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Research Article

Stability in Terms of Two Measures for Nonlinear Impulsive Systems on Time Scales

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We investigate some stability problems in terms of two measures for nonlinear dynamic systems on time scales with fixed moments of impulsive effects. Sufficient conditions for (uniform) stability, (uniform) asymptotic stability, and instability in terms of two measures are derived by using the method of Lyapunov functions. Our results include the existing results as special cases when the time scale reduces to the set of real numbers. Particularly, our results provide stability criteria for impulsive discrete systems in terms of two measures, which have not been investigated extensively. Two examples are presented to illustrate the efficiency of the proposed results.

1. Introduction

It is well known that the theory of impulsive differential equations provides a general framework for mathematical modeling of many real world phenomena [1, 2]. In particular, it serves as an adequate mathematical tool for studying evolution processes that are subjected to abrupt changes in their states. At the present time, the qualitative theory of such equations has been extensively studied. Many results on the stability and boundedness of their solutions are obtained [1–4]. Due to the needs of applications, the concepts of Lyapunov stability have given rise to many new notions, for example, partial stability, conditional stability, eventual stability, practical stability, and so on. A notion which unifies and includes the above concepts of stability is the notion of stability in terms of two measures which was initialed by Movchan [5]. Since the publication of Salvadori's paper [6], this unified theory in terms of two measures became popular. For a systematic introduction to the theory of stability in terms of two measures, refer to [7].

On the other hand, a theory of time scales or calculus on measure chains was introduced by Hilger in his Ph.D. thesis [8] in 1988, with the purpose of incorporating both the existing theory of dynamic systems on continuous and discrete time scales, namely, time scale as arbitrary closed subset of real numbers, and extending the existing theory to dynamic systems on generalized hybrid (continuous/discrete) time scales. The theory of time scales recently has gained much attention and is undergoing rapid development. Recently, various work has been done on the stability problem of dynamic systems on time scales [9–14]. For more details about the theory of time scales, refer to [15–17].

Motivated by the above discussion, in this paper, we will consider the stability problems in terms of two measures for impulsive systems on time scales. Several new stability criteria and instability criteria are obtained by using the method of Lyapunov functions. As far as we know, there are very few studies on stability analysis of impulsive discrete systems in terms of two measures. Moreover, our results can be applied to impulsive systems on other time scales in addition to the set of integers and the set of real numbers.

The rest of this paper is organized as follows. In Section 2, we introduce some basic knowledge of dynamic systems on time scales. In Section 3, we formulate the problem and present several definitions of stability and instability in terms of two measures. In Section 4, several

(h_0, h)-stability and instability criteria are established by employing the Lyapunov function approach. For illustration of our results, two examples are shown in Section 5. Finally, some conclusions are drawn in Section 6.

2. Preliminaries

In this section, we briefly introduce some basic definitions and results concerning time scales for later use.

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^+$ be the set of nonnegative real numbers, $\mathbb{Z}$ be the set of integers, $\mathbb{Z}^+$ be the set of nonnegative integers, $\mathbb{N} = \{1, 2, \ldots\}$, and $\mathbb{T}$ be an arbitrary nonempty closed subset of $\mathbb{R}$. We assume that $\mathbb{T}$ is a topological space with relative topology induced from $\mathbb{R}$. Then, $\mathbb{T}$ is called a time scale.

Definition 1. The mappings $\sigma, \theta : \mathbb{T} \to \mathbb{T}$ defined as

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$
$$\theta(t) = \sup\{s \in \mathbb{T} : s < t\} \quad (1)$$

are called forward and backward jump operators, respectively.

A nonmaximal element $t \in \mathbb{T}$ is called right-scattered (rs) if $\sigma(t) > t$ and right-dense (rd) if $\sigma(t) = t$. A nonminimal element $t \in \mathbb{T}$ is called left-scattered (ls) if $\theta(t) < t$ and left-dense (ld) if $\theta(t) = t$. If $\mathbb{T}$ has a ls maximum m, then we define $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$, otherwise, $\mathbb{T}^k = \mathbb{T}$.

Definition 2. The graininess function $\mu : \mathbb{T} \to \mathbb{R}^+$ is defined by

$$\mu(t) = \sigma(t) - t. \quad (2)$$

Definition 3. For $y : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, one defines the delta derivative $y^{\Delta}(t)$ of $y(t)$, to be the number (when it exists) with the property that for any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \bigcap \mathbb{T}$ for some $\delta > 0$) such that

$$\left| y(\sigma(t)) - y(s) - y^{\Delta}(t)(\sigma(t) - s) \right| \le \varepsilon |\sigma(t) - s|, \quad \forall s \in U. \quad (3)$$

A function $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous if it is continuous at rd points in $\mathbb{T}$ and its left-side limits exist at ld points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{\mathrm{rd}} = C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$. If f is continuous at each rd point and each ld point, f is said to be continuous function on $\mathbb{T}$. If $a, b \in \mathbb{T}$, then one defines the interval $[a, b]$ on $\mathbb{T}$ by $[a, b] := \{t \in \mathbb{T} : a \le t \le b\}$. Open intervals and half-open intervals can be defined similarly.

Definition 4. Let $f \in C_{\mathrm{rd}}$. A function $g : \mathbb{T} \to \mathbb{R}$ is called the antiderivative of f on $\mathbb{T}$ if it is differentiable on $\mathbb{T}$ and satisfies $g^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}$. In this case, one defines

$$\int_a^t f(s)\,\Delta s = g(t) - g(a), \quad a, t \in \mathbb{T}. \quad (4)$$

One says that a function $p : \mathbb{T} \to \mathbb{R}$ is regressive if $1 + \mu(t)p(t) \ne 0$ for all $t \in \mathbb{T}$. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{\mathrm{rd}}\mathscr{R} = C_{\mathrm{rd}}\mathscr{R}(\mathbb{T}, \mathbb{R})$, and the set of all positively regressive elements of $C_{\mathrm{rd}}\mathscr{R}$ is denoted by $C_{\mathrm{rd}}\mathscr{R}^+ = C_{\mathrm{rd}}\mathscr{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in C_{\mathrm{rd}}\mathscr{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Definition 5. If $p \in C_{\mathrm{rd}}\mathscr{R}$, then one defines the exponential function on time scale $\mathbb{T}$ by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\,\Delta\tau \right), \quad \text{for } t, s \in \mathbb{T}, \quad (5)$$

where the cylinder transformation

$$\xi_h(z) = \begin{cases} \dfrac{\mathrm{Log}(1 + hz)}{h}, & h \ne 0, \\ z, & h = 0, \end{cases} \quad (6)$$

where Log is the principal logarithm function.

It is known that $x(t) = e_p(t, t_0)$ is the unique solution of the initial value problem $x^{\Delta}(t) = p(t)x(t)$, $x(t_0) = 1$.

Remark 6. Let $\alpha \in C_{\mathrm{rd}}\mathscr{R}$ be a constant. If $\mathbb{T} = \mathbb{Z}$, then $e_\alpha(t, t_0) = (1 + \alpha)^{t - t_0}$ for all $t \in \mathbb{T}$. If $\mathbb{T} = \mathbb{R}$, then $e_\alpha(t, t_0) = e^{\alpha(t - t_0)}$ for all $t \in \mathbb{T}$.

Definition 7. One says that a function $m : \mathbb{T} \to \mathbb{R}$ is right-nondecreasing at a point $t \in \mathbb{T}$ provided

(i) if t is rs, then $m(\sigma(t)) \ge m(t)$;

(ii) if t is rd, then there is a neighborhood U of t such that

$$m(s) \ge m(t), \quad \forall s \in U \text{ with } s > t. \quad (7)$$

Similarly, one says that m is right-nonincreasing if above in (i) $m(\sigma(t)) \le m(t)$ and in (ii) $m(s) \le m(t)$. If m is right-nondecreasing (right-nonincreasing) at every $t \in \mathbb{T}$, one says that m is right-nondecreasing (right-nonincreasing) on $\mathbb{T}$.

Lemma 8. *Let $m \in C(\mathbb{T}, \mathbb{R})$. Then $m(t)$ is right-nondecreasing (right-nonincreasing) on $\mathbb{T}$ if and only if $D^+ m^{\Delta}(t) \ge 0$ ($D^+ m^{\Delta}(t) \le 0$) for every $t \in \mathbb{T}$, where*

$$D^+ m^{\Delta}(t) = \begin{cases} \dfrac{m(\sigma(t)) - m(t)}{\mu(t)}, & \sigma(t) > t, \\ \limsup\limits_{s \to t^+} \dfrac{m(s) - m(t)}{s - t}, & \sigma(t) = t. \end{cases} \quad (8)$$

Proof. The condition is obviously necessary. Let us prove that it is sufficient. We only assume $D^+ m^{\Delta}(t) \ge 0$ for $t \in \mathbb{T}$ as the second statement can be shown similarly.

