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Research Article

Algorithms for Some Euler-Type Identities for Multiple Zeta Values

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Multiple zeta values are the numbers defined by the convergent series $\zeta(s_1, s_2, \ldots, s_k) = \sum_{n_1>n_2>\cdots>n_k>0}(1/n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k})$, where s_1, s_2, $\ldots$, s_k are positive integers with $s_1 > 1$. For $k \le n$, let $E(2n, k)$ be the sum of all multiple zeta values with even arguments whose weight is $2n$ and whose depth is k. The well-known result $E(2n, 2) = 3\zeta(2n)/4$ was extended to $E(2n, 3)$ and $E(2n, 4)$ by Z. Shen and T. Cai. Applying the theory of symmetric functions, Hoffman gave an explicit generating function for the numbers $E(2n, k)$ and then gave a direct formula for $E(2n, k)$ for arbitrary $k \le n$. In this paper we apply a technique introduced by Granville to present an algorithm to calculate $E(2n, k)$ and prove that the direct formula can also be deduced from Eisenstein's double product.

1. Introduction

The multiple zeta sums,

$$\zeta(s_1, s_2, \ldots, s_m) = \sum_{n_1>n_2>\cdots>n_k>0} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_m^{s_m}}, \tag{1}$$

are also called Euler-Zagier sums, where $s_1, s_2, \ldots, s_m$ are positive integers with $s_1 \ge 2$. Clearly, the Riemann zeta function $\zeta(s)$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s > 1, \tag{2}$$

is the case $m = 1$ in (1). The multiple zeta functions have attracted considerable interest in recent years.

For Riemann's zeta function $\zeta(s)$, Euler proved the following identity:

$$\sum_{\substack{j_1+j_2=n \\ j_1, j_2 \ge 1}} \zeta(2j_1)\zeta(2j_2) = \frac{2n+1}{2}\zeta(2n). \tag{3}$$

Recently, some identities similar to (3) have also been established. Given two positive integers n and k (suppose $n \ge k$), define a number $Z(n, k)$ by

$$Z(n, k) = \sum_{\substack{j_1+j_2+\cdots+j_k=n \\ j_1, j_2, \ldots, j_k \ge 1}} \zeta(2j_1)\zeta(2j_2)\cdots\zeta(2j_k). \tag{4}$$

Then, for $k \in \{3, 4, \ldots, 9\}$, the value of $Z(n, k)$ is known [1–5].

Following [6], for $k \le n$, let $E(2n, k)$ be the sum of all multiple zeta values with even arguments whose weight is $2n$ and whose depth is k; that is,

$$E(2n, k) = \sum_{\substack{j_1+\cdots+j_k=n \\ j_1, j_2, \ldots, j_k \ge 1}} \zeta(2j_1, 2j_2, \ldots, 2j_k). \tag{5}$$

In [7], Gangl et al. proved the following identities:

$$E(2n, 2) = \frac{3}{4}\zeta(2n), \quad \text{for } n \ge 2, \tag{6}$$

$$\sum_{r=1}^{n-1} \zeta(2r+1, 2n-2r-1) = \frac{1}{4}\zeta(2n), \quad \text{for } n \ge 2. \tag{7}$$

Recently, using harmonic shuffle relations, Shen and Cai proved the following results in [8]:

$$E(2n,3)=\frac{5}{8}\zeta(2n)-\frac{1}{4}\zeta(2)\zeta(2n-2),\quad \text{for } n\geq 3,$$
$$E(2n,4)=\frac{35}{64}\zeta(2n)-\frac{5}{16}\zeta(2)\zeta(2n-2),\quad \text{for } n\geq 4. \tag{8}$$

In [6], applying the theory of symmetric functions, Hoffman established the generating function for the numbers $E(2n,k)$. He proved that

$$1+\sum_{n\geq k\geq 1}E(2n,k)t^n s^k=\frac{\sin\left(\pi\sqrt{1-s}\sqrt{t}\right)}{\sqrt{1-s}\sin\left(\pi\sqrt{t}\right)}. \tag{9}$$

Based on this generating function, some formulas for $E(2n,k)$ for arbitrary $n\geq k$ are given. For example, Hoffman obtained that

$$E(2n,k)=\frac{\zeta(2n)}{2^{2(k-1)}}\binom{2k-1}{k}-\sum_{j=1}^{[(k-1)/2]}\frac{\zeta(2j)\zeta(2n-2j)}{2^{2k-3}(2j+1)B_{2j}}\binom{2k-2j-1}{k}, \tag{10}$$

where B_{2j} is the $2j$th Bernoulli number.

