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THE STATIONARY LOCAL SOJOURN TIME IN SINGLE SERVER TANDEM QUEUES WITH RENEWAL INPUT

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We start from an earlier paper evaluating the overall sojourn time to derive the *local sojourn time* in stationary regime, in a single server tandem queue of $(m+1)$ stages with renewal input. The successive service times of a customer may or may not be mutually dependent, and are governed by a general distribution which may be different at each sage.

Key words: Tandem Queue, First Come-First Served Discipline, Renewal Input.

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1. Introduction

We consider the stochastic behavior of a single server tandem queue with $(m+1)$ successive queues and with a *renewal* customer arrival process. For a customer, the successive service times may or may not be different and may or may not be mutually dependent. They are governed, at each stage, by a general distribution which may be different. Customers enter the queue of a given stage immediately on terminating their service at the previous stage. No limitation is placed on the length of the waiting time with buffers being supposed of infinite capacity. Service discipline at each stage is "*first come-first served*" (FC-FS). We wish to derive the distribution function of the *local sojourn time* in *stationary regime*.

For practical applications, we will consider the case of Poisson arrivals at the first stage. More particularly, we will present the calculations in the case of several packet traffic streams, with each traffic stream packet length (i.e., service duration) being constant (i.e., deterministic).

For this paper, we will refer to our earlier publications (see Le Gall [1-3]).

2. Notation and Assumptions

The number of successive queues is $(m+1)$. *At stage* $\boldsymbol{k}$ $(k=1,2,\ldots,m+1)$, for the nth customer, we define the following times:

a. *queueing delay* (due to queueing process): $\boldsymbol{w_n^k}$;
b. *service time*: $\boldsymbol{T_n^k}$;
c. *sojourn time*: $\boldsymbol{s_n^k} = w_n^k + T_n^k$;
d. *interarrival interval* [between customers $(n-1)$ and n]: $\boldsymbol{Y_{n-1}^k}$; and
e. *occasional idle period* [during Y_{n-1}^k]: $\boldsymbol{e_n^k}$.

At the first queue, the interarrival interval between arrivals $(n-1)$ and n is: $\boldsymbol{Y_{n-1}^1 \equiv Y_{n-1}}$. The arrival process is a *renewal* process with successive intervals $\boldsymbol{Y_{n-1}}$ being independent and identically distributed with distribution function $\boldsymbol{F_0(t)}$. Service times $\boldsymbol{T_n^k}$ (for $\boldsymbol{n}$ given) are assumed to be independent of the arrival process with distribution function $\boldsymbol{F_k(t)}$ at stage $\boldsymbol{k}$, which is the same for all customers. We introduce the Laplace-Stieltjes (L-S) transforms, for $Re(z) \geq 0$:

$$E\exp(-zY_{n-1}) = \varphi_0(z) = \int_0^\infty e^{-zt} \cdot dF_0(t),$$
$$E\exp(-zT_n^k) = \varphi_k(z) = \int_0^\infty e^{-zt} \cdot dF_k(t). \tag{1}$$

We may note the relations (at stage $\boldsymbol{k}$):

$$Y_{n-1}^k = T_n^{k-1} + e_n^{k-1},$$
$$w_n^k = \text{Max}[0, s_{n-1}^k - Y_{n-1}^k] = \text{Max}[0, s_{n-1}^k - (T_n^{k-1} + e_n^{k-1})]. \tag{2}$$

We let:

$$T_n^1 + \ldots + T_n^m = \boldsymbol{T'_n(m)};$$
$$s_n^2 + \ldots + s_n^{m+1} = \boldsymbol{S_n^m}. \tag{3}$$

Finally, in *stationary regime*, we let:

a. *arrival rate*: $\lambda = [EY_{n-1}]^{-1}$;
b. *load* (i.e., traffic intensity): $\boldsymbol{\rho_k} = \lambda \cdot E[T_n^k]$. (4)

3. Preliminary Results in Tandem Queues

3.1 The General Recurrence Relation

In Le Gall [1], Theorem (A.1), we presented the following properties:

Theorem 1: (Recurrence relation) *If at stage* $\boldsymbol{k}$ $(k=2\ldots m+1)$ *we have the relation*:

$$T_n^{k-1} \leq s_{n-1}^k, \tag{5}$$

the following recurrence relation using (3) *is then satisfied for* ***arbitrary*** *arrival processes* (*excluding simultaneous arrivals*):

$$(T_n^1 + w_n^2) + \ldots + (T_n^m + w_n^{m+1}) = \text{Max}[T'_n(m), S_{n-1}^m - e_n^1]. \tag{6}$$

Property 1: (Busy Period not Broken Up) Equation (5) means that the *busy period is not broken up* at stage k, during the busy period at stage $(k-1)$. It follows that, for the occasional idle periods e_n^k, we have: $e_n^1 = \ldots = e_n^{m+1}$. Finally, only $\boldsymbol{e_n^1}$ appears in (6).

