq \overline{\mathrm{apr}}''_P(X). \end{aligned} \quad (80)$$

By Proposition 16 and Lemmas 23, 24, and 25, we have Theorem 27.

Theorem 27. *Let f_A be a soft set over U, and let $P = (U, f_A)$ be a soft approximation space. Then, the following properties hold for any $X \in 2^U$.*

(1) *If f_A is full, then*

$$\begin{aligned} \underline{apr}'''_P(X) &\subseteq \underline{apr}'_P(X) \subseteq \underline{apr}''_P(X) \subseteq X \\ &\subseteq \overline{apr}''_P(X) \subseteq \overline{apr}'_P(X) \subseteq \overline{apr}'''_P(X). \end{aligned} \quad (81)$$

(2) *If f_A is full and keeping union, then*

$$\begin{aligned} \underline{apr}'''_P(X) &\subseteq \underline{apr}'_P(X) \subseteq \underline{apr}''_P(X) \subseteq \underline{apr}_P(X) \\ &\subseteq X \subseteq \overline{apr}''_P(X) \subseteq \overline{apr}'_P(X) \\ &\subseteq \overline{apr}'''_P(X) \subseteq \overline{apr}_P(X). \end{aligned} \quad (82)$$

3.4. The Relationship between Soft Rough Approximation Operators and Pawlak Rough Approximation Operators. In this section, we shall explore the relationship between soft rough approximation operators and Pawlak rough approximation operators.

Definition 28. Let R be an equivalence relation on U. Define a mapping $f_R : A \to 2^U$ by

$$f_R(a) = [a]_R \quad (83)$$

for each $a \in A$, where $A = U$. Then, $(f_R)_A$ is called the soft set induced by R on U.

Theorem 29 (see [10]). *Let R be an equivalence relation on U, let $(f_R)_A$ be the soft set induced by R on U, and let $P = (U, (f_R)_A)$ be a soft approximation space. Then, for each $X \in 2^U$,*

$$\underline{R}(X) = \underline{apr}_P(X), \qquad \overline{R}(X) = \overline{apr}_P(X). \quad (84)$$

Thus, in this case, $X \in 2^U$ is a Pawlak rough set if and only if X is a soft P-rough set.

By Proposition 16 and Lemmas 23 and 25, we have Theorem 30.

Theorem 30. *Let f_A be a partition soft set over U, and let $P = (U, f_A)$ be a soft approximation space. Then, the following properties hold for any $X \in 2^U$.*

(1) $\underline{R_f}(X) = \underline{apr}_P(X) = \underline{apr}'_P(X) = \underline{apr}''_P(X) = \underline{apr}'''_P(X)$;

(2) $\overline{R_f}(X) = \overline{apr}_P(X) = \overline{apr}'_P(X) = \overline{apr}''_P(X) = \overline{apr}'''_P(X)$,

where $\underline{R_f}(X)$ and $\overline{R_f}(X)$ are the Pawlak rough approximations of X.

Corollary 31. *Let f_A be a full soft set over U, and let $P = (U, f_A)$ be a soft approximation space. Then,*

(1) *every soft P'''-definable set is a soft P'-definable set.*

(2) *every soft P'-definable set is a soft P''-definable set.*

Remark 32. Theorems 29 and 30 illustrate that Pawlak's rough set models can be viewed as a special case of soft rough sets.

Remark 33. Example 4.6 in [10] illustrates that a soft rough approximation is a worth considering alternative to the rough approximation. Soft rough sets could provide a better approximation than rough sets do.

4. The Relationships between Soft Sets and Topologies

Let f_A be a soft set over U, and let $P = (U, f_A)$ be a soft approximation space. Denote

$$\begin{aligned} \tau_f &= \left\{X \in 2^U : X = \underline{\mathrm{apr}}_P(X)\right\}, \\ \sigma_f &= \left\{X \in 2^U : X = \overline{\mathrm{apr}}_P(X)\right\}; \\ \tau'_f &= \left\{X \in 2^U : X = \underline{\mathrm{apr}}'_P(X)\right\}, \\ \sigma'_f &= \left\{X \in 2^U : X = \overline{\mathrm{apr}}'_P(X)\right\}; \\ \tau''_f &= \left\{X \in 2^U : X = \underline{\mathrm{apr}}''_P(X)\right\}, \\ \sigma''_f &= \left\{X \in 2^U : X = \overline{\mathrm{apr}}''_P(X)\right\}; \\ \tau'''_f &= \left\{X \in 2^U : X = \underline{\mathrm{apr}}'''_P(X)\right\}, \\ \sigma'''_f &= \left\{X \in 2^U : X = \overline{\mathrm{apr}}'''_P(X)\right\}. \end{aligned} \quad (85)$$

4.1. The First Sort of Topologies Induced by a Soft Set and Related Results. By Propositions 13 and 14, we have Theorem 34.

Theorem 34. *Let f_A be a full and keeping intersection or a partition soft set over U and let $P = (U, f_A)$ be a soft approximation space. Then τ_f is a topology on U.*

Remark 35. Let f_A be a full and keeping union soft set over U, and let $P = (U, f_A)$ be an soft approximation space. Then, by Proposition 14(5), $\sigma_f = \{\emptyset, U\}$ is a indiscrete topology on U.

The following theorem gives the structure of the first sort of topologies induced by a soft set.

Theorem 36. *Let f_A be a full and keeping intersection soft set over U, let $P = (U, f_A)$ be a soft approximation space, and let τ_f be the topology induced by f_A on U. Then,*

(1)

$$\left\{\overline{apr}_P(X) : X \in 2^U\right\} \subseteq \tau_f = \left\{\underline{apr}_P(X) : X \in 2^U\right\}; \quad (86)$$

(2)

$$\tau_f \supseteq \{f(a) : a \in A\}; \quad (87)$$

(3) *if f_A is topological, then*

$$\tau_f = \{f(a) : a \in A\}; \quad (88)$$

(4) *$\underline{apr}_P$ is an interior operator of τ_f.*

Proof. (1) By Proposition 13, we have

$$\left\{\overline{\mathrm{apr}}_P(X) : X \in 2^U\right\} \subseteq \tau_f. \quad (89)$$

Obviously,

$$\tau_f \subseteq \left\{\underline{\mathrm{apr}}_P(X) : X \in 2^U\right\}. \quad (90)$$

Let $Y \in \{\underline{\mathrm{apr}}_P(X) : X \in 2^U\}$. Then, $Y = \underline{\mathrm{apr}}_P(X)$ for some $X \in 2^U$. By Proposition 13, $\underline{\mathrm{apr}}_P(\underline{\mathrm{apr}}_P(X)) = \underline{\mathrm{apr}}_P(X)$. This implies $Y \in \tau_f$. Thus,

$$\tau_f \supseteq \left\{\underline{\mathrm{apr}}_P(X) : X \in 2^U\right\}. \quad (91)$$

Hence,

$$\left\{\overline{\mathrm{apr}}_P(X) : X \in 2^U\right\} \subseteq \tau_f = \left\{\underline{\mathrm{apr}}_P(X) : X \in 2^U\right\}. \quad (92)$$

(2) For each $a \in A$, by Proposition 13,

$$\underline{\mathrm{apr}}_P(f(a)) = \bigcup\left\{f(a') : a' \in A,\ f(a') \subseteq f(a)\right\} \subseteq f(a). \quad (93)$$

Then, $f(a) = \underline{\mathrm{apr}}_P(f(a))$. So $f(a) \in \tau_f$. Thus,

$$\{f(a) : a \in A\} \subseteq \tau_f. \quad (94)$$

(3) Suppose that $X \in \tau_f$. If $X = \emptyset$, by f_A is topological, there exists $a \in A$ such that $X = f(a)$. If $X \neq \emptyset$, for each $x \in X$, $X = \underline{\mathrm{apr}}_P(X)$, there exists $a_x \in A$ such that $x \in f(a_x) \subseteq X$. Then,

$$X = \bigcup_{x\in X}\{x\} \subseteq \bigcup_{x\in X} f(a_x) \subseteq X. \quad (95)$$

So, $X = \bigcup_{x\in X} f(a_x)$. Since f_A is keeping union, then

$$\bigcup_{x\in X} f(a_x) = f(a) \quad \text{for some } a \in A. \quad (96)$$

This implies $X \in \{f(a) : a \in A\}$. Thus, $\tau_f \subseteq \{f(a) : a \in A\}$.