In this paper we use a technique introduced by Granville [9] to present an elementary recursion algorithm to calculate $E(2n,k)$, we also give some direct formula for $E(n,k)$ for arbitrary $n\geq k$. Our algorithm may be of some interest if we note that it is obtained through an elementary analytic method and that the statement of the algorithm is fairly simple.

2. Statements of the Theorems

Theorem 1. *Let N denote a positive integer. Let $a_0^{(N)}, a_1^{(N)}, a_2^{(N)},\ldots,$ be a series of numbers defined by*

$$\prod_{\substack{r=1\\ r\neq N}}^{\infty}\left(1+\frac{x}{r^2-N^2}\right)=a_0^{(N)}+a_1^{(N)}x+\cdots+a_{k-1}^{(N)}x^{k-1}+\cdots. \tag{11}$$

Then, for any two positive integers n and k with $n\geq k$, one has

$$E(2n,k)=\sum_{N=1}^{\infty}\frac{a_{k-1}^{(N)}}{N^{2n-2k+2}}. \tag{12}$$

Theorem 2. *Given a positive integer N, we have*

$$\prod_{\substack{r=1\\ r\neq N}}^{\infty}\left(1+\frac{x}{r^2-N^2}\right)=2N^2(-1)^{N+1}\frac{\sin\left(\pi\sqrt{N^2-x}\right)}{\pi x\sqrt{N^2-x}}. \tag{13}$$

When k is not large, we may use the following recursion algorithm to calculate $a_k^{(N)}$ then use Theorem 1 to get the formula for $E(2n,k)$.

Theorem 3. *The coefficients $a_0^{(N)}, a_1^{(N)},\ldots,a_k^{(N)},\ldots,$ can be calculated recursively by the following formulas:*

$$a_0^{(N)}=1;\qquad ka_k^{(N)}=-\sum_{j=0}^{k-1}b_{k-j}a_j^{(N)},\quad \text{for } k\geq 1, \tag{14}$$

where $b_1,b_2,\ldots,$ are the numbers defined by

$$b_j\triangleq\sum_{\substack{r=1\\ r\neq N}}^{\infty}\frac{1}{\left(N^2-r^2\right)^j},\quad \forall j=1,2,3,\ldots. \tag{15}$$

In [6], Hoffman established an interesting result [6, Lemma 1.3] to obtain his formula (10) for $E(2n,k)$. This lemma might be deduced from the theory of Bessel functions. Using the expressions for the Bessel functions of the first kind with a half integer index, we may deduce from the generating function (13) a direct formula for $a_k^{(N)}$.

Theorem 4. *For $k\geq 1$, one has*

$$a_{k-1}^{(N)}=\frac{\pi^{k-1}}{k!(2N)^{k-1}}\left\{\sin\left(\frac{k\pi}{2}\right)\sum_{j=0}^{[k/2]}\frac{(-1)^j(k+2j)!}{(2j)!(k-2j)!}\cdot\frac{1}{(2N\pi)^{2j}}-\cos\left(\frac{k\pi}{2}\right)\times\sum_{j=0}^{[(k-1)/2]}\frac{(-1)^j(k+2j+1)!}{(2j+1)!(k-2j-1)!}\cdot\frac{1}{(2N\pi)^{2j+1}}\right\}, \tag{16}$$

$$a_{k-1}^{(N)}=\frac{1}{2^{2k-2}k!}\sum_{j=0}^{[(k-1)/2]}\frac{(-1)^j(2\pi)^{2j}(2k-1-2j)!}{(k-1-2j)!(2j+1)!}\cdot\frac{1}{N^{2k-2-2j}}. \tag{17}$$

To deduce (17) from (16), we only need to write the expression of $a_{k-1}^{(N)}$, respectively, according to whether k is odd or even, and use $[(k-1)/2]-j$ (if k is odd) or $[k/2]-j$ (if k is even) to replace j. In the two cases, we will get the expression (17) for $a_{k-1}^{(N)}$. By Theorem 1, we have

$$E(2n,k)=\sum_{j=0}^{[(k-1)/2]}\frac{(-1)^j\pi^{2j}(2k-1-2j)!}{2^{2k-2-2j}k!(k-1-2j)!(2j+1)!}\zeta(2n-2j), \tag{18}$$

which reproduces Hoffman's formula (10).

3. Proofs of the Theorems

Proof of Theorem 1. The left side of (12) is

$$\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 1}} \zeta(2j_1,2j_2,\ldots,2j_k)$$
$$= \sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 1}} \sum_{n_1>n_2>\cdots>n_k>0} \frac{1}{n_1^{2j_1} n_2^{2j_2}\cdots n_k^{2j_k}} \tag{19}$$
$$= \sum_{n_1>n_2>\cdots>n_k>0} \sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 1}} \frac{1}{n_1^{2j_1} n_2^{2j_2}\cdots n_k^{2j_k}}.$$

The second sum in (19) is the coefficient of x^{2n} in the formal power series

$$\sum_{j=1}^{\infty}\left(\frac{x^2}{n_1^2}\right)^j \sum_{j=1}^{\infty}\left(\frac{x^2}{n_2^2}\right)^j \cdots \sum_{j=1}^{\infty}\left(\frac{x^2}{n_k^2}\right)^j$$
$$= \frac{x^{2k}}{(n_1^2-x^2)(n_2^2-x^2)\cdots(n_k^2-x^2)} \tag{20}$$
$$= \sum_{j=1}^{k}\left(\frac{x^{2k}}{n_j^2-x^2}\prod_{\substack{1\leq m\leq k\\ m\neq j}}\frac{1}{n_m^2-n_j^2}\right).$$

It follows that the coefficient of x^{2n} earlier is

$$\sum_{j=1}^{k}\left(\frac{1}{n_j^{2n-2k+2}}\prod_{\substack{1\leq m\leq k\\ m\neq j}}\frac{1}{n_m^2-n_j^2}\right). \tag{21}$$

Hence, the sum (19) is

$$\sum_{n_1>n_2>\cdots>n_k>0}\sum_{j=1}^{k}\frac{1}{n_j^{2n-2k+2}}\prod_{\substack{1\leq m\leq k\\ m\neq j}}\frac{1}{n_m^2-n_j^2}. \tag{22}$$

Now, consider the function

$$f_N(x) = \prod_{\substack{r=1\\ r\neq N}}\left(1+\frac{x}{r^2-N^2}\right). \tag{23}$$

We partition $f_N(x)$ into two parts. Let

$$\prod_{r>N}\left(1+\frac{x}{r^2-N^2}\right) = P_0^{(N)} + P_1^{(N)}x + P_2^{(N)}x^2 + \cdots,$$
$$\prod_{1\leq r<N}\left(1+\frac{x}{r^2-N^2}\right) = Q_0^{(N)} + Q_1^{(N)}x + Q_2^{(N)}x^2 + \cdots. \tag{24}$$

Then, we have $P_0^{(N)} = Q_0^{(N)} \triangleq 1$, $Q_m^{(N)} = 0$, for all $m \geq N$, and

$$P_{j-1}^{(N)} = \sum_{n_1>n_2>\cdots>n_{j-1}>N}\frac{1}{(n_1^2-N^2)\cdots(n_{j-1}^2-N^2)}, \quad \forall 1<j\leq k,$$
$$Q_{k-j}^{(N)} = \sum_{N>n_{j+1}>n_{j+2}>\cdots>n_k\geq 1}\frac{1}{(n_{j+1}^2-N^2)\cdots(n_k^2-N^2)}, \quad \forall 1\leq j<k. \tag{25}$$