Property 2: (Includes Intermediate Arrivals) An intermediate arrival at stage $\boldsymbol{k}$ may be considered as an arrival at stage 1 with $T_n^1 = \ldots = T_n^{k-1} = 0$. Equation (5) being satisfied, (6) includes the case of *intermediate arrivals.*

Property 3: (Includes Successive Service Times *Mutually Independent when Heavy Traffic*) If the load (i.e., traffic intensity) is high enough to increase w_{n-1}^k such that (5) is almost always satisfied during busy periods, the above theorem can be applied to the case of *mutually independent successive service times.*

Finally, (6) may be of general use.

3.2 The Equivalent Packet Tandem Queue

In Le Gall [1] and [2], we have seen that the local sojourn time distribution is practically defined by the two first moments, and finally by $\text{Var}T'_n(m)$. Consequently, in Le Gall [2] we introduced the parameter $\boldsymbol{m_0}$, with

$$\text{Var}(m_0 \cdot T_n^{m+1}) = \text{Var}T'_n(m), \tag{7}$$

and excluding the case of T_n^{m+1} and $T'_n(m)$ constant. In this way, the local sojourn distribution is practically the same as for a *single server packet tandem queue* corresponding to *identical, successive service times*: $T_n^1 = \ldots = T_n^{m_0} = T_n^{m_0+1}$, if $\boldsymbol{m_0}$ is an integer. From (7), (6) becomes, with (3):

$$S_n^{m_0} = \text{Max}[m_0 \cdot T_n^{m_0+1}, S_{n-1}^{m_0} - e_n^1]. \tag{8}$$

Finally, due to (5), *the local sojourn time distribution is practically given by the equivalent packet tandem queue* above for $(\boldsymbol{m_0}+1)$ *stages.* When $\boldsymbol{m_0}$ is not an integer, the distribution function may be used with this new value $\boldsymbol{m_0}$.

When service times are highly varying, we have: $[ET_n^{m+1}]^2 << E[T_n^{m+1}]^2$ and $[ET'_n(m)]^2 << E[T'_n(m)]^2$. It follows: $\text{Var}T_n^{m+1} \cong E[T_n^{m+1}]^2$ and $\text{Var}T'_n(m) \cong E[T'_n(m)]^2$. Consequently, (7) yields

$$m_0^2 = \frac{E[T'_n(m)]^2}{E[T_n^{m+1}]^2}. \tag{9}$$

3.3 The Agglutination Phenomenon

For the equivalent packet tandem queue, (2) gives:

$$s_n^{m+1} = \text{Max}[T_n^{m+1}, s_{n-1}^{m+1} - e_n^m]. \tag{10}$$

During a busy period, at stages $(m+1)$ and m, we have $e_n^m = 0$, and consequently $s_n^{m+1} = s_{n-1}^{m+1} = \ldots = s_{n_1}^{m+1}$, where $\boldsymbol{n_1}$ corresponds to the arbitrary customer initiating the busy period at these successive stages. Thus, the *local sojourn time appears to be constant during any busy period*, and equal to the sojourn time of the customer initiating the busy period. In the case of the arrival of a new customer with a service duration higher than $s_{n_1}^{m+1}$, the busy period is broken up, and this new customer initiates a new busy period. From stage to stage, busy periods initiated by long service durations tend to amalgamate, with busy periods initiated by short service durations, leading to some increase of the sojourn time. Finally, *from stage to stage, the agglutination phenomenon is amplified, with the local sojourn time of short service durations being equal to long service durations* corresponding to customers with no local queueing delay. As a consequence, with (5) of *busy periods not broken up, the product form theory cannot exist.*

We may note that, in Sections 3.1-3.3, the arrival process (at stage 1) is not necessarily restricted to a renewal process. This process may be arbitrary in stationary regime.