By (1), $\tau_f \supseteq \{f(a) : a \in A\}$.

Hence,

$$\tau_f = \{f(a) : a \in A\}. \quad (97)$$

(4) It suffices to show that

$$\underline{\mathrm{apr}}_P(X) = \mathrm{int}(X) \quad \text{for each } X \in 2^U. \quad (98)$$

By (1), $\underline{\mathrm{apr}}_P(X) \in \tau_f$. By Proposition 13, $\underline{\mathrm{apr}}_P(X) \subseteq X$. Thus

$$\underline{\mathrm{apr}}_P(X) \subseteq \mathrm{int}(X). \quad (99)$$

Conversely, for each $Y \in \tau_f$ with $Y \subseteq X$, we have $Y = \underline{\mathrm{apr}}_P(Y) \subseteq \underline{\mathrm{apr}}_P(X)$ by Proposition 13. Thus,

$$\mathrm{int}(X) = \cup\left\{Y : Y \in \tau_f,\ Y \subseteq X\right\} \subseteq \underline{\mathrm{apr}}_P(X). \quad (100)$$

Hence,

$$\underline{\mathrm{apr}}_P(X) = \mathrm{int}(X). \quad (101)$$

□

Definition 37. Let τ be a topology on U. Put $\tau = \{U_a : a \in A\}$, where A is the set of indexes. Define a mapping $f_\tau : A \to 2^U$ by $f_\tau(a) = U_a$ for each $a \in A$. Then, the soft set $(f_\tau)_A$ over U is called the soft set induced by τ on U.

Definition 38. Let (U, μ) be a topological space. If there exists a full and keeping intersection or a partition soft set f_A over U such that $\tau_f = \mu$, then (U, μ) is called a soft approximating space.

The following proposition can easily be proved.

Proposition 39. *(1) Let τ be a topology on U, and let $(f_\tau)_A$ be the soft set induced by τ on U. Then, $(f_\tau)_A$ is a full, keeping intersection, and keeping union soft set over U.*

(2) Let τ_1 and τ_2 be two topologies on U, and let $(f_{\tau_1})_{A_1}$ and $(f_{\tau_2})_{A_2}$ be two soft sets induced, respectively, by τ_1 and τ_2 on U. If $\tau_1 \subseteq \tau_2$, then

$$(f_{\tau_1})_{A_1} \tilde{\subset} (f_{\tau_2})_{A_2}. \quad (102)$$

Theorem 40. *Let τ be a topology on U, let $(f_\tau)_A$ be the soft set induced by τ on U, and let τ_{f_τ} be the topology induced by $(f_\tau)_A$ on U. Then, $\tau = \tau_{f_\tau}$.*

Proof. Put $\tau = \{U_a : a \in A\}$; then, $f_\tau : A \to 2^U$ is a mapping, where $f_\tau(a) = U_a$ for each $a \in A$. By Proposition 39, $(f_\tau)_A$ is full, keeping intersection, and keeping union.

By Theorem 36, $\tau_{f_\tau} = \{f_\tau(a) : a \in A\}$.

Hence, $\tau_{f_\tau} = \tau$. □

Corollary 41. *Every topological space on the initial universe is a soft approximating space.*

Theorem 42. *Let (U, τ) be a topological space. Then, there exists a full, keeping intersection, and keeping union soft set f_A over U such that*

$$\underline{apr}_P(X) = \text{int}(X) \quad \text{for each } X \in 2^U, \tag{103}$$

where $P = (U, f_A)$ is a soft approximation space.

Proof. Put $\tau = \{U_a : a \in A\}$, where A is the set of indexes. Define a mapping $f : A \to 2^U$ by

$$f(a) = U_a \quad \text{for each } a \in A. \tag{104}$$

By Proposition 39, f_A is full, keeping intersection, and keeping union.

Let $X \in 2^U$. For each $x \in \underline{\text{apr}}_P(X)$, $x \in f(a) \subseteq X$ for some $a \in A$. Then, $x \in U_a \subseteq X$ with $U_a \in \tau$. This implies $x \in \text{int}(X)$.

Conversely, for each $x \in \text{int}(X)$, there exists an open neighborhood W of x in U such that $W \subseteq X$. So, $W = U_a$ for some $a \in A$. This implies $x \in f(a) \subseteq X$. Thus, $x \in \underline{\text{apr}}_P(X)$.

Hence, $\underline{\text{apr}}_P(X) = \text{int}(X)$. □

Theorem 43. *Let f_A be a full and keeping intersection soft set over U, let τ_f be the topology induced by f_A on U, and let $(f_{\tau_f})_B$ be the soft set induced by τ_f on U. Then,*

(1)

$$f_A \tilde{\subset} \left(f_{\tau_f}\right)_B. \tag{105}$$

(2) *If f_A is topological, then*

$$f_A = \left(f_{\tau_f}\right)_B. \tag{106}$$

Proof. (1) By Theorem 36, $\tau_f \supseteq \{f(a) : a \in A\}$. Denote

$$\tau_f = \{U_a : a \in B\}, \quad \text{where } A \subseteq B, \qquad U_a = f(a) \quad \text{for each } a \in A. \tag{107}$$

Thus f_{τ_f} is a mapping given by $f_{\tau_f} : B \to 2^U$, where $f_{\tau_f}(a) = U_a$ for each $a \in B$.

Hence, $f_A \tilde{\subset} (f_{\tau_f})_B$.

(2) Since f_A is topological, then by Theorem 36, $A = B$.

Hence,

$$f_A = \left(f_{\tau_f}\right)_B. \tag{108}$$

□

4.2. The Second Sort of Topologies Induced by a Soft Set. By Proposition 19, we have Theorem 44.

Theorem 44. *Let f_A be a full soft set over U, and let $P = (X, f_A)$ be a soft approximation space. Then, τ'_f is a topology on U.*

Definition 45. Let τ be a topology on U.

(1) τ is called an Alexandrov topology on U, if τ is closed for arbitrary intersections.

(2) (U, τ) is called an Alexandrov space, if τ is an Alexandrov topology on U.

(3) τ is called a pseudodiscrete topology on U, if $A \in \tau$ if and only if $U - A \in \tau$.

Obviously, every pseudodiscrete topology is an Alexandrov topology.

The following theorem gives the structure of the second sort of topologies induced by a soft set.

Theorem 46. *Let f_A be a full soft set over U, and let τ'_f be the topology induced by f_A on U. Then, (U, τ'_f) is an Alexandrov space.*

Proof. By Proposition 16, R_f is reflexive and symmetric. Then, by Proposition 5 in [16], τ'_f is a pseudodiscrete topology on U.

Thus,

$$\left(U, \tau'_f\right) \text{ is an Alexandrov space.} \tag{109}$$

□

4.3. The Third Sort of Topologies Induced by a Soft Set

Example 47. Let f_A be a full soft set over U and let $P = (U, f_A)$ be a soft approximation space in Example 26.

Let $Y = \{h_2, h_4\}$, and $Z = \{h_1, h_2, h_3, h_5\}$. We have

$$\underline{\text{apr}}''_P(Y) = Y, \qquad \underline{\text{apr}}''_P(Z) = Z, \qquad \underline{\text{apr}}''_P(Y \cap Z) = \emptyset \neq Y \cap Z. \tag{110}$$

Thus, τ''_f is not a topology on U.

By Proposition 21, we have Theorem 48.

Theorem 48. *Let f_A be a full soft set over U, and let $P = (U, f_A)$ be a soft approximation space. Then, τ'''_f is a topology on U.*

4.4. The Relationships among Three Sorts of Topologies Induced by a Soft Set and Related Results. By Theorem 27, we have Theorem 49, which illustrates relationships among three sorts of topologies induced by a soft set.

Theorem 49. *(1) If f_A is a full soft set over U, then*

$$\tau_f''' \subseteq \tau_f'. \quad (111)$$

(2) If f_A is a full and keeping intersection soft set over U, then

$$\tau_f''' \subseteq \tau_f' \subseteq \tau_f. \quad (112)$$

(3) If f_A is a partition soft set over U, then

$$\tau_f = \tau_f' = \tau_f'''. \quad (113)$$

By Theorem 36, Proposition 39 and Theorem 49, we have Theorem 50.

Theorem 50. *Let τ be a topology on U, let $(f_\tau)_A$ be the soft set induced by τ on U, and let τ_{f_τ}, τ'_{f_τ} and τ'''_{f_τ} be the topology induced, respectively, by $(f_\tau)_A$ on U. Then*

$$\tau = \tau_{f_\tau} \supseteq \tau'_{f_\tau} \supseteq \tau'''_{f_\tau}. \quad (114)$$

By Proposition 39 and Theorems 43 and 49, we have Theorem 51.

Theorem 51. *(1) If f_A is a full soft set over U, let τ_f' and τ_f''' be the topologies induced, respectively, by f_A on U, and let $(f_{\tau_f'})_C$ (resp., $(f_{\tau_f'''})_D$) be the soft set induced by τ_f' (resp., τ_f''') on U. Then,*

$$\left(f_{\tau_f'}\right)_C \widetilde{\supset} \left(f_{\tau_f'''}\right)_D. \quad (115)$$

(2) Let f_A be a full and keeping intersection soft set over U, let τ_f, τ_f', and τ_f''' be the topologies induced respectively by f_A on U and let $(f_{\tau_f})_B$ (resp., $(f_{\tau_f'})_C$, $(f_{\tau_f'''})_D$) be the soft set induced by τ_f (resp., τ_f', τ_f''') on U. Then,

(a)

$$\left(f_{\tau_f}\right)_B \widetilde{\supset} f_A, \qquad \left(f_{\tau_f}\right)_B \widetilde{\supset} \left(f_{\tau_f'}\right)_C \widetilde{\supset} \left(f_{\tau_f'''}\right)_D. \quad (116)$$

(b) If f_A is keeping union, then

$$f_A = \left(f_{\tau_f}\right)_B \widetilde{\supset} \left(f_{\tau_f'}\right)_C \widetilde{\supset} \left(f_{\tau_f'''}\right)_D. \quad (117)$$

(3) Let f_A be a partition soft set over U, let τ_{f_τ}, τ'_{f_τ} and τ'''_{f_τ} be the topologies induced, respectively, by f_A on U and let $(f_{\tau_f})_B$ (resp., $(f_{\tau_f'})_C$, $(f_{\tau_f'''})_D$) be the soft set induced by τ_f (resp., τ_f', τ_f''') on U. Then,

$$\left(f_{\tau_f}\right)_B \widetilde{\supset} f_A, \qquad \left(f_{\tau_f}\right)_B = \left(f_{\tau_f'}\right)_C = \left(f_{\tau_f'''}\right)_D. \quad (118)$$

Example 52. Let f_A be a partition soft set over U in Example 11, let τ_f be the topology induced by f_A on U, and let $(f_{\tau_f})_B$ be the soft set induced by τ_f on U. We have

$$\begin{aligned}
&\tau_f = \{U_a : a \in B\}, \\
&\text{where } B = \{a_1, a_2, \ldots, a_{16}\}, \\
&U_{a_1} = f(a_1) = \{h_1, h_2\}, \\
&U_{a_2} = f(a_2) = \{h_5\}, \\
&U_{a_3} = f(a_3) = \{h_3\}, \\
&U_{a_4} = f(a_4) = \{h_4\}, \\
&U_{a_5} = \{h_3, h_4\}, \\
&U_{a_6} = \{h_3, h_5\}, \\
&U_{a_7} = \{h_4, h_5\}, \\
&U_{a_8} = \{h_1, h_2, h_3\}, \\
&U_{a_9} = \{h_1, h_2, h_4\}, \\
&U_{a_{10}} = \{h_1, h_2, h_5\}, \\
&U_{a_{11}} = \{h_3, h_4, h_5\}, \\
&U_{a_{12}} = \{h_1, h_2, h_3, h_4\}, \\
&U_{a_{13}} = \{h_1, h_2, h_3, h_5\}, \\
&U_{a_{14}} = \{h_1, h_2, h_4, h_5\}, \\
&U_{a_{15}} = \emptyset, \\
&U_{a_{16}} = U.
\end{aligned} \quad (119)$$

Obviously,

$$f_A \widetilde{\subset} \left(f_{\tau_f}\right)_B, \qquad f_A \widetilde{\not\supseteq} \left(f_{\tau_f}\right)_B. \quad (120)$$

Thus,

$$f_A \neq \left(f_{\tau_f}\right)_B. \quad (121)$$

5. The Related Properties of Soft Rough Sets

In this section, four sorts of soft rough sets based on four pairs of soft rough approximations are investigated.

For $\mathscr{A}, \mathscr{B} \subseteq 2^U$, we denote

$$\mathscr{A} \setminus \mathscr{B} = \left\{X \in 2^U : X \in \mathscr{A},\ X \notin \mathscr{B}\right\}. \quad (122)$$

Lemma 53 (see [10]). *Let f_A be a soft set over U and let $P = (U, f_A)$ be a soft approximation space. Then for each $X \in 2^U$,*

$$X \text{ is a soft } P\text{-rough set} \Longleftrightarrow \overline{apr}_P(X) \subseteq X. \quad (123)$$

By Corollary 31, we have the following Lemma 54.

Lemma 54. *Let f_A be a full soft set over U, and let $P = (U, f_A)$ be a soft approximation space. Then,*

(1) *every soft P'-rough set is a soft P'''-rough set.*
(2) *every soft P''-rough set is a soft P'-rough set.*

By Theorem 27, we have Lemma 55.

Lemma 55. *Let f_A be a full and keeping union soft set over U, and let $P = (U, f_A)$ be a soft approximation space. If $X \in 2^U$ is a soft P-definable set, then,*

$$X \in \sigma_f''' \subseteq \sigma_f' \subseteq \sigma_f''. \tag{124}$$

The following theorem gives the structure of soft rough sets.

Theorem 56. *Let f_A be a soft set over U, and let $P = (U, f_A)$ be a soft approximation space. Denote*

$$\begin{aligned} \Omega &= \left\{X \in 2^U : X \text{ is a soft } P\text{-rough set}\right\}, \\ \Omega' &= \left\{X \in 2^U : X \text{ is a soft } P'\text{-rough set}\right\}, \\ \Omega'' &= \left\{X \in 2^U : X \text{ is a soft } P''\text{-rough set}\right\}, \\ \Omega''' &= \left\{X \in 2^U : X \text{ is a soft } P'''\text{-rough set}\right\}. \end{aligned} \tag{125}$$

(1) *If f_A is full, then*

(a)

$$\sigma_f = 2^U \setminus \Omega; \tag{126}$$

(b)

$$\Omega'' \subseteq \Omega' \subseteq \Omega'''; \tag{127}$$

(c)

$$2^U \setminus \tau_f' \subseteq \Omega', \qquad 2^U \setminus \tau_f'' \subseteq \Omega'', \qquad 2^U \setminus \tau_f''' \subseteq \Omega'''; \tag{128}$$

(d)

$$\begin{aligned} \tau_f' \cap \sigma_f' &= 2^U \setminus \Omega', \\ \tau_f'' \cap \sigma_f'' &= 2^U \setminus \Omega'', \\ \tau_f''' \cap \sigma_f''' &= 2^U \setminus \Omega'''. \end{aligned} \tag{129}$$

(2) *If f_A is full and keeping union, then*

(a) $2^U \setminus \sigma_f'' \subseteq 2^U \setminus \sigma_f' \subseteq 2^U \setminus \sigma_f''' \subseteq 2^U \setminus \sigma_f \subseteq \Omega$;
(b) $\sigma_f = \{\emptyset, U\} = 2^U \setminus \Omega$.

(3) *If f_A is full and keeping intersection, then*

(a) $\sigma_f = 2^U \setminus \Omega \subseteq \tau_f$.

Proof. These hold by Proposition 14 and Lemmas 53, 54, and 55. □

Theorem 57. *Let f_A be a soft set over U and let $P = (U, f_A)$ be a soft approximation space. Then, for $X \in 2^U$, one has*

$$\begin{aligned} Pos_P'(U - X) &= Neg_P'(X), \\ Pos_P''(U - X) &= Neg_P''(X), \\ Pos_P'''(U - X) &= Neg_P'''(X). \end{aligned} \tag{130}$$

Proof. This is obtained from Propositions 19, 20, and 21. □

By Proposition 14 and Theorem 27, we have Theorem 58.

Theorem 58. *Let f_A be a soft set over U, and let $P = (U, f_A)$ be a soft approximation space. Then, for $X \in 2^U$, one has the following.*

(1) *If f_A be a full, then*

$$Bnd_P''(X) \subseteq Bnd_P'(X) \subseteq Bnd_P'''(X). \tag{131}$$

(2) *If f_A is full and keeping union, then*

(a) $\overline{apr}_P''(X) - X \subseteq \overline{apr}_P'(X) - X \subseteq \overline{apr}_P'''(X) - X \subseteq Bnd_P(X)$;
(b) $Neg_P(X) = \emptyset$ *and* $Bnd_P(X) = U - Pos_P(X)$ *where* $X \neq \emptyset$.

Remark 59. Theorem 58 illustrates that soft P''-rough sets could provide a better approximation than soft P'-rough sets do and soft P'-rough sets could provide a better approximation than soft P'''-rough sets do.

6. A Correspondence Relationship

In this section, we give a one-to-one correspondence relationship in order to reveal the broad application prospect of soft sets.

Definition 60 (see [17]). Let U be a finite set of objects, let A be a finite set of attributes, and let I be a binary relation on from U to A. The triple (U, A, I) is called a formal context.

Let (U, A, I) be a formal context. For $u \in U$ and $a \in A$, uRa, which is also written as $(u, a) \in R$, means that the object u possesses the attribute a.

Denote

$$\{a\}^* = \{u \in U : uIa\}, \qquad \{u\}^* = \{a \in A : uIa\}. \tag{132}$$

Definition 61. Let FC $= (U, A, I)$ be a formal context. Define a mapping $f_{\mathrm{FC}} : A \to 2^U$ by

$$f_{\mathrm{FC}}(a) = \{a\}^* \tag{133}$$

for each $a \in A$. Then, $(f_{\mathrm{FC}})_A = (f_{\mathrm{FC}}, A)$ is called the soft set over U induced by FC. We denote $(f_{\mathrm{FC}})_A$ by S_{FC}.

Definition 62. Let $S = f_A$ be a soft set over U. Define a binary relation $I_S \in 2^U \times A$ by

$$(u, a) \in I_S \Longleftrightarrow u \in f(a) \tag{134}$$

for each $u \in U$ and each $a \in A$. Then, (U, A, I_S) is called the formal context induced by S. We denote (U, A, I_S) by FC_S.

Lemma 63. *Let $FC = (U, A, I)$ be a formal context, let S_{FC} be the soft set induced by FC, and let $FC_{S_{FC}}$ be the formal context induced by S_{FC}. Then,*

$$FC = FC_{S_{FC}}. \tag{135}$$

Proof. Obviously, $\mathrm{FC}_{S_{\mathrm{FC}}} = (U, A, I_{S_{\mathrm{FC}}})$.

For each $(u, a) \in U \times A$,

$$(u, a) \in I_{S_{\mathrm{FC}}} \Longleftrightarrow u \in f_{\mathrm{FC}}(a). \tag{136}$$

For each $a \in A$,

$$f_{\mathrm{FC}}(a) = \{a\}^* = \{u \in U : uIa\} = \{u \in U : (u, a) \in I\}. \tag{137}$$

Then,

$$u \in f_{\mathrm{FC}}(a) \Longleftrightarrow (u, a) \in I. \tag{138}$$

Thus, for each $(u, a) \in U \times A$, $(u, a) \in I_{S_{\mathrm{FC}}} \Leftrightarrow (u, a) \in I$. This implies $I_{S_{\mathrm{FC}}} = I$.

Hence,

$$\mathrm{FC} = \mathrm{FC}_{S_{\mathrm{FC}}}. \tag{139}$$

□

Lemma 64. *Let $S = f_A$ be a soft set over U, let FC_S be the formal context induced by S, and let*

$$S_{FC_S} \tag{140}$$

be the soft set induced by FC_S. Then,

$$S = S_{FC_S}. \tag{141}$$

Proof. Obviously, $S_{\mathrm{FC}_S} = (f_{\mathrm{FC}_S}, A)$.

Since $\mathrm{FC}_S = (U, A, I_S)$, then

$$f_{\mathrm{FC}_S}(a) = \{u \in U : (u, a) \in I_S\} \quad \text{for each } a \in A. \tag{142}$$

So, $(u, a) \in I_S \Leftrightarrow u \in f(a)$.

Obviously, $f(a) = \{u \in U : u \in f(a)\}$ for each $a \in A$.

Thus, for each $a \in A$, $f_{\mathrm{FC}}(a) = f(a)$.

Hence,

$$S = S_{\mathrm{FC}_S}. \tag{143}$$

□

Theorem 65. *Let*

$$\Gamma = \{S = f_A : S \text{ is a soft set over } U\},$$
$$\Sigma = \{FC = (U, A, I) : FC \text{ is a formal context}\}. \tag{144}$$

Then, there exists a one-to-one correspondence between Γ and Σ.

Proof. Two mappings $k : \Sigma \rightarrow \Gamma$ and $l : \Gamma \rightarrow \Sigma$ are defined as follows:

$$k(\mathrm{FC}) = S_{\mathrm{FC}}, \quad \text{for any } \mathrm{FC} = (U, A, I) \in \Sigma,$$
$$l(S) = \mathrm{FC}_S, \quad \text{for any } S = f_A \in \Gamma. \tag{145}$$

By Lemma 63,

$$l \circ k = i_\Sigma, \tag{146}$$

where $l \circ k$ is the composition of k and G, and i_Σ is the identity mapping on Σ.

By Lemma 64,

$$k \circ l = i_\Gamma, \tag{147}$$

where $l \circ F$ is the composition of l and k, and i_Γ is the identity mapping on Γ.

Hence, F and G are both a one-to-one correspondence. This prove that there exists a one-to-one correspondence between Σ and Γ. □

Remark 66. Theorem 65 illustrates that we can do formal concept analysis for soft sets or do soft analysis for formal contexts.

7. Conclusions

In this paper, we investigated soft rough approximation operators and the problems of combing soft sets with soft rough sets and topologies. Four pairs of soft rough approximation operators were considered, and their properties were given. Four sorts of soft rough sets are defined by using four pairs of soft rough approximation operators, and Pawlak's rough set models can be viewed as a special case of soft rough sets. We researched relationships among soft sets, soft rough sets and topologies, obtained the structure of soft rough sets, and revealed that every topological space on the initial universe is a soft approximating space. We may mention that soft rough sets can be used in object evaluation and group decision making. It should be noted that the use of soft rough sets could, to some extent, automatically reduce the noise factor caused by the subjective nature of the expert's evaluation. We will investigate these problems in future papers.

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