If t is rs, then

$$D^+ m^{\Delta}(t) = \frac{m(\sigma(t)) - m(t)}{\mu(t)} \ge 0, \quad (9)$$

and hence $m(\sigma(t)) \ge m(t)$.

Let now t to be rd, and N be a neighborhood of t. We need to show that $m(s) \ge m(t)$ for $s > t$ with $s \in N$. This follows directly from Lemma 1.1.1 in [7].

Thus the proof of the lemma is complete. □

3. Problem Formulation

Consider the following nonlinear impulsive system on time scale $\mathbb{T}$:

$$x^{\Delta}(t) = f(t,x), \quad t \neq t_k,\ t \in \mathbb{T},$$
$$\Delta x(t) = I_k(x(t)), \quad t = t_k,\ k \in \mathbb{N}, \tag{10}$$
$$x(t_0^+) = x_0$$

under the following assumptions.

(a) $\mathbb{T}$ is a time scale with $t_0 \geq 0$ as minimal element and no maximal element.

(b) $\{t_k\} \in \mathbb{T}$, $t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim_{k\to\infty} t_k = \infty$.

(c) $x \in \mathbb{R}^n$ and $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$. If t_k is rd point, $x(t_k^+)$ denotes the right limit of x at t_k; if t_k is rs point, $x(t_k^+)$ denotes the state of x at t_k with the impulse. If t_k is ld point, $x(t_k^-)$ denotes the left limit of x at t_k with $x(t_k^-) = x(t_k)$ if t_k is ls point. Here, we assume that $x(t_k^-) = x(t_k)$.

(d) $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in $(t_{k-1}, t_k] \times \mathbb{R}^n$ for $k \in \mathbb{N}$, $f(t,0) = 0$, and for each $x \in \mathbb{R}^n$, $k \in \mathbb{N}$, $\lim_{(t,y)\to(t_k^+,x)} f(t,y) = f(t_k^+, x)$;

(e) $I_k : \mathbb{R}^n \to \mathbb{R}^n$ and $I_k(0) = 0$.

Throughout this paper, we denote by $x(t) = x(t;t_0,x_0)$ the solution of system (10) satisfying initial condition $x(t_0^+) = x_0$. Obviously, system (10) admits the trivial solution. Moreover, f is assumed to satisfy necessary assumptions so that the following initial value problems:

$$x^{\Delta} = f(t,x), \quad t \in [t_0, t_1],$$
$$x(t_0) = x_0,$$
$$x^{\Delta} = f(t,x), \quad t \in (t_k, t_{k+1}], \tag{11}$$
$$x(t_k^+) = x(t_k) + I_k(x(t_k)),$$

have unique solutions $x_0(t)$, $t \in [t_0, t_1]$, and $x_k(t)$, $t \in (t_k, t_{k+1}]$, $k \in \mathbb{N}$, respectively (e.g., see [17] for existence and uniqueness results for dynamical systems on time scales.). Thus, if we define

$$x(t;t_0,x_0) = \begin{cases} x_0(t), & t \in [t_0,t_1] \\ x_1(t), & t \in (t_1,t_2] \\ \vdots & \vdots \\ x_k(t), & t \in (t_k,t_{k+1}] \\ \vdots & \vdots \end{cases} \tag{12}$$

then it is easy to see that $x(t;t_0,x_0)$ is the unique solution of system (10).

Let us list the classes of functions and definitions for convenience.

$PC = \{\delta : \mathbb{T} \to \mathbb{R}^+$, continuous on $(t_{k-1}, t_k]$ and $\lim_{t\to t_k^+} \delta(t) = \delta(t_k^+)$ exists for $k \in \mathbb{N}\}$;

$\mathscr{K} = \{\delta \in C(\mathbb{R}^+, \mathbb{R}^+)$, strictly increasing and $\delta(0) = 0\}$;

$PC\mathscr{K} = \{\delta : \mathbb{T} \times \mathbb{R}^+ \to \mathbb{R}^+, \delta(\cdot,u) \in PC$ for each $u \in \mathbb{R}^+$ and $\delta(t,\cdot) \in \mathscr{K}$ for each $t \in \mathbb{T}\}$;

$\Gamma = \{h : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^+, h(\cdot,x) \in PC$ for each $x \in \mathbb{R}^n$, $h(t,\cdot) \in C(\mathbb{R}^n, \mathbb{R}^+)$ for each $t \in \mathbb{T}$ and $\inf h(t,x) = 0\}$;

$\nu_0 = \{V : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^+$, continuous on $(t_{k-1}, t_k] \times \mathbb{R}^n$, $k \in \mathbb{N}$, and for all $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$, $\lim_{(t,y)\to(t_k^+,x)} V(t,y) = v(t_k^+, x)$ exists$\}$.

For $V \in \nu_0$, $(t,x) \in (t_{k-1}, t_k] \times \mathbb{R}^n$, $k \in \mathbb{N}$, we define the upper right-hand Dini delta derivative of $V(t,x)$ relative to (10) as follows:

$$D^+V^{\Delta}(t,x) = \begin{cases} \dfrac{V(\sigma(t), x(\sigma(t))) - V(t, x(t))}{\mu(t)}, & \sigma(t) > t, \\ \limsup\limits_{s\to t^+} \dfrac{V(s, x(t) + (s-t) f(t,x(t))) - V(t,x(t))}{s-t}, & \sigma(t) = t. \end{cases} \tag{13}$$

Definition 9. Let $h_0, h \in \Gamma$. Then one says that

(i) h_0 is finer than h if there exists a constant $\rho > 0$ and a function $\varphi \in \mathscr{K}$ such that $h_0(t,x) < \rho$ implies $h(t,x) \leq \varphi(h_0(t,x))$;

(ii) h_0 is weakly finer than h if there exists a constant $\rho > 0$ and a function $\varphi \in PC\mathscr{K}$ such that $h_0(t,x) < \rho$ implies $h(t,x) \leq \varphi(t, h_0(t,x))$.

Definition 10. Let $V \in \nu_0$ and $h_0, h \in \Gamma$. Then $V(t,x)$ is said to be

(i) h-positive definite if there exist a $\rho > 0$ and a function $b \in \mathscr{K}$ such that $h(t,x) < \rho$ implies $b(h(t,x)) \leq V(t,x)$;

(ii) h_0-decrescent if there exist a $\rho > 0$ and a function $a \in \mathscr{K}$ such that $h_0(t,x) < \rho$ implies $V(t,x) \leq a(h_0(t,x))$;

(iii) h_0-weakly decrescent if there exist a $\rho > 0$ and a function $a \in PC\mathscr{K}$ such that $h_0(t,x) < \rho$ implies $V(t,x) \leq a(t, h_0(t,x))$.

Definition 11. The impulsive system (10) is said to be

(S_1) (h_0,h)-stable, if for each $\varepsilon > 0$, $t_0 \in \mathbb{T}$, there exists a $\delta = \delta(t_0,\varepsilon) > 0$ such that $h_0(t_0,x_0) < \delta$ implies $h(t,x(t)) < \varepsilon$, $t \geq t_0$ for any solution $x(t) = x(t;t_0,x_0)$ of (10);

(S_2) (h_0,h)-uniformly stable, if the δ in (S_1) is independent of t_0;

(S_3) (h_0,h)-attractive, if for each $\varepsilon > 0$, $t_0 \in \mathbb{T}$, there exist two positive constants $\delta = \delta(t_0,\varepsilon)$ and $T = T(t_0,\varepsilon)$ such that $h_0(t_0,x_0) < \delta$ implies $h(t,x(t)) < \varepsilon$, $t \geq t_0 + T$;

(S_4) (h_0, h)-uniformly attractive, if (S_3) holds with δ and T being independent of t_0;

(S_5) (h_0, h)-asymptotically stable, if (S_1) and (S_3) hold simultaneously;

(S_6) (h_0, h)-uniformly asymptotically stable, if (S_2) and (S_4) hold together;

(S_7) (h_0, h)-unstable, if (S_1) fails to hold.

4. Main Results

Let us establish, in this section, sufficient conditions for (h_0, h)-(uniform) stability, (h_0, h)-(uniform) asymptotic stability, and (h_0, h)-instability properties of impulsive systems (10) in the following subsections, respectively. Let

$$S(h, \rho) = \{(t, x) \in \mathbb{T} \times \mathbb{R}^n : h(t, x) < \rho\}. \quad (14)$$

4.1. (h_0,h)-(Uniform) Stability

Theorem 12. *Assume that*

(i) $h, h_0 \in \Gamma$, *and* h_0 *is weakly finer than* h;

(ii) $V \in \nu_0$, $V(t, x)$ *is* h*-positive definite on* $S(h, \rho)$, h_0*-weakly decrescent, locally Lipschitz in* x *for* $t \in \mathbb{T}$ *which is rd, and*

$$D^+ V^{\Delta}(t, x) \le \varsigma_k V(t, x), \quad t \in (t_k, t_{k+1}), \quad (t, x) \in S(h, \rho), \quad (15)$$

where $\varsigma_k \ge 0$, $k \in \mathbb{Z}^+$;

(iii) *there exists* $\varsigma \ge \varsigma_k > 0$, *such that*

$$V(t_k^+, x_k + I_k(x_k)) \le \varsigma_k V(t_k, x_k), \quad (t_k, x_k) \in S(h, \rho), \quad (16)$$

where $x_k = x(t_k)$, $k \in \mathbb{N}$;

(iv) $\sup_{k \in \mathbb{Z}^+}\{\prod_{i=0}^{k} \varsigma_i e_{\varsigma_i}(t_{i+1}, t_i)\} = M < \infty$ *where* $\varsigma_0 = 1$;

(v) *there exists a constant* ρ_1, $0 < \rho_1 < \rho$, *such that* $h(t, x) < \rho_1$ *implies* $h(\sigma(t), x(\sigma(t))) < \rho$;

(vi) *there exists a constant* ρ_0, $0 < \rho_0 < \rho_1$, *such that* $h(t_k, x_k) < \rho_0$ *implies* $h(t_k^+, x_k + I_k(x_k)) < \rho_1$.

Then system (10) *is* (h_0, h)*-stable.*

Proof. Since $V(t, x)$ is h_0-weakly decrescent, there exist a constant $\delta_0 > 0$ and a function $a \in PC\mathcal{K}$ such that

$$V(t, x) \le a(t, h_0(t, x)), \quad \text{if } h_0(t, x) < \delta_0. \quad (17)$$

There exists, in view of (ii), a function $b \in \mathcal{K}$ such that

$$b(h(t, x)) \le V(t, x), \quad \text{if } h(t, x) < \rho. \quad (18)$$

By (i), there exist $\delta_1 > 0$ and $\varphi \in PC\mathcal{K}$ such that

$$h(t, x) \le \varphi(t, h_0(t, x)), \quad \text{if } h_0(t, x) < \delta_1. \quad (19)$$

Let $\varepsilon \in (0, \rho_0)$ and $t_0 \in \mathbb{T}$ be given. There exists $\delta_2 = \delta_2(t_0, \varepsilon)$ such that

$$\varphi(t_0, \delta_2) < \rho_0, \quad \max\{1, M\} a(t_0, \delta_2) < b(\varepsilon). \quad (20)$$

Choose $\delta = \min\{\delta_0, \delta_1, \delta_2\}$. Let $(t_0, x_0) \in \mathbb{T} \times \mathbb{R}^n$ such that $h_0(t_0, x_0) < \delta$ and $x(t) = x(t; t_0, x_0)$ be any solution of (10). Then, from (17) to (20), we get

$$b(h(t_0, x_0)) \le V(t_0, x_0) \le a(t_0, h_0(t_0, x_0)) < b(\varepsilon), \quad (21)$$

which implies $h(t_0, x_0) < \varepsilon$.

We now claim that, for every solution $x(t) = x(t; t_0, x_0)$ of (10), $h_0(t_0, x_0) < \delta$ implies

$$h(t, x(t)) < \varepsilon, \quad t \ge t_0. \quad (22)$$

If this is not true, then there exist a solution $x(t)$ with $h_0(t_0, x_0) < \delta$ and a $t^* > t_0$ such that $t_k < t^* \le t_{k+1}$, for some k, satisfying

$$h(t^*, x(t^*)) \ge \varepsilon, \quad h(t, x(t)) < \varepsilon \quad \text{for } t \in [t_0, t_k]. \quad (23)$$

Since $0 < \varepsilon < \rho_0$, it follows form condition (vi) that

$$h(t_k^+, x_k^+) := h(t_k^+, x_k + I_k(x_k)) < \rho_1, \quad (24)$$

where $x_k^+ = x(t_k^+)$ and $h(t_k, x_k) < \varepsilon$. Next, we will show that there exists a t^0, $t_k < t^0 \le t^*$, such that

$$\varepsilon \le h(t^0, x(t^0)) < \rho, \quad h(t, x(t)) < \rho \quad \text{for } t \in [t_0, t^0]. \quad (25)$$

To do this, we consider the following two cases:

(1) there exists a t_*, $t_k < t_* \le t^*$, such that $h(t_*, x(t_*)) \ge \rho$;

(2) $h(t, x(t)) < \rho$ for all $t \in (t_k, t^*]$.

Case 1. Let $\bar{t} = \inf\{t \in (t_k, t^*], h(t, x(t)) \ge \rho\}$. As $h(t_k^+, x_k^+) < \rho_1 < \rho$, we know that $t_k < \bar{t} \le t^*$. If $\bar{t}$ is left-dense, from the selection of $\bar{t}$, we know that there exists a left-hand neighborhood $U_\epsilon = (\bar{t} - \epsilon, \bar{t}) \subset (t_k, t^*]$ for some $\epsilon > 0$, such that $\rho_1 < h(t, x(t)) < \rho$ for all $t \in U_\epsilon$. Then, we can choose $t^0 \in U_\epsilon$.

If $\bar{t}$ is left-scattered, from the selection of $\bar{t}$ and $h(t_k^+, x_k^+) < \rho_1$, we know that $t_k < \theta(\bar{t}) < t^*$ and $h(\theta(\bar{t}), x(\theta(\bar{t}))) < \rho$. Here, we claim that $h(\theta(\bar{t}), x(\theta(\bar{t}))) \ge \rho_1$. If this is not true, that is, $h(\theta(\bar{t}), x(\theta(\bar{t}))) < \rho_1$, form condition (v), we know that

$$h(\bar{t}, x(\bar{t})) = h(\sigma(\theta(\bar{t})), x(\sigma(\theta(\bar{t})))) < \rho, \quad (26)$$

which is a contradiction. Thus, $\rho_1 \le h(\theta(\bar{t}), x(\theta(\bar{t}))) < \rho$. Then, we can choose $t^0 = \theta(\bar{t})$.

Case 2. If $h(t, x(t)) < \rho$ for all $t \in (t_k, t^*]$, then we can choose $t^0 = t^*$.

Hence, we can find a t^0, $t_k < t^0 \le t^*$, such that (25) holds.

For $t_0 \le t \le t^0$ and by conditions (ii) and (iii), we obtain

$$D^+V^{\Delta}(t,x) \le c_i V(t,x), \quad t \in (t_i, t_{i+1}), \ t \le t^0, \tag{27}$$

$$V(t_i^+, x_i + I_i(x_i)) \le \varsigma_i V(t_i, x_i), \quad t_0 < t_i < t^0. \tag{28}$$

By (27), we will show that

$$V(t,x) \le e_{c_0}(t,t_0) V(t_0,x_0), \quad t \in [t_0, t_1]. \tag{29}$$

To do this, we apply the induction principle ([17], Theorem 1.7) on $[t_0, t_1]$ to the statement

$$A(t): V(t,x) \le e_{c_0}(t,t_0) V(t_0,x_0). \tag{30}$$

(1) The statement $A(t_0)$ is true since $e_{c_0}(t_0,t_0)V(t_0,x_0) = V(t_0,x_0)$.

(2) Let t be rs and $A(t)$ be true. We have to prove that $A(\sigma(t))$ is true.

By the definition of upper right-hand derivative, we see that

$$\begin{aligned} D^+V^{\Delta}(t,x) &= \frac{V(\sigma(t), x(\sigma(t))) - V(t, x(t))}{\mu(t)} \\ &\le c_0 V(t,x), \end{aligned} \tag{31}$$

then

$$\begin{aligned} V(\sigma(t), x(\sigma(t))) &\le (1 + c_0\mu(t)) V(t,x) \\ &= e_{c_0}(\sigma(t), t) V(t,x) \\ &\le e_{c_0}(\sigma(t), t) e_{c_0}(t,t_0) V(t_0,x_0) \\ &= e_{c_0}(\sigma(t), t_0) V(t_0,x_0) \end{aligned} \tag{32}$$

which implies that $A(\sigma(t))$ is true.

(3) Let t be rd, $A(t)$ be true and N be a neighborhood of t. We need to show that $A(s)$ is true for $s > t$, $s \in N$. By (27) and Remark 6, we get

$$\begin{aligned} V(s, x(s)) &\le e^{c_0(s-t)} V(t,x) \\ &\le e^{c_0(s-t)} e_{c_0}(t,t_0) V(t_0,x_0) \\ &= e_{c_0}(s,t) e_{c_0}(t,t_0) V(t_0,x_0) \\ &= e_{c_0}(s,t_0) V(t_0,x_0) \end{aligned} \tag{33}$$

which implies that $A(s)$ is true.

(4) Let t be ld and $A(s)$ be true for all $s < t$. We need to show that $A(t)$ is true. By the continuous property of function V and the exponential function, it follows that

$$\begin{aligned} V(t,x) &= \lim_{s \to t^-} V(s, x(s)) \\ &\le \lim_{s \to t^-} e_{c_0}(s,t_0) V(t_0,x_0) \\ &= e_{c_0}(t,t_0) V(t_0,x_0) \end{aligned} \tag{34}$$

which implies that $A(t)$ is true.