3.4 The Overall Distribution in Case of a Renewal Input

Now, we come back to the case of a *renewal input*, and we consider only the *equivalent packet tandem queue* (i.e., $T_n^{m+1} = T_n^1$). We let, *in stationary regime*, for $Re(z) \geq 0$:

$$\text{Prob}(w_n^1 = 0) = Q_1,$$

$$E\exp(-zs_n^1) = \Phi_1(z), \tag{10}$$

$$\varphi_1(z;t) = \int_0^t e^{-z\alpha} \cdot dF_1(\alpha),$$

$$\text{Prob}(S_n^m < t) = S_m(t),$$

where S_n^m is defined by (3). To present the following expressions, we will use Cauchy contour integrals along the imaginary axis in the complex plane $\boldsymbol{u}$. If the contour (followed from the bottom to the top) is to the right of the imaginary axis (the contour being closed at infinity to the right), we write $\int_{+0}$. If the contour is to the left of the imaginary axis, we write $\int_{-0}$. We introduce:

$$\boldsymbol{Q_1(t)} = \text{Exp}\left\{\frac{-1}{2\pi i}\int_{-0} \log[1 - \varphi_0(-u)\cdot\varphi_1(u,t)]\cdot\frac{du}{u}\right\}. \tag{11}$$

In Le Gall [3], and *with* (5) *of busy periods not broken up*, we gave the following expression for the distribution function of the *overall sojourn time*, from stages 2 to $(m+1)$:

$$\boldsymbol{S_m(t)} = \frac{1}{2\pi i} \cdot \int_{+0} \frac{\varphi_0(-u) \cdot \Phi_1(u)}{Q_1} \cdot S_m(t,u) \cdot \frac{du}{u}, \tag{12}$$

with

$$S_m(t,u) = v_0\left(\frac{t}{m}\right) \cdot \left[v\left(\frac{t}{m}, u\right)\right]^m, \quad v_0(t) = \frac{Q_1}{Q_1(t)} \cdot F_1(t)$$
$$v(t,u) = \mathrm{Exp}\left\{ -u \int_t^{\infty} \frac{1 - F_1(v)}{Q_1(v)} \cdot dv \right\}, \tag{13}$$

by using (1), (10) and (11).

3.5 The Unitary Sojourn Time

In the *stationary regime*, it may be useful to introduce the concept of *unitary sojourn time* $\boldsymbol{U(m)}$ (i.e., the *overall* sojourn time *divided by* $\boldsymbol{m}$ [number of equivalent stages]). From (12) and (13), and assuming (5) *of busy periods not broken up*, the distribution function of $\boldsymbol{U(m)}$ is:

$$\boldsymbol{U_m(t)} = \frac{1}{2\pi i} \int_{+0} \frac{\varphi_0(-u) \cdot \Phi_1(u)}{Q_1} \cdot u_m(t,u) \cdot \frac{du}{u}, \tag{14}$$

with

$$\boldsymbol{u_m(t,u)} = v_0(t) \cdot [v(t,u)]^m, \tag{15}$$

or

$$\boldsymbol{u_m(t,u)} = u_1(t,u) \cdot [v(t,u)]^{m-1}, \tag{16}$$

where $\boldsymbol{v_0(t)}$ and $\boldsymbol{u_1(t,u)}$, defined by (13), *relates only to the case* $m = 1$. Here, $\boldsymbol{u_m(t,u)}$ is the distribution function of the random variable $\boldsymbol{U(m,u)}$. In other words, we have the stochastic expression:

$$\boldsymbol{U(m)} = \frac{1}{2\pi i} \cdot \int_{+0} \frac{\varphi_0(-u) \cdot \Phi_1(u)}{Q_1} \cdot U(m,u) \cdot \frac{du}{u}. \tag{17}$$

We note that $\boldsymbol{v_0(t)}$ and $\boldsymbol{v(t,u)}$ are the distribution functions of $\boldsymbol{V_0}$ and $\boldsymbol{V_j(u)}$, $j = 1, \ldots, m$, respectively. From (15), we write:

$$\boldsymbol{U(m,u)} = \mathrm{Min}_{j=1\ldots m}[V_0, V_j(u)]. \tag{18}$$

Finally, from (18) we deduce the stochastic expression giving the *overall sojourn time*:

$$S_n^m = m \cdot U(m) = \frac{1}{2\pi i} \cdot \int_{+0} \frac{\varphi_0(-u) \cdot \Phi_1(u)}{Q_1} \cdot [m \cdot U(m,u)] \cdot \frac{du}{u}, \tag{19}$$

which has $\boldsymbol{U_m\left(\frac{t}{m}\right)}$, from (14), for its distribution function.

We now consider the case of Poisson input. Since the arrival rate, for an arbitrary customer, is λ, we have:

$$\varphi_0(z) = \frac{\lambda}{\lambda + z}, \quad \rho = \lambda \cdot E(T_n) \quad (\rho < 1). \tag{20}$$

In the integrand of (19) there is just one pole for $Re(u) > 0: u = \lambda$. Equations (14) and (15) