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Classroom Note

Fourier Method for Laplace Transform Inversion

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Abstract. A method is described for inverting the Laplace transform. The performance of the Fourier method is illustrated by the inversion of the test functions available in the literature. Results are shown in the tables.

Keywords: Inversion of Laplace tranform, ill-posedness, eigen-expension, gamma function, quadrature, Mellin transform, Fourier transform.

1. Introduction

During the past few decades, methods based on integral transforms, in particular, the Laplace transforms, are being increasingly employed in mathematics, physics, mechanics and other engineering sciences. Laplace transforms have a wide variety of applications in the solution of differential, integral and difference equations. To solve such equations by Laplace transform, one applies the Laplace transform to the equation, obtaining an equation for the transform of the required function. The latter equation is usually considerably simpler than the initial equation and its solution is often a function of quite simple structure. One must then derive the solution of the original equation from its Laplace transform, that is invert the Laplace transform.

In the terminology of ill-posed problems, the Laplace transform is a severely ill-posed problem. Unfortunately many problems of physical interest lead to Laplace transforms whose inverses are not readily expressed in

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terms of tabulated functions, although there exist extensive tables of transforms and their inverses. It is highly desirable, therefore, to have methods for appropriate numerical inversion.

The notion of ill-posedness is usually attributed to Hadamard [9]. A modern treatment of the concept appears in Tikhonov and Arsenin [22]. In an ill-posed inverse problem, a classical least squares, minimum distance or cross-validation solution may not be uniquely defined. Moreover the sensitivity of such solutions to slight perturbations in the data is often unacceptably large.

Ill-posed inverse problems have become a recurrent theme in modern sciences; see, for example, crystallography (Grunbaum [8]), Geophysics (Aki and Richards [2]), medical electrocardiograms (Franzone et al [7]), meteorology (Smith [20]), radio astronomy (Jayens [10]), reservoir engineering (Karavaris and Seinfeld [11]) and tomography (Vardi et al [24]). Corresponding to this broad spectrum of fields of applications, there is a wide literature on different kinds of inversion algorithms, that is techniques for solving the inverse problems.

The basic principle common to all such methods is as follows: seek a solution that is consistent both with the observed data and prior notions about the physical behavior of the phenomenon under study. Different practical problems have led to unique strategies for implementation of this principle such as the method of regularization (Tikhonov and Arsenin [22]), (Varah [23]), maximum entropy (Jaynes [10], Mead [15]), quasi-reversibility (Lattes and Lions [12]) and cross-validation (Wahba [25]).

Regularization methods have also been discussed by (Varah [23], Essah and Delves [6]) and by (Bertero [3])); other methods are also available in the literature for the numerical inversion of Laplace transform which have been described by (Norden [16]) and (Salzer [19]). However no single method gives optimum results for all purposes and for all occasions. For a detailed bibliography, the reader is referred to (Piessens and Pissens and Branders, [17, 18]). Several methods and a comparison is given by (Davis [4]) and (Talbot [21]).

Laplace Transform an Incorrectly Posed Problem

The problem of the recovery of a real function $f(t), \quad t \geq 0$, given its Laplace transform

$$\int_0^{\infty} e^{-st} f(t) dt = g(s) \tag{1.1}$$

for real values of s, is an ill-posed problem and, therefore, affected by numerical instability.

2. McWhirter and Pike's Method for Laplace Transform Inversion

Under the assumptions that $\int_0^{\infty} |g(s)| s^{-1/2} ds$ and $\int_0^{\infty} |f(t)| t^{-1/2} dt$ are finite, McWhirter and Pike [13, 14] show that the solution $f(t)$ of equation (1.1) may be represented in terms of a continuous eigen-expansion as follows:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{1}{\lambda_\omega} \left\{ \psi_\omega^+(t) + i\psi_\omega^-(t) \right\} \int_0^{\infty} g(s) \left\{ \psi_\omega^+(s) + i\psi_\omega^-(s) ds \right\} ds \tag{2.1}$$

where $\psi_\omega^{\pm}(s)$ are the real and imaginary parts of

$$\frac{\sqrt{\Gamma\left(\frac{1}{2} + i\omega\right)} s^{-\frac{1}{2} - i\omega}}{\sqrt{\pi \left|\Gamma\left(\frac{1}{2} + i\omega\right)\right|}} \tag{2.2}$$

and the eigenvalues λ_ω are real:

$$\lambda_\omega = \left| \Gamma\left(\frac{1}{2} + i\omega\right) \right| = \sqrt{\frac{\pi}{\cosh(\pi\omega)}}. \tag{2.3}$$

Here $\Gamma(z)$ is the complex Gamma function (see, e.g. [1, 5]).

In order to approximate (2.1) numerically, McWhirter and Pike replace the semi-infinite interval $[0, \infty)$ by the finite interval $[L_1, L_2]$, where $0 < L_1 << 1$ and $\infty > L_2 >> 1$. By introducing a spacing $H = \frac{2\pi}{T}$, where $T = \log L_2 - \log L_1$, and a discrete spectrum $\omega_n = nH$, they replace the integral (2.1) by the finite sum

$$f_N(t) = \frac{1}{2} \frac{a_0^+}{\lambda_0^+} \psi_0^+(t) + \sum_{n=1}^{N-1} \left\{ \frac{a_n^+}{\lambda_n^+} \psi_{\omega_n}^+(t) + \frac{a_n^-}{\lambda_n^-} \psi_{\omega_n}^-(t) \right\} \tag{2.4}$$

where

$$\left.\begin{aligned} a_n^{\pm} &= H\kappa_n^{\pm}\int_0^\infty g(s)s^{-\frac{1}{2}-i\omega_n}ds, \qquad n\neq 0,\\ a_0^{+} &= H\kappa_0^{+}\int_0^\infty g(s)s^{-1/2}ds, \end{aligned}\right\} \tag{2.5}$$

$$\kappa_n^{+} + i\kappa_n^{-} = \sqrt{\frac{\Gamma\left(\frac{1}{2}+i\omega_n\right)}{\pi\left|\Gamma\left(\frac{1}{2}+i\omega_n\right)\right|}}, \tag{2.6}$$

and $\lambda_{\omega_n}^{\pm} = \pm\lambda_{\omega_n}$.

The ill-posedness of the problem reflected by the very rapid decay of λ_{ω_n} with increasing n. Thus the inclusion of too many terms in the expansion (2.4) leads to large oscillations in $f_N(t)$, whereas too few terms do not give a sufficiently accurate solution. McWhirter and Pike [14] evaluate the coefficients $a_n^{\pm}$ in (2.5) by quadrature and determine N in (2.4) by trial and error.

3. Our Method

We are interested in finding

$$a_n = H\kappa_n\int_0^\infty g(s)s^{-\frac{1}{2}i\omega_n}ds \tag{3.1}$$

where κ_n is complex as defined earlier, ω_n is real and a_n are the complex coefficients to be determined. We use the notations as $\sim$ represents Mellin transform, $\wedge$ denotes Fourier transform. Consider

$$\tilde{g}(\lambda) = \int_0^\infty s^{i\lambda-1}g(s)ds \tag{3.2}$$

which is the Mellin transform of $g(s)$, λ being complex. From (3.1) and (3.2) we obtain

$$a_n = H\kappa_n\tilde{g}\left(-\omega_n - \frac{1}{2}i\right). \tag{3.3}$$

Now consider

$$\tilde{g}(\lambda) = \int_{-\infty}^\infty e^{i\lambda t}g\left(e^{-t}\right)dt \tag{3.4}$$

which is a well-known relationship between MTs and FTs, obtained by substituting for $s = e^{-t}$ in equation (3.2).

From (3.3) and (3.4)

$$a_n = H\kappa_n \int_{-\infty}^{\infty} e^{i\omega_n t}\left[e^{-\frac{1}{2}t} g\left(e^{-t}\right)\right] dt \tag{3.5}$$

which can be written as

$$a_n = H\kappa_n \overline{\hat{G}(\omega_n)} \tag{3.6}$$

where $\hat{G}(\omega) = \int_{-\infty}^{\infty} G(t)e^{-i\omega t}dt$ and $G(t) = e^{-\frac{1}{2}t}g(e^{-t})$.

From Abramowitz and Stegum [1].

$$\Gamma\left(\frac{1}{2} + i\omega_n\right) = \left|\Gamma\left(\frac{1}{2} + i\omega_n\right)\right|^2 \sum_{m=1}^{\infty} c_m \left(\frac{1}{2} - i\omega_n\right)^m$$

where the coefficients c_m are given to 7 decimal places in Table 1. Thus

$$\kappa_n = \sqrt{\frac{\Gamma\left(\frac{1}{2} + i\omega_n\right)}{\pi\left|\Gamma\left(\frac{1}{2} + i\omega_n\right)\right|}} = \sqrt{\frac{\left|\Gamma\left(\frac{1}{2} + i\omega_n\right)\right|}{\pi}} \left[\sum_{n=1}^{\infty} c_m \left(\frac{1}{2} - i\omega_n\right)^m\right]^{1/2}$$

and

$$a_n = H\kappa_n \int_{-\infty}^{\infty} e^{i\omega_n t} G(t) dt. \tag{3.7}$$

Table 1

m	c_m	m	c_m
1	1.0000000	11	0.0001280
2	0.5772156	12	−0.0000201
3	−0.6558780	13	−0.0000012
4	−0.0420026	14	0.0000011
5	0.1665386	15	−0.0000002
6	−0.0421977	16	−0.0000000
7	−0.0096219	17	0.0000000
8	0.0072189	18	0.0000000
9	−0.0011651	19	0.0000000
10	−0.0002152	20	0.0000000

Having written a_n in the form (3.7) it is sometimes possible, when $g(t)$ is given analytically, to evaluate a_n exactly from tables of Fourier transforms (see [1, 5]). This has the advantage of removing quadrature errors from the coefficients in the expansion (2.4) which are amplified by small eigenvalues.

4. Numerical Examples

Example 1. McWhirter and Pike [14]

$$g(s) = \frac{1}{(1+s)^2}, \quad s \geq 0, \; f(t) = te^{-t}, \quad t \geq 0.$$

We have

$$a_n = H\kappa_n \int_{-\infty}^{\infty} e^{i\omega_n t} \frac{e^{-\frac{1}{2}t}}{(1+e^{-t})^2} dt.$$

For reasons of comparison with McWhirter and Pike [14] we choose $H = 0.136$ and we tabulate the error in the numerical solution (2.4) versus N in Table 2. The optimal N is clearly 24.

Table 2

N	$\|f - f_N\|_2$
16	5.725×10^{-2}
20	2.877×10^{-2}
24	1.956×10^{-2}
28	2.433×10^{-2}
32	2.851×10^{-2}
36	2.980×10^{-2}
40	2.995×10^{-2}
44	2.996×10^{-2}
48	2.998×10^{-2}

Table 3

N	$\|f - f_N\|_2$
16	4.872×10^{-2}
20	1.873×10^{-2}
24	3.734×10^{-2}
28	4.425×10^{-2}
32	4.892×10^{-2}
36	4.932×10^{-2}
40	4.965×10^{-2}
44	4.983×10^{-2}
48	5.121×10^{-2}

Example 2. Varah [23]

$$\begin{aligned} g(s) &= \frac{\frac{1}{2}}{s\left(s+\frac{1}{2}\right)}, \qquad s \geq 0 \\ f(t) &= 1 - e^{-t/2}, \qquad t \geq 0 \end{aligned}$$

we have

$$a_n = H\kappa_n \int_{-\infty}^{\infty} \frac{e^{i\omega_n t}\frac{1}{2}e^{-\frac{1}{2}t}}{e^{-t}\left(e^{-t}+\frac{1}{2}\right)}.$$

For $H = 0.136$ and $N = 20$, the numerical solution obtained is the best giving the least error norm and is exceedingly better than Varah's solution.

5. Conclusion

Our method worked very well over both the test problems and the results obtained are shown in Tables 2 and 3. The method is easy to understand as compared with other more technical methods and yields equally good results.

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