Hence, we conclude that (29) is true.

Similarly, we can prove that

$$V(t,x) \le e_{c_i}(t,t_i) V(t_i^+, x(t_i^+)), \quad \text{for } t \in (t_i, t_{i+1}], \ t \le t^0. \tag{35}$$

Then, by (28), (35), and (20), we obtain

$$\begin{aligned} V(t^0, x(t^0)) &\le e_{c_k}(t^0, t_k) V(t_k^+, x(t_k^+)) \\ &\le e_{c_k}(t^0, t_k) \varsigma_k V(t_k, x(t_k)) \\ &\le e_{c_k}(t^0, t_k) \varsigma_k e_{c_{k-1}}(t_k, t_{k-1}) V(t_{k-1}^+, x(t_{k-1}^+)) \\ &\le \cdots \\ &\le e_{c_k}(t^0, t_k) \varsigma_k e_{c_{k-1}}(t_k, t_{k-1}) \varsigma_{k-1} \cdots e_{c_1}(t_2, t_1) \\ &\quad \times \varsigma_1 e_{c_0}(t_1, t_0) V(t_0, x_0) \\ &\le \prod_{i=0}^{k} \varsigma_i e_{c_i}(t_{i+1}, t_i) V(t_0, x_0) \le MV(t_0, x_0) \\ &\le Ma(t_0, h_0(t_0, x_0)) \le Ma(t_0, \delta) < b(\varepsilon) \end{aligned} \tag{36}$$

that is, $V(t^0, x(t^0)) < b(\varepsilon)$. Thus, by (18) and (25),

$$b(\varepsilon) \le b(h(t^0, x(t^0))) \le V(t^0, x(t^0)) < b(\varepsilon) \tag{37}$$

which is a contradiction. Therefore (22) is true and system (10) is (h_0, h)-stable. □

Theorem 13. *Assume that all conditions of Theorem 12 hold with the following changes:*

(i)* h_0 *is finer than* h;

(ii)* $V(t,x)$ *is* h_0-*decrescent.*

Then, system (10) *is* (h_0, h)-*uniformly stable.*

Proof. From conditions (i)* and (ii)*, the number δ in the proof of Theorem 12 can be chosen independent of t_0. Then following the same reasoning of Theorem 12, we can get the (h_0, h)-uniform stability of system (10). The details are omitted. □

If $c_k \equiv 0$ in condition (ii) of the previous theorems, then the Lyapunov function V is monotone along the solutions of system (10) in each impulsive intervals. In this case, we have the following conservative result.

Corollary 14. *Assume that*

(i) *$h_0, h \in \Gamma$, $V \in \nu_0$, $V(t,x)$ is h-positive definite on $S(h,\rho)$, locally Lipschtiz in x for each $t \in \mathbb{T}$ which is rd, and $D^+V^{\Delta}(t,x) \le 0$ for $t \ne t_k$ and $(t,x) \in S(h,\rho)$;*

(ii) *$V(t_k^+, x_k + I_k(x_k)) - V(t_k, x_k) \le d_k V(t_k, x_k)$ for $(t_k, x_k) \in S(h,\rho)$, where $d_k \ge 0$ and $\sum_{i=1}^{\infty} d_i < \infty$,* *and conditions (v), (vi) of Theorem 12 hold. Then*

(A) *if, in addition, h_0 is weakly finer than h, and $V(t,x)$ is h_0-weakly decrescent, then system* (10) *is (h_0,h)-stable;*

(B) *if, in addition, h_0 is finer than h, and $V(t,x)$ is h_0-decrescent, then system* (10) *is (h_0,h)-uniformly stable.*

Proof. Notice

$$\sup_{k\in\mathbb{Z}^+}\left\{\prod_{i=0}^{k}\varsigma_i e_{c_i}(t_{i+1},t_i)\right\} = \sup_{k\in\mathbb{Z}^+}\left\{\prod_{i=0}^{k}(1+d_i)\right\} \le \prod_{i=1}^{\infty}(1+d_i) < \infty, \tag{38}$$

where $d_0 = 0$. Then, by Theorems 12 and 13, the result holds. □

Remark 15. The continuous version of Corollary 14 with $d_k \equiv 0$, $k \in \mathbb{N}$, can be found in [7], while the discrete one for impulsive discrete systems is brand new, and the discrete version of Corollary 14 with $V(t_k^+, x_k + I_k(x_k)) = V(t_k, x(t_k))$, $k \in \mathbb{N}$, reduces to Theorems 3.1 and 3.2 in [18] for discrete systems with no impulses.

4.2. (h_0,h)-(Uniform) Asymptotic Stability

Theorem 16. *Assume that conditions (v), (vi) of Theorem 12 and condition (i) of Corollary 14 hold and the following conditions are satisfied:*

(i) *h_0 is weakly finer than h, and $V(t,x)$ is h_0-weakly decrescent;*

(ii) *$V(t_k^+, x_k + I_k(x_k)) - V(t_k, x_k) \le -\lambda_k \psi(V(t_k, x_k))$ for $(t_k, x_k) \in S(h,\rho)$, where $\lambda_k \ge 0$, $\sum_{k=1}^{\infty}\lambda_k = \infty$, $\psi : \mathbb{R}^+ \to \mathbb{R}^+$, $\psi(0) = 0$ and $\psi(s) > 0$ if $s > 0$.*

Then system (10) *is (h_0,h)-asymptotically stable.*

Proof. By Theorem 12, system (10) is (h_0,h)-stable. Thus, for $\rho > 0$, there exists $\delta = \delta(t_0,\rho) > 0$ such that $h_0(t_0,x_0) < \delta$ implies $h(t,x) < \rho$, $t \ge t_0$. To prove the theorem, it remains to show that $\lim_{t\to\infty} h(t,x(t)) = 0$.

Let $m(t) = V(t,x(t))$. Then it follows from assumptions that $m(t)$ is right-nonincreasing and bounded from below, and consequently $\lim_{t\to\infty} m(t) = \omega \ge 0$ exists. If $\omega > 0$ for some solution $x(t) = x(t; t_0, x_0)$ of (10), we let $\gamma = \min_{\omega\le s\le m(t_0)} \psi(s)$. Then, by condition (ii), we have

$$m(t_k^+) - m(t_k) \le -\lambda_k \psi(m(t_k)) \le -\lambda_k \gamma. \tag{39}$$

Thus we obtain from (39) that

$$\begin{aligned} m(t_k^+) &\le m(t_k) - \lambda_k\gamma \le m(t_{k-1}^+) - \lambda_k\gamma \\ &\le m(t_{k-1}) - \lambda_{k-1}\gamma - \lambda_k\gamma \\ &\le \cdots \\ &\le m(t_1) - \gamma\sum_{j=1}^{k}\lambda_j, \end{aligned} \tag{40}$$

which implies, in view of the assumption $\sum_{j=1}^{k}\lambda_j = \infty$, that $\lim_{k\to\infty} m(t_k^+) = -\infty$. This is a contradiction. Thus we must have $\omega = 0$ and consequently $\lim_{t\to\infty} h(t,x(t)) = 0$. Hence system (10) is (h_0,h)-attractive and the proof is complete. □

In the following theorems, two auxiliary functions of class ν_0 are used to investigate the (h_0,h)-asymptotic stability property of system (10).

Theorem 17. *Let conditions (v), (vi) of Theorem 12 hold and assume that*

(i) *$h_0, h \in \Gamma$ and h_0 is weakly finer than h;*

(ii) *there exists a function $V \in \nu_0$ such that $V(t,x)$ is locally Lipschtiz in x for each $t \in \mathbb{T}$ which is rd, h-positive definite on $S(h,\rho)$, h_0-weakly decrescent and*

$$D^+V^{\Delta}(t,x) \le -c(W(t,x)), \quad t \ne t_k,\ (t,x) \in S(h,\rho), \tag{41}$$

where $c \in \mathscr{K}$, $W \in \nu_0$;

(iii) *$V(t_k^+, x_k + I_k(x_k)) - V(t_k, x_k) \le d_k V(t_k, x_k)$, $(t_k, x_k) \in S(h,\rho)$, where $d_k \ge 0$ and $\sum_{i=1}^{\infty} d_i < \infty$;*

(iv) *$W(t,x)$ is h-positive definite on $S(h,\rho)$, locally Lipschtiz in x for every $t \in \mathbb{T}$ which is rd and*

$$\begin{aligned} &D^+W^{\Delta}(t,x) \le 0, \quad t \ne t_k,\ (t,x) \in S(h,\rho), \\ &W(t_k^+, x_k + I_k(x_k)) - W(t_k, x_k) \\ &\quad \le \bar{d}_k W(t_k, x_k), \quad (t_k, x_k) \in S(h,\rho), \end{aligned} \tag{42}$$

where $\bar{d}_k \ge 0$ and $\sum_{i=1}^{\infty}\bar{d}_i < \infty$.

Then system (10) *is (h_0,h)-asymptotically stable.*

Proof. From Corollary 14, it follows that system (10) is (h_0,h)-stable. Thus, for $\rho > 0$, there exists $\delta = \delta(t_0,\rho) > 0$ such that $h_0(t_0,x_0) < \delta$ implies $h(t,x) < \rho$, $t \ge t_0$. To prove the theorem, it remains to show that for every solution $x(t)$ of (10) with $h_0(t_0,x_0) < \delta$, $\lim_{t\to\infty} h(t,x) = 0$.

Suppose that this is not true. Then there exists a sequence $\{\xi_i\}_{i=1}^{\infty}$ diverging to ∞ as $i \to \infty$ and such that $h(\xi_i, x(\xi_i)) \ge r$ $(i \in \mathbb{N})$ for some positive number r. From condition (iv), we know that there exists a function $b \in \mathscr{K}$ such that $b(h(t,x)) \le W(t,x)$, if $h(t,x) < \rho$. Then

$$W(\xi_i, x(\xi_i)) > b(r), \quad i \in \mathbb{N}. \tag{43}$$

For any given $t \in \mathbb{T}$, there exists a $i > 0$ such that $t \in (\xi_i, \xi_{i+1}]$. Then for ξ_{i+1}, there exists a $k > 0$ such that $\xi_{i+1} \in (t_k, t_{k+1}]$. If $t_k < t \le t_{k+1}$, from (43) and condition (iv), we have

$$W(t,x) \ge W(\xi_{i+1}, x(\xi_{i+1})) > b(r) \ge \frac{b(r)}{M_0}, \tag{44}$$

where $M_0 = \prod_{j=1}^{\infty}(1+\overline{d}_j) < \infty$. If $t_{k-1} < t \le t_k$, we have

$$W(t,x) \ge W(t_k, x_k) \ge \frac{1}{1+\overline{d}_k} W(t_k^+, x(t_k^+)) > \frac{1}{1+\overline{d}_k} b(r) \ge \frac{b(r)}{M_0}. \tag{45}$$

Following this procedure, we conclude that

$$W(t,x) > \frac{b(r)}{M_0}, \quad t \in \mathbb{T}. \tag{46}$$

Let

$$L(t,x) = V(t,x) + \int_{t_k}^{t} c(W(s,x(s)))\Delta s, \quad t \in (t_k, t_{k+1}], \ k \in \mathbb{Z}^+. \tag{47}$$

Then, by condition (ii),

$$D^+L^{\Delta}(t,x) = D^+V^{\Delta}(t,x) + c(W(t,x)) \le 0, \quad t \ne t_k, \tag{48}$$

which implies

$$V(t,x) \le V(t_k^+, x(t_k^+)) - \int_{t_k}^{t} c(W(s,x(s)))\Delta s, \quad t \in (t_k, t_{k+1}], \tag{49}$$

for $k \in \mathbb{Z}^+$.

Hence, for $\xi_{i+1} \in (t_k, t_{k+1}]$, we obtain, form (46), (49), and condition (iii),

$$\begin{aligned} V(\xi_{i+1}, x(\xi_{i+1})) &\le V(t_k^+, x(t_k^+)) - \int_{t_k}^{\xi_{i+1}} c(W(s,x(s)))\Delta s \\ &\le (1+d_k) V(t_k, x(t_k)) - \int_{t_k}^{\xi_{i+1}} c(W(s,x(s)))\Delta s \\ &\le (1+d_k)\left(V(t_{k-1}^+, x(t_{k-1}^+)) - \int_{t_{k-1}}^{t_k} c(W(s,x(s)))\Delta s\right) - \int_{t_k}^{\xi_{i+1}} c(W(s,x(s)))\Delta s \\ &\le (1+d_k)(1+d_{k-1}) V(t_{k-1}, x(t_{k-1})) - \int_{t_{k-1}}^{\xi_{i+1}} c(W(s,x(s)))\Delta s \\ &\le \cdots \\ &\le \prod_{j=1}^{k}(1+d_j) V(t_0,x_0) - \int_{t_0}^{\xi_{i+1}} c(W(s,x(s)))\Delta s \\ &\le MV(t_0,x_0) - c\left(\frac{b(r)}{M_0}\right)(\xi_{i+1}-t_0) \to -\infty, \quad \text{for } i \to \infty, \end{aligned} \tag{50}$$

where $M = \prod_{j=1}^{\infty}(1+d_j) < \infty$. This is a contradiction, hence $\lim_{t\to\infty} h(t,x) = 0$. Theorem 17 is proved. □

In Theorem 17, the function may have a special form. In the case when $W(t,x) = V(t,x)$ and $d_k = \overline{d}_k$, $k \in \mathbb{N}$, we deduce the following corollary.

Corollary 18. *Let conditions (v), (vi) of Theorem 12 hold and assume that*

(i) *$h_0, h \in \Gamma$ and h_0 is weakly finer than h;*

(ii) *there exist $c \in \mathscr{K}$, and function $V \in \nu_0$ such that $V(t,x)$ is locally Lipschitz in x for each $t \in \mathbb{T}$ which is rd, h-positive definite on $S(h,\rho)$, h_0-weakly decrescent and*

$$D^+V^{\Delta}(t,x) \le -c(V(t,x)), \quad t \ne t_k, \ (t,x) \in S(h,\rho); \tag{51}$$

(iii) *$V(t_k^+, x_k + I_k(x_k)) - V(t_k, x_k) \le d_k V(t_k, x_k)$, $(t_k, x_k) \in S(h,\rho)$, where $d_k \ge 0$ and $\sum_{i=1}^{\infty} d_i < \infty$.*

Then system (10) *is (h_0,h)-asymptotically stable.*

Theorem 19. *Assume that all conditions of Theorem 13 hold. Suppose further, that there exists a function $W \in \nu_0$ such that $W(t,x)$ is locally Lipschitz in x for each $t \in \mathbb{T}$ which is rd, and the following conditions hold:*

(i) *$D^+W^{\Delta}(t,x) \le -p(t)c(h_0(t,x)) + q(t)$, $t \ne t_k$, $(t,x) \in S(h,\rho)$, where $c \in \mathscr{K}$, $p, q \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$, $\int_{t_0}^{\infty} p(\tau)\Delta\tau = \infty$, and $\int_{t_0}^{\infty} q(\tau)\Delta\tau < \infty$;*

(ii) *$W(t_k^+, x_k + I_k(x_k)) - W(t_k, x_k) \le \overline{d}_k W(t_k, x_k)$, $(t_k, x_k) \in S(h,\rho)$, where $\overline{d}_k \ge 0$ and $\sum_{i=1}^{\infty} \overline{d}_i < \infty$.*

Then system (10) *is (h_0,h)-attractive.*

Proof. By Theorem 13, system (10) is (h_0,h)-uniformly stable. Thus, for $\rho > 0$, there exists a $\delta_0 = \delta_0(\rho) > 0$ such that $h_0(t_0,x_0) < \delta_0$ implies that $h(t,x) < \rho$, $t \ge t_0$, where $x(t) = x(t;t_0,x_0)$ is any solution of (10).

Let $\varepsilon \in (0,\rho)$ be given, $\delta = \delta(\varepsilon) > 0$ be the same as defined in the definition of (h_0,h)-uniform stability, and $h_0(t_0,x_0) < \delta_0$. We claim that there exists a $t^* \ge t_0$ such that

$$h_0(t^*, x(t^*)) < \delta. \tag{52}$$

If this is not true, then $h_0(t,x) \ge \delta$ for all $t \ge t_0$.

Let

$$L(t,x) = W(t,x) + \int_{t_k}^{t} p(\tau) c(h_0(\tau, x(\tau)))\Delta\tau - \int_{t_k}^{t} q(\tau)\Delta\tau, \quad t \in (t_k, t_{k+1}], \tag{53}$$

for $k \in \mathbb{Z}^+$. By condition (i), we obtain

$$D^+L^{\Delta}(t,x) = D^+W^{\Delta}(t,x) + p(t)h_0(t,x) - q(t) \le 0, \quad t \ne t_k, \tag{54}$$

which implies that, for $t \in (t_k, t_{k+1}]$, $k \in \mathbb{Z}^+$,

$$L(t,x) \le L(t_k^+, x(t_k^+)). \tag{55}$$

Then, it follows from (55) and condition (ii) that, for $t \in (t_k, t_{k+1}]$,

$$\begin{aligned} W(t,x) &\le W(t_k^+, x(t_k^+)) - \int_{t_k}^{t} p(\tau) c(h_0(\tau, x(\tau)))\Delta\tau \\ &\quad + \int_{t_k}^{t} q(\tau)\Delta\tau \\ &\le (1+\bar{d}_k) W(t_k, x(t_k)) \\ &\quad - \int_{t_k}^{t} p(\tau) c(h_0(\tau, x(\tau)))\Delta\tau + \int_{t_k}^{t} q(\tau)\Delta\tau \\ &\le (1+\bar{d}_k) \\ &\quad \times \Bigg(W(t_{k-1}^+, x(t_{k-1}^+)) \\ &\qquad - \int_{t_{k-1}}^{t_k} p(\tau) c(h_0(\tau, x(\tau)))\Delta\tau \\ &\qquad + \int_{t_{k-1}}^{t_k} q(\tau)\Delta\tau \Bigg) \\ &\quad - \int_{t_k}^{t} p(\tau) c(h_0(\tau, x(\tau)))\Delta\tau + \int_{t_k}^{t} q(\tau)\Delta\tau \\ &\le (1+\bar{d}_k)(1+\bar{d}_{k-1}) W(t_{k-1}, x(t_{k-1})) \\ &\quad - \int_{t_{k-1}}^{t} p(\tau) c(h_0(\tau, x(\tau)))\Delta\tau \\ &\quad + (1+\bar{d}_k) \int_{t_{k-1}}^{t} q(\tau)\Delta\tau \\ &\le \cdots \\ &\le \prod_{i=1}^{k} (1+\bar{d}_i) W(t_0, x_0) - \int_{t_0}^{t} p(\tau) c(h_0(\tau, x(\tau)))\Delta\tau \\ &\quad + \prod_{i=2}^{k} (1+\bar{d}_i) \int_{t_0}^{t} q(\tau)\Delta\tau \\ &\le M_0 \left(W(t_0, x_0) + \int_{t_0}^{t} q(\tau)\Delta\tau \right) - c(\delta) \int_{t_0}^{t} p(\tau)\Delta\tau, \end{aligned} \tag{56}$$

where $M_0 = \prod_{i=1}^{\infty}(1+\bar{d}_i) < \infty$. Then, (56) implies that $W(t,x) \to -\infty$, for $t \to \infty$. This contradiction shows that (52) is true, and hence,

$$h(t,x) < \varepsilon, \quad t \ge t^*. \tag{57}$$

Thus we conclude that system (10) is (h_0, h)-attractive. □

Remark 20. When $\mathbb{T} = \mathbb{Z}$, and $d_k = \bar{d}_k = 0$, $k \in \mathbb{N}$, Theorem 19 contains Theorem 3.4 in [18] for discrete systems without impulse effects.

Next, we will give two results on uniform asymptotic stability in terms of two measures.

Theorem 21. *Let all the conditions of Theorem 19 and the following additional conditions hold:*

(i) *$W(t,x)$ is h_0-decrescent, and $p(t) \equiv p$ (p is a positive constant), for $t \in \mathbb{T}$;*

(ii) *there exists a constant $\tau > 0$ such that*

$$\{t + k\tau : t \in \mathbb{T}, k \in \mathbb{N}\} \subset \mathbb{T}. \tag{58}$$

Then system (10) is uniformly asymptotically stable.

Proof. Since $W(t,x)$ is h_0-decrescent, there exist $\delta_1 > 0$ and a function $b \in \mathscr{K}$ such that

$$W(t,x) \le b(h_0(t,x)), \quad \text{if } h_0(t,x) < \delta_1. \tag{59}$$

By Theorem 19, system (10) is (h_0, h)-uniformly stable. Thus, there exists a $\delta_0 = \delta_0(\rho) \in (0, \delta_1)$ such that $h_0(t_0, x_0) < \delta_0$ implies $h(t,x) < \rho$, $t \ge t_0$, for any solution $x(t) = x(t; t_0, x_0)$ of (10).

Let $\varepsilon \in (0, \rho)$ be given and $\delta = \delta(\varepsilon) > 0$ be the same as defined in the definition of (h_0, h)-uniformly stability. Let $m > 0$ be the smallest integer such that

$$m > \frac{M_0 (b(\delta_0) + N)}{p\tau c(\delta)}, \tag{60}$$

where $M_0 = \prod_{k=1}^{\infty}(1+\bar{d}_k) < \infty$ and $N = \int_{t_0}^{\infty} q(s)\Delta s < \infty$.

Choose $T = m\tau$ and let $x(t) = x(t; t_0, x_0)$ be any solution of (10) with $h_0(t_0, x_0) < \delta_0$. We claim that there exists a $t^* \in [t_0, t_0 + T]$ such that $h_0(t^*, x(t^*)) < \delta$. If this is not true, then $h_0(t,x) \ge \delta$ for all $t^* \in [t_0, t_0 + T]$. By (56), (60), and condition (ii), we have

$$\begin{aligned} &W(t_0 + T, x(t_0 + T)) \\ &\quad \le M_0 W(t_0, x_0) - pc(\delta) T + M_0 \int_{t_0}^{t_0+T} q(s)\Delta s \\ &\quad \le M_0 b(\delta_0) - mc(\delta) p\tau + M_0 N < 0 \end{aligned} \tag{61}$$

which is a contradiction. Thus, our claim is true and by the uniform stability we have

$$h(t,x) < \varepsilon, \qquad t \ge t_0 + T \ge t^*. \tag{62}$$

Hence, system (10) is (h_0, h)-uniformly attractive. This completes the proof. □

Theorem 22. *Let conditions (v), (vi) of Theorem 12 hold and assume that*

(i) *$h_0, h \in \Gamma$ and h_0 is finer than h;*

(ii) *there exist a* $c \in \mathscr{K}$*, and function* $V \in \nu_0$ *such that* $V(t,x)$ *is locally Lipschtiz in* x *for each* $t \in \mathbb{T}$ *which is rd,* h*-positive definite on* $S(h,\rho)$*,* h_0*-decrescent and*

$$D^+V^{\Delta}(t,x) \le -c(h_0(t,x)),\quad t \ne t_k,\quad (t,x)\in S(h,\rho); \tag{63}$$

(iii) $V(t_k^+, x_k + I_k(x_k)) - V(t_k, x_k) \le d_k V(t_k, x_k)$, $(t_k, x_k) \in S(h,\rho)$, *where* $d_k \ge 0$ *and* $\sum_{i=1}^{\infty} d_i < \infty$.

Then system (10) *is* (h_0,h)*-uniformly asymptotically stable.*

Proof. Since $V(t,x)$ is h_0-decrescent, there exist a constant δ_0 and a function $a \in \mathscr{K}$ such that

$$V(t,x) \le a(h_0(t,x)),\quad \text{if } h_0(t,x) < \delta_0. \tag{64}$$

The fact that $V(t,x)$ is h-positive definite on $S(h,\rho)$ implies that there exists a function $b \in \mathscr{K}$ such that

$$b(h(t,x)) \le V(t,x),\quad \text{if } h(t,x) < \rho. \tag{65}$$

It follows from Corollary 14 that system (10) is (h_0,h)-uniformly stable. Thus for $\rho > 0$, there exist a $\delta_1 = \delta_1(\rho) \in (0,\delta_0)$ such that $h_0(t_0,x_0) < \delta_1$ implies

$$h(t,x) < \rho,\quad t \ge t_0. \tag{66}$$

By the choice of δ_1, we get

$$V(t_0,x_0) \le a(h_0(t_0,x_0)) \le a(\delta_1). \tag{67}$$

To prove the theorem, it is enough to show that system (10) is (h_0,h)-uniformly attractive.

Given $0 < \varepsilon < \rho$, let $\delta = \delta(\varepsilon) > 0$ be the same as defined in the definition of (h_0,h)-uniformly stability. Then for any solution $x(t) = x(t;t_0,x_0)$ of system (10) with $h_0(t_0,x_0) < \delta_1$, we claim that there exists a $T = T(\varepsilon) > Ma(\delta_1)/c(\delta)$ such that, for some $t^* \in [t_0, t_0+T]$,

$$h_0(t^*, x(t^*)) < \delta, \tag{68}$$

where $M = \prod_{k=1}^{\infty}(1+d_k) < \infty$. Suppose that this is false. Then for any $T > Ma(\delta_1)/c(\delta)$ there exists a solution $x(t) = x(t;t_0,x_0)$ of (10) satisfying $h_0(t_0,x_0) < \delta_1$, such that

$$h_0(t,x(t)) \ge \delta,\quad t \in [t_0, t_0+T]. \tag{69}$$

By setting

$$L(t,x) = V(t,x) + \int_{t_k}^{t} c(h_0(s,x(s)))\Delta s,\quad t \in (t_k, t_{k+1}],\ k \in \mathbb{Z}^+, \tag{70}$$

and condition (ii), we have

$$D^+L^{\Delta}(t,x) = D^+V^{\Delta}(t,x) + c(h_0(t,x)) \le 0,\quad t \ne t_k, \tag{71}$$

which implies $L(t,x) \le L(t_k^+, x(t_k^+))$, for $t \in (t_k, t_{k+1}]$. Then, for $t \in (t_k, t_{k+1}]$, we get

$$\begin{aligned} V(t,x) &\le V(t_k^+, x(t_k^+)) - \int_{t_k}^{t} c_0(s)\Delta s \\ &\le (1+d_k)V(t_k, x(t_k)) - \int_{t_k}^{t} c_0(s)\Delta s \\ &\le (1+d_k)\left(V(t_{k-1}^+, x(t_{k-1}^+)) - \int_{t_{k-1}}^{t_k} c_0(s)\Delta s\right) \\ &\quad - \int_{t_k}^{t} c_0(s)\Delta s \\ &\le (1+d_k)(1+d_{k-1})V(t_{k-1}^+, x(t_{k-1}^+)) \\ &\quad - \int_{t_{k-1}}^{t} c_0(s)\Delta s \\ &\le V(t_0,x_0)\prod_{i=1}^{k}(1+d_i) - \int_{t_0}^{t} c_0(s)\Delta s \\ &\le MV(t_0,x_0) - \int_{t_0}^{t} c_0(s)\Delta s, \end{aligned} \tag{72}$$

where $c_0(t) := c(h_0(t,x))$. Hence, for $t = t_0+T$, we obtain

$$\begin{aligned} V(t,x(t))|_{t=t_0+T} &\le MV(t_0,x_0) - \int_{t_0}^{t_0+T} c(h_0(s,x(s)))\Delta s \\ &\le Ma(\delta_1) - c(\delta)T < 0 \end{aligned} \tag{73}$$

which is a contradiction. Hence there exists a number $t^* \in (t_0, t_0+T]$ such that $h_0(t^*, x(t^*)) < \delta$. Then for $t \ge t^*$, thus for $t \ge t_0+T$ as well, we have

$$h(t,x) < \varepsilon, \tag{74}$$

that is, $h(t,x) < \varepsilon$ holds for $t \ge t_0+T$ which means that system (10) is uniformly attractive. Theorem 22 is proved. □

4.3. (h_0,h)-Instability

Theorem 23. *Assume that*

(i) $h_0, h \in \Gamma$, $V \in \nu_0$, $V(t,x)$ *is locally Lipschtiz in* x *for each* $t \in \mathbb{T}$ *which is rd,* h*-positive definite on* $S(h,\rho)$*, and*

$$D^+V^{\Delta}(t,x) \ge 0,\quad t \ne t_k,\ (t,x)\in S(h,\rho), \tag{75}$$

and for any $s \ge t_0$ *and* $\alpha > 0$*, there is* $\beta > 0$ *such that* $V(t,x) \ge \alpha$ *for* $t \ge s$ *implies* $h(t,x) \ge \beta$ *for* $t \ge s$*;*

(ii) *for* $(t_k, x_k) \in S(h,\rho)$*,* $k \in \mathbb{N}$*,*

$$\begin{aligned} &V(t_k^+, x(t_k^+)) - V(t_k, x_k) \\ &\quad \ge \lambda_k \psi(V(t_k, x_k)), \end{aligned} \tag{76}$$

where $\lambda_k \geq 0$ *with* $\sum_{k=1}^{\infty} \lambda_k = \infty$, $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is nondecreasing and* $\psi(0) = 0$, $\psi(s) > 0$ *for* $s > 0$.

Then system (10) *is* (h_0, h)*-unstable.*

Proof. Let us assume on the contrary that system (10) is (h_0, h)-stable. Then for $0 < \varepsilon < \rho$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $h_0(t_0, x_0) < \delta$ implies that $h(t, x) < \varepsilon$ for $t \geq t_0$.

By setting $L(t) = V(t, x)$, we know from condition (i) and (ii) that $L(t)$ is right-nondecreasing for $t \geq t_0$. Then, $L(t) \geq L(t_0)$, if $t \geq t_0$. Thus, it follows from condition (i) that there exists a $\beta_0 = \beta_0(L(t_0)) > 0$ such that $h(t, x) \geq \beta_0$ for $t \geq t_0$. Since $V(t, x)$ is h-positive definite on $S(h, \rho)$, there exists $b \in \mathscr{K}$ such that $b(h(t, x)) \leq V(t, x)$, if $h(t, x) < \rho$. Then, we have

$$L(t_k) \geq b(h(t_k, x_k)) \geq b(\beta_0), \quad k \in \mathbb{N}. \tag{77}$$

From condition (ii) and (77), we have

$$\begin{aligned} L(t_k^+) - L(t_{k-1}^+) &\geq L(t_k^+) - L(t_k) \\ &\geq \lambda_k \psi(L(t_k)) \geq \lambda_k \psi(b(\beta_0)), \end{aligned} \tag{78}$$

and so

$$\begin{aligned} L(t_k^+) &\geq L(t_{k-1}^+) + \lambda_k \psi(b(\beta_0)) \\ &\geq L(t_{k-2}^+) + (\lambda_{k-1} + \lambda_k) \psi(b(\beta_0)) \\ &\geq \cdots \\ &\geq L(t_1) + \psi(b(\beta_0)) \sum_{i=1}^{k} \lambda_i \to \infty, \quad \text{as } k \to \infty, \end{aligned} \tag{79}$$

which implies that, for a given number $M \gg 0$, there exists a $s \geq t_0$ such that $L(t) \geq M$ for $t \geq s$. Thus by condition (i), there is a $\beta = \beta(M) > 0$ such that $h(t, x) \geq \beta$ for $t \geq s$ which is a contradiction to the (h_0, h)-stability. Therefore, system (10) is (h_0, h)-unstable. □

Theorem 24. *Assume that*

(i) $h_0, h \in \Gamma$, $V \in \nu_0$, $V(t, x)$ *is locally Lipschtiz in* x *for each* $t \in \mathbb{T}$ *which is rd,* h*-decrescent on* $S(h, \rho)$, *and*

$$\begin{gathered} -g(t) c(V(t, x)) \leq D^+ V^{\Delta}(t, x) \leq 0, \\ t \neq t_k, \ (t, x) \in S(h, \rho), \end{gathered} \tag{80}$$

where $g \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ *and* $c \in \mathscr{K}$;

(ii) *there exist* $\psi_k : \mathbb{R}^+ \to \mathbb{R}^+$, $k \in \mathbb{N}$, *such that* $\psi_k(s) > s$, $\psi_k(s) - s \geq \psi_k(t) - t$ *for* $t \geq s \geq 0$, *and*

$$\begin{gathered} V(t_k^+, x_k + I_k(x_k)) \geq \psi_k(V(t_k, x_k)), \\ (t_k, x_k) \in S(h, \rho), \ k \in \mathbb{N}; \end{gathered} \tag{81}$$

(iii) *for any* $u > 0$,

$$-\int_{t_k}^{t_{k+1}} g(\tau) \Delta\tau + \frac{\psi_k(u) - u}{c(u)} \geq r_k, \quad k \in \mathbb{N}, \tag{82}$$

where $r_k \geq 0$ *and* $\sum_{i=1}^{\infty} r_k = \infty$.

Then system (10) *is* (h_0, h)*-unstable.*

Proof. For the sake of contradiction, we assume that system (10) is (h_0, h)-stable. Then for $0 < \varepsilon < \rho$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $h_0(t_0, x_0) < \delta$ implies $h(t, x) < \varepsilon$ for $t \geq t_0$. Since $V(t, x)$ is h-decrescent on $S(h, \rho)$, there exists a function $a \in \mathscr{K}$ such that

$$V(t, x) \leq a(h(t, x)) < a(\rho), \quad \text{if } h(t, x) < \rho. \tag{83}$$

Let $m(t) = V(t, x)$ and $L(t) = V(t, x) + \int_{t_k}^{t} g(\tau) c(V(\tau, x(\tau))) \Delta\tau$, for $t \in (t_k, t_{k+1}]$, $k \in \mathbb{Z}^+$. Then it follows from condition (i) that

$$D^+ L^{\Delta}(t) = D^+ V^{\Delta}(t, x) + g(t) c(V(t, x)) \geq 0, \quad t \neq t_k, \tag{84}$$

which implies

$$L(t) \geq L(t_k^+), \quad t \in (t_k, t_{k+1}]. \tag{85}$$

Since $D^+ V^{\Delta}(t, x) \leq 0$, it follows from (85), that

$$\begin{aligned} m(t_{k+1}) - m(t_k^+) &\geq -\int_{t_k}^{t_{k+1}} g(\tau) c(m(\tau)) \Delta\tau \\ &\geq -c(m(t_k^+)) \int_{t_k}^{t_{k+1}} g(\tau) \Delta\tau. \end{aligned} \tag{86}$$

From condition (ii), we have

$$\begin{aligned} m(t_k^+) - m(t_k) &\geq \psi_k(m(t_k)) - m(t_k) \\ &\geq \psi_k(m(t_k^+)) - m(t_k^+) \end{aligned} \tag{87}$$

which, together with (86) and condition (iii), yields

$$\begin{aligned} m(t_{k+1}) - m(t_k) &= m(t_{k+1}) - m(t_k^+) + m(t_k^+) - m(t_k) \\ &\geq \psi_k(m(t_k^+)) - m(t_k^+) \\ &\quad - c(m(t_k^+)) \int_{t_k}^{t_{k+1}} g(\tau) \Delta\tau \\ &\geq c(m(t_k^+)) r_k \geq c(m(t_k)) r_k \\ &\geq c(m(t_1)) r_k, \quad k \in \mathbb{N}. \end{aligned} \tag{88}$$

Thus,

$$m(t_{k+1}) \geq m(t_1) + c(m(t_1)) \sum_{i=1}^{k} r_i \longrightarrow \infty, \quad \text{as } k \longrightarrow \infty, \tag{89}$$

which contradicts (83). Therefore, system (10) is (h_0, h)-unstable and the proof is complete. □

5. Examples

In this section, as applications of the above-derived theoretical criteria, two representative examples are given as follows.