Consider the sum (22). For $j \in \{1,2,\ldots,k\}$, we treat each sum in (22) with respect to n_j as follows:

$$\sum_{n_1>n_2>\cdots>n_k>0}\frac{1}{n_j^{2n-2k+2}}\prod_{\substack{1\leq m\leq k\\ m\neq j}}\frac{1}{n_m^2-n_j^2}$$
$$= \sum_{n_j=k-j+1}^{\infty}\left[\frac{1}{n_j^{2n-2k+2}} \times\left(\sum_{n_1>n_2>\cdots>n_{j-1}>n_j}\prod_{1\leq m<j}\frac{1}{n_m^2-n_j^2}\right) \times\left(\sum_{n_j>n_{j+1}>\cdots>n_k\geq 1}\prod_{j<m\leq k}\frac{1}{n_m^2-n_j^2}\right)\right]$$
$$= \sum_{n_j=k-j+1}^{\infty}\frac{1}{n_j^{2n-2k+2}}P_{j-1}^{(n_j)}Q_{k-j}^{(n_j)}$$
$$= \sum_{N=k-j+1}^{\infty}\frac{1}{N^{2n-2k+2}}P_{j-1}^{(N)}Q_{k-j}^{(N)}$$
$$= \sum_{N=1}^{\infty}\frac{1}{N^{2n-2k+2}}P_{j-1}^{(N)}Q_{k-j}^{(N)}. \tag{26}$$

In the last step, N begins with 1 since $Q_{k-j}^{(N)} = 0$ for $1 \leq N < k-j$.

It follows that the sum (22) becomes that

$$\sum_{N=1}^{\infty}\frac{P_0^{(N)}Q_{k-1}^{(N)} + P_1^{(N)}Q_{k-2}^{(N)} + \cdots + P_{k-1}^{(N)}Q_0^{(N)}}{N^{2n-2k+2}}. \tag{27}$$

Clearly, the sum $P_0^{(N)}Q_{k-1}^{(N)} + P_1^{(N)}Q_{k-2}^{(N)} + \cdots + P_{k-1}^{(N)}Q_0^{(N)}$ in (27) is the coefficient of x^{k-1} in the Cauchy product of

$$\left[P_0^{(N)}+P_1^{(N)}x+P_2^{(N)}x^2+\cdots\right]\cdot\left[Q_0^{(N)}+Q_1^{(N)}x+Q_2^{(N)}x^2+\cdots\right]; \tag{28}$$

that is, it is the coefficient of x^{k-1} in the power series

$$f_N(x) = a_0^{(N)} + a_1^{(N)}x + \cdots + a_{k-1}^{(N)}x^{k-1} + \cdots. \tag{29}$$

Therefore, the sum (27) is

$$\sum_{N=1}^{\infty} \frac{a_{k-1}^{(N)}}{N^{2n-2k+2}}. \tag{30}$$

The proof is completed. □

Remark 5. If we take x to be a complex variable, then the series

$$\sum_{\substack{r=1\\ r\neq N}}^{\infty} \frac{x}{r^2 - N^2} \tag{31}$$

is absolutely and uniformly convergent for x in any compact set in the complex plane; thus, the function

$$f_N(x) = \prod_{\substack{r=1\\ r\neq N}}^{\infty} \left(1 + \frac{x}{r^2 - N^2}\right) \tag{32}$$

is analytic in the complex plane. Hence, it may be expanded as a Taylor series.

Proof of Theorem 2. First we recall Euler's classical formula

$$\sin(\pi z) = \pi z \prod_{r=1}^{\infty} \left(1 - \frac{z^2}{r^2}\right), \quad z \in \mathbb{C}. \tag{33}$$

Similar to Euler's formula, Eisenstein studied a product of two variables and proved that for $(\omega, z) \in (\mathbb{C}\backslash\mathbb{Z})\times\mathbb{C}$ the following formula holds (see [10, page 17]):

$$\begin{aligned}\frac{\sin[\pi(\omega - z)]}{\sin(\pi\omega)} &= \prod_{r=-\infty}^{\infty} \left(1 - \frac{z}{r+\omega}\right)\\ &\triangleq \lim_{n\to\infty} \prod_{r=-n}^{n} \left(1 - \frac{z}{r+\omega}\right).\end{aligned} \tag{34}$$

Let $N \geq 1$ be temporarily fixed. By (34), for $\omega \notin \mathbb{Z}$ we have

$$\begin{aligned}&\prod_{\substack{r=1\\ r\neq N}}^{\infty} \left(1 - \frac{z^2 - 2\omega z}{r^2 - \omega^2}\right)\\ &\quad= \frac{\sin[\pi(\omega - z)]}{\sin(\pi\omega)} \left(1 - \frac{z}{\omega}\right)^{-1} \left(1 - \frac{z^2 - 2\omega z}{N^2 - \omega^2}\right)^{-1}.\end{aligned} \tag{35}$$

Now, let $\omega \to N$. We get

$$\prod_{\substack{r=1\\ r\neq N}}^{\infty} \left(1 - \frac{z^2 - 2Nz}{r^2 - N^2}\right) = 2N^2(-1)^N \frac{\sin[\pi(N - z)]}{\pi(N - z)(z^2 - 2Nz)}. \tag{36}$$

We write $z^2 - 2Nz = -x$. Or equivalently, let $z = N \pm \sqrt{N^2 - x}$. Then, we get

$$\prod_{\substack{r=1\\ r\neq N}}^{\infty} \left(1 + \frac{x}{r^2 - N^2}\right) = 2N^2(-1)^{N+1} \frac{\sin\left(\pi\sqrt{N^2 - x}\right)}{\pi x\sqrt{N^2 - x}}. \tag{37}$$

□

Proof of Theorem 3. Taking logarithms of both sides of (32), we get that

$$\log(f_N(x)) = \sum_{\substack{r=1\\ r\neq N}}^{\infty} \log\left(1 - \frac{x}{N^2 - r^2}\right). \tag{38}$$

By Remark 5, the series may be differentiated term-by-term; hence, we have

$$\begin{aligned}\frac{f_N'(x)}{f_N(x)} &= \sum_{\substack{r=1\\ r\neq N}}^{\infty} \frac{-1}{N^2 - r^2 - x}\\ &= -\sum_{\substack{r=1\\ r\neq N}}^{\infty} \sum_{j=0}^{\infty} \frac{x^j}{(N^2 - r^2)^{j+1}}\\ &= -\sum_{j=0}^{\infty} \left(x^j \sum_{\substack{r=1\\ r\neq N}}^{\infty} \frac{1}{(N^2 - r^2)^{j+1}} \right)\\ &= -\sum_{j=0}^{\infty} b_{j+1} x^j,\end{aligned} \tag{39}$$

where we denote

$$b_{j+1} = \sum_{\substack{r=1\\ r\neq N}}^{\infty} \frac{1}{(N^2 - r^2)^{j+1}}, \quad \text{for } j = 0, 1, 2, \ldots. \tag{40}$$

The order of the summation can be changed since the series $\sum_{r=1, r\neq N}^{\infty} (-1/(N^2 - r^2 - x))$ is dominated by $\sum_{n=1}^{\infty} (L/n^2)$ for some positive constant L. From (39), we get that

$$f_N'(x) = -f_N(x) \sum_{j=0}^{\infty} b_{j+1} x^j, \tag{41}$$

or

$$\sum_{k=1}^{\infty} k a_k^{(N)} x^{k-1} = -\left(\sum_{i=0}^{\infty} a_i^{(N)} x^i\right) \left(\sum_{j=0}^{\infty} b_{j+1} x^j\right). \tag{42}$$

Write out the Cauchy product in the right side of (42), then compare the coefficient of x^{k-1} on both sides. We get that

$$k a_k^{(N)} = -\sum_{\substack{i+j=k-1\\ i,j\geq 0}} b_{j+1} a_i^{(N)} = -\sum_{j=0}^{k-1} b_{k-j} a_j^{(N)}. \tag{43}$$

□

Proof of Theorem 4. We now study the the generating function

$$f_N(x) = 2N^2(-1)^{N+1} \frac{\sin\left(\pi\sqrt{N^2 - x}\right)}{\pi x\sqrt{N^2 - x}}. \tag{44}$$

We may use L'Hospital's rule to verify that

$$\lim_{x\to 0} 2N^2(-1)^{N+1} \frac{\sin\left(\pi\sqrt{N^2 - x}\right)}{\pi x\sqrt{N^2 - x}} = 1. \tag{45}$$

Now we expand out $f_N(x)$. We have

$$f_N(x) = \frac{2N^2(-1)^{N+1}}{x}\sum_{m=0}^{\infty}\frac{(-1)^m\pi^{2m}\left(N^2-x\right)^m}{(2m+1)!}$$
$$= \frac{2N^2(-1)^{N+1}}{x}\sum_{m=0}^{\infty}\frac{(-1)^m\pi^{2m}}{(2m+1)!}$$
$$\times\sum_{k=0}^{m}\binom{m}{k}N^{2m-2k}(-1)^k x^k \tag{46}$$
$$= 2N^2(-1)^{N+1}\sum_{k=0}^{\infty}\frac{(-1)^k}{N^{2k}}$$
$$\times\sum_{m=k}^{\infty}\frac{(-1)^m(N\pi)^{2m}}{(2m+1)!}\binom{m}{k}x^{k-1}.$$

By (11) and (13), we have

$$a_{k-1}^{(N)} = 2N^{2-2k}(-1)^{N+k+1}$$
$$\times\sum_{m=k}^{\infty}\frac{(-1)^m(N\pi)^{2m}}{(2m+1)!}\binom{m}{k},\quad \forall k=1,2,3,\ldots. \tag{47}$$

Consider the function

$$\frac{\sin\left(N\pi\sqrt{x}\right)}{N\pi\sqrt{x}} = \sum_{m=0}^{\infty}\frac{(-1)^m(N\pi)^{2m}}{(2m+1)!}x^m. \tag{48}$$

Clearly, the sum in (47) can be rewritten as

$$\sum_{m=k}^{\infty}\frac{(-1)^m(N\pi)^{2m}}{(2m+1)!}\binom{m}{k} = \frac{1}{k!}\left(\frac{\sin\left(N\pi\sqrt{x}\right)}{N\pi\sqrt{x}}\right)_x^{(k)}\Bigg|_{x=1}, \tag{49}$$

where $()_x^{(k)}$ means the kth derivative of a function with respect to x.

We denote $\phi(x) = \sin\sqrt{x}/\sqrt{x}$. Then, we have

$$\phi\left(N^2\pi^2 t\right) = \frac{\sin\left(N\pi\sqrt{t}\right)}{N\pi\sqrt{t}}, \tag{50}$$

and, hence,

$$\frac{d^k\phi\left(N^2\pi^2 t\right)}{dt^k} = (N\pi)^{2k}\frac{d^k\phi(x)}{dx^k}\bigg|_{x=N^2\pi^2 t},\quad \forall k=0,1,2,\ldots, \tag{51}$$

which implies that

$$\left(\frac{\sin\left(N\pi\sqrt{t}\right)}{N\pi\sqrt{t}}\right)_t^{(k)}\Bigg|_{t=1} = (N\pi)^{2k}\left(\frac{\sin\sqrt{x}}{\sqrt{x}}\right)_x^{(k)}\Bigg|_{x=(N\pi)^2}. \tag{52}$$

Finally, from (47) (49) we get that

$$a_{k-1}^{(N)} = \frac{2N^2\pi^{2k}(-1)^{N+k+1}}{k!}\left(\frac{\sin\sqrt{x}}{\sqrt{x}}\right)_x^{(k)}\Bigg|_{x=(N\pi)^2},\quad k\ge 1. \tag{53}$$

We may apply Hoffman's result [6, Lemma 1.3] to get the direct formula for

$$\left(\frac{\sin\sqrt{x}}{\sqrt{x}}\right)_x^{(k)}\Bigg|_{x=(N\pi)^2}. \tag{54}$$

Here, we use some simple properties of the Bessel functions of the first kind to give its direct expression.

Lemma 6. *Let $k \ge 0$ be an integer and let $x > 0$. Then one has*

$$\frac{d^k}{dx^k}\left(\frac{\sin\sqrt{x}}{\sqrt{x}}\right) = \sqrt{\frac{\pi}{2}}(-1)^k 2^{-k}x^{-(2k+1)/4}J_{k+1/2}\left(\sqrt{x}\right), \tag{55}$$

where $J_{k+1/2}(x)$ denotes the Bessel function of the first kind of index $k+1/2$.

The Bessel functions with a half-integer index can be represented by elementary functions. The following lemma is well known.

Lemma 7. *Let $k \ge 0$ be an integer, and let $x > 0$. Then, one has*

$$J_{k+1/2}(x)$$
$$= \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{x}}\left\{\sin\left(x-\frac{k\pi}{2}\right)\sum_{j=0}^{[k/2]}\frac{(-1)^j(k+2j)!}{(2j)!(k-2j)!}\right.$$
$$\cdot\frac{1}{(2x)^{2j}} + \cos\left(x-\frac{k\pi}{2}\right)$$
$$\left.\times\sum_{j=0}^{[(k-1)/2]}\frac{(-1)^j(k+2j+1)!}{(2j+1)!(k-2j-1)!}\cdot\frac{1}{(2x)^{2j+1}}\right\}. \tag{56}$$

From Lemmas 6 and 7, and (53), we get that

$$a_{k-1}^{(N)}$$
$$= \frac{\pi^{k-1}}{k!(2N)^{k-1}}\left\{\sin\left(\frac{k\pi}{2}\right)\sum_{j=0}^{[k/2]}\frac{(-1)^j(k+2j)!}{(2j)!(k-2j)!}\right.$$
$$\cdot\frac{1}{(2N\pi)^{2j}} - \cos\left(\frac{k\pi}{2}\right) \tag{57}$$
$$\times\sum_{j=0}^{[(k-1)/2]}\frac{(-1)^j(k+2j+1)!}{(2j+1)!(k-2j-1)!}$$
$$\left.\cdot\frac{1}{(2N\pi)^{2j+1}}\right\}.$$

This completes the proof of Theorem 4. □

4. Examples

The direct formula for $a_k^{(N)}$ can be found from Theorem 4. However, we would like to use Theorem 3 to present some concrete examples to show how to calculate $a_k^{(N)}$ for small k. The difficult part of the recursion formula (14) is for $j \geq 1$ to calculate the sum

$$b_j = \sum_{\substack{r=1 \\ r\neq N}}^{\infty} \frac{1}{(N^2 - r^2)^j} = \left(\frac{1}{2N}\right)^j \sum_{\substack{r=1 \\ r\neq N}}^{\infty} \left[\frac{1}{N-r} + \frac{1}{N+r}\right]^j$$
$$= \lambda^j \sum_{\substack{r=1 \\ r\neq N}}^{\infty} [A_r + B_r]^j, \tag{58}$$

where we denote $A_r = 1/(N-r)$, $B_r = 1/(N+r)$, and $\lambda = 1/2N$.

It follows from $A_r B_r = \lambda[A_r + B_r]$ that

$$[A_r + B_r]^2 = [A_r^2 + B_r^2] + 2\lambda [A_r + B_r],$$
$$[A_r + B_r]^3 = [A_r^3 + B_r^3] + 3\lambda [A_r^2 + B_r^2] + 6\lambda^2 [A_r + B_r]. \tag{59}$$

Generally, we can use induction on j to prove that if for $j \geq 2$ we have gotten some positive integers $c_1, c_2, \ldots, c_{j-1}$ such that

$$[A_r + B_r]^j = [A_r^j + B_r^j] + c_1 \lambda [A_r^{j-1} + B_r^{j-1}] + \cdots + c_{j-1} \lambda^{j-1} [A_r + B_r], \tag{60}$$

then the expression for $[A_r + B_r]^{j+1}$ is

$$\begin{aligned}[A_r + B_r]^{j+1} &= [A_r^{j+1} + B_r^{j+1}] + (1 + c_1) \lambda [A_r^j + B_r^j] \\ &\quad + (1 + c_1 + c_2) \lambda^2 [A_r^{j-1} + B_r^{j-1}] \\ &\quad + \cdots + (1 + c_1 + c_2 + \cdots + c_{j-1}) \lambda^{j-1} [A_r^2 + B_r^2] \\ &\quad + 2(1 + c_1 + c_2 + \cdots + c_{j-1}) \lambda^j [A_r + B_r].\end{aligned} \tag{61}$$

Note that if $i \geq 2$ is an even integer, then we have

$$\begin{aligned}\sum_{\substack{r=1 \\ r\neq N}}^{\infty} [A_r^i + B_r^i] &= \sum_{1\leq r<N} \left[\frac{1}{(N-r)^i} + \frac{1}{(N+r)^i}\right] \\ &\quad + \sum_{r>N} \left[\frac{1}{(N-r)^i} + \frac{1}{(N+r)^i}\right] \\ &= 2\zeta(i) - \frac{1+2^i}{(2N)^i} = 2\zeta(i) - (1 + 2^i) \lambda^i.\end{aligned} \tag{62}$$

Similarly, if $i \geq 1$ is an odd integer, then we have

$$\sum_{\substack{r=1 \\ r\neq N}}^{\infty} [A_r^i + B_r^i] = -\frac{1+2^i}{(2N)^i} = -(1 + 2^i) \lambda^i. \tag{63}$$

From (58)–(63), we get that

$$\begin{aligned}b_1 &= -a_1^{(N)} = \lambda \sum_{\substack{r=1 \\ r\neq N}}^{\infty} [A_r + B_r] = -3\lambda^2, \\ b_2 &= \lambda^2 \sum_{\substack{r=1 \\ r\neq N}}^{\infty} \{[A_r^2 + B_r^2] + 2\lambda [A_r + B_r]\} \\ &= \lambda^2 \cdot \{[2\zeta(2) - 5\lambda^2] - 6\lambda^2\} \\ &= -11\lambda^4 + 2\zeta(2) \lambda^2, \\ b_3 &= \lambda^3 \sum_{\substack{r=1 \\ r\neq N}}^{\infty} \{[A_r^3 + B_r^3] + 3\lambda [A_r^2 + B_r^2] + 6\lambda^2 [A_r + B_r]\} \\ &= \lambda^3 \cdot \{-9\lambda^3 + 3\lambda [2\zeta(2) - 5\lambda^2] - 18\lambda^3\} \\ &= -42\lambda^6 + 6\zeta(2)\lambda^4, \\ b_4 &= \lambda^4 \sum_{\substack{r=1 \\ r\neq N}}^{\infty} \{[A_r^4 + B_r^4] + 4\lambda [A_r^3 + B_r^3] \\ &\quad + 10\lambda^2 [A_r^2 + B_r^2] \\ &\quad + 20\lambda^3 [A_r + B_r]\} \\ &= -163\lambda^8 + 20\zeta(2) \lambda^6 + 2\zeta(4) \lambda^4.\end{aligned} \tag{64}$$

From formula (14), we get that

$$\begin{aligned}a_2^{(N)} &= -\frac{1}{2} \{a_0^{(N)} b_2 + a_1^{(N)} b_1\} \\ &= -\frac{1}{2} \{[-11\lambda^4 + 2\zeta(2)\lambda^2] - (3\lambda^2)^2\} \\ &= 10\lambda^4 - \zeta(2) \lambda^2 = \frac{5}{8N^4} - \frac{\zeta(2)}{4N^2}, \\ a_3^{(N)} &= -\frac{1}{3} \{a_0^{(N)} b_3 + a_1^{(N)} b_2 + a_2^{(N)} b_1\} \\ &= -\frac{1}{3} \{[-42\lambda^6 + 6\zeta(2) \lambda^4] + 3\lambda^2 [-11\lambda^4 + 2\zeta(2) \lambda^2] \\ &\quad -3\lambda^2 [10\lambda^4 - \zeta(2) \lambda^2]\} \\ &= 35\lambda^6 - 5\zeta(2) \lambda^4 = \frac{35}{64N^6} - \frac{5\zeta(2)}{16N^4}, \\ a_4^{(N)} &= -\frac{1}{4} \{a_0^{(N)} b_4 + a_1^{(N)} b_3 + a_2^{(N)} b_2 + a_3^{(N)} b_1\} \\ &= 126\lambda^8 - 21\zeta(2) \lambda^6 - \frac{1}{2}\zeta(4) \lambda^4 + \frac{1}{2}\zeta^2(2) \lambda^4 \\ &= \frac{63}{128N^8} - \frac{21\zeta(2)}{64N^6} + \frac{3\zeta(4)}{64N^4}.\end{aligned} \tag{65}$$

For $k = 2, 3, 4$, using $a_1^{(N)}$, $a_2^{(N)}$, and $a_3^{(N)}$ in formula (12), respectively, we will get identities (6) and (8). Moreover, for $n \geq 5$, we have

$$E(2n, 5) = \frac{63}{128}\zeta(2n) - \frac{21}{64}\zeta(2)\zeta(2n-2) + \frac{3}{64}\zeta(4)\zeta(2n-4). \tag{66}$$

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