Example 25. Consider the system

$$\begin{aligned} x^{\Delta}(t) &= \frac{y(t)}{2(1+x^2(t))} - x(t), \quad t \neq t_k, \\ y^{\Delta}(t) &= \frac{x(t)}{2(1+y^2(t))} - y(t), \quad t \neq t_k, \\ \Delta x(t_k) &= \frac{3}{2^k} x(t_k), \quad k \in \mathbb{N}, \\ \Delta y(t_k) &= \frac{5}{2^k} y(t_k), \quad k \in \mathbb{N}, \end{aligned} \tag{90}$$

on time scale $\mathbb{T}$.

Let $V(x, y) = x^2 + y^2$. Then we have

$$\begin{aligned} & V^{\Delta}(x, y) \\ &= (x^2 + y^2)^{\Delta} \\ &= x^{\Delta} x + x^{\sigma} x^{\Delta} + y^{\Delta} y + y^{\sigma} y^{\Delta} \\ &= x^{\Delta}(2x + \mu x^{\Delta}) + y^{\Delta}(2y + \mu y^{\Delta}) \\ &= 2x\left(\frac{y}{2(1+x^2)} - x\right) + 2y\left(\frac{x}{2(1+y^2)} - y\right) \\ &\quad + \mu\left[\left(\frac{y}{2(1+x^2)} - x\right)^2 + \left(\frac{x}{2(1+y^2)} - y\right)^2\right] \\ &= \frac{(1-\mu)xy(1+x^2) + (\mu/4)y^2}{(1+x^2)^2} + (\mu - 2)(x^2 + y^2) \\ &\quad + \frac{(1-\mu)xy(1+y^2) + (\mu/4)x^2}{(1+y^2)^2} \\ &\leq \frac{(1/2)(1-\mu)(x^2+y^2)(1+x^2) + (\mu/4)(x^2+y^2)}{(1+x^2)^2} \\ &\quad + (\mu - 2)(x^2 + y^2) \\ &\quad + \frac{(1/2)(1-\mu)(x^2+y^2)(1+y^2) + (\mu/4)(x^2+y^2)}{(1+y^2)^2} \\ &\leq \frac{1}{2}(\mu - 2) V(x, y), \quad t \neq t_k, \end{aligned} \tag{91}$$

where $x^{\sigma}(t) = x(\sigma(t))$, and

$$\begin{aligned} & V(x(t_k^+), y(t_k^+)) \\ &= x^2(t_k^+) + y^2(t_k^+) \\ &= \left(1 + \frac{3}{2^k}\right)^2 x^2(t_k) + \left(1 + \frac{5}{2^k}\right)^2 y^2(t_k) \\ &\leq V(x(t_k), y(t_k)) + \frac{1}{2^{k-5}} V(x(t_k), y(t_k)), \quad k \in \mathbb{N}, \end{aligned} \tag{92}$$

that is,

$$V(x(t_k^+), y(t_k^+)) - V(x(t_k), y(t_k)) \leq d_k V(x(t_k), y(t_k)), \tag{93}$$

where $d_k = 1/2^{k-5}$ and $\prod_{k=1}^{\infty} d_k < \infty$.

If $\mu(t) \leq 2$, then by (91) we have

$$V^{\Delta}(x, y) \leq 0. \tag{94}$$

Then for $h(x, y) = |x|$ and $h_0(x, y) = \sqrt{x^2 + y^2}$, we have

$$h^2(x, y) \leq V(t, x) \leq h_0^2(x, y). \tag{95}$$

From Corollary 14, we conclude that system (90) is (h_0, h)-uniformly stable. It should be noted that (h_0, h)-stability in this case implies, by the choice of h and h_0, that the trivial solution of (90) is partially stable with respect to x.

If, for a given positive constant $m \leq 1$, $\mu(t) \leq 2(1 - m)$, then we have

$$V^{\Delta}(x, y) \leq -mV(x, y). \tag{96}$$

Then for $h(x, y) = 2|xy|$ and $h_0(x, y) = x^2 + y^2$, we have

$$h(x, y) \leq V(x, y) \leq h_0(x, y). \tag{97}$$

By Corollary 18, we conclude that system (90) is (h_0, h)-asymptotically stable. Moreover, since $h_0 = V$, we conclude, by Theorem 22, that system (90) is (h_0, h)-uniformly asymptotically stable.

Example 26. Consider the following system [14]

$$\begin{aligned} x_1^{\Delta} &= -(e^{-t} + 1)x_1 - \frac{x_1 x_2^2}{1 + x_2^2}, \quad t \neq t_k, \\ x_2^{\Delta} &= -(e^{-t} + 1)x_2 - \frac{x_1^2 x_2}{1 + x_1^2}, \quad t \neq t_k, \\ \Delta x_1 &= \frac{1}{2} x_1, \quad t = t_k, \ k \in \mathbb{N}, \\ \Delta x_2 &= \frac{1}{2} x_2, \quad t = t_k, \ k \in \mathbb{N}, \end{aligned} \tag{98}$$

on time scale $\mathbb{T}$ such that $\mu(t) \geq 2/(e^{-t} + 1)$, for all $t \in \mathbb{T}$.

Let $V(t,x) = x_1^2 + x_2^2$, then we get

$$\begin{aligned} V^{\Delta}(t,x) &= x_1^{\Delta}x_1 + x_1^{\sigma}x_1^{\Delta} + x_2^{\Delta}x_2 + x_2^{\sigma}x_2^{\Delta} \\ &= x_1^{\Delta}x_1 + \left(x_1 + \mu x_1^{\sigma}\right)x_1^{\Delta} + x_2^{\Delta}x_2 + \left(x_2 + \mu x_2^{\sigma}\right)x_2^{\Delta} \\ &= 2x_1^{\Delta}x_1 + 2x_2^{\Delta}x_2 + \mu\left[\left(x_1^{\Delta}\right)^2 + \left(x_2^{\Delta}\right)^2\right], \end{aligned} \tag{99}$$

where $x_i^{\sigma} = x_i(\sigma(t)), i = 1, 2$. Substituting (98) into (99) yields

$$\begin{aligned} V^{\Delta}(t,x) = &\left[\mu\left(e^{-t}+1\right)^2 - 2\left(e^{-t}+1\right)\right]\left(x_1^2 + x_2^2\right) \\ &+ \frac{\mu x_1^2 x_2^4}{\left(1+x_2^2\right)^2} + \frac{\mu x_1^4 x_2^2}{\left(1+x_1^2\right)^2} \\ &+ 2\left[\mu\left(e^{-t}+1\right) - 1\right]\left(\frac{x_1^2x_2^2}{1+x_1^2} + \frac{x_1^2x_2^2}{1+x_2^2}\right) \\ \geq &\frac{x_1^2x_2^4}{\left(1+x_2^2\right)^2} + \frac{x_1^4x_2^2}{\left(1+x_1^2\right)^2} \geq 0. \end{aligned} \tag{100}$$

When $t = t_k$, we have

$$\begin{aligned} V\left(t_k^+, x\left(t_k^+\right)\right) &= x_1^2\left(t_k^+\right) + x_2^2\left(t_k^+\right) = \frac{9}{4}x_1^2\left(t_k\right) + \frac{9}{4}x_2^2\left(t_k\right) \\ &= V\left(t_k, x\left(t_k\right)\right) + \lambda_k\psi\left(V\left(t_k, x\left(t_k\right)\right)\right), \end{aligned} \tag{101}$$

where $\lambda_k \equiv 5/4$ and $\psi(s) = s$.

For $h(t,x) = |x_1| + |x_2|$ and $h_0 \in \Gamma$, we have

$$h^2(t,x) = \left(|x_1| + |x_2|\right)^2 = x_1^2 + x_2^2 + 2x_1x_2 \geq V(t,x). \tag{102}$$

Then, for any $s \geq t_0$ and $\alpha > 0$, $V(t,x) \geq \alpha$ implies that $h(t,x) \geq \beta = \sqrt{\alpha}$. Hence, all the conditions of Theorem 23 are satisfied, and system (98) is (h_0, h)-unstable.

6. Conclusions

We have generalized the concepts of stability in terms of two measures relative to ordinary impulsive systems to impulsive systems on time scales. By employing the approach of Lyapunov function, we have established some criteria ensuring stability and instability in terms of two measures for nonlinear impulsive systems on time scales. Two examples have been worked out to demonstrate the main results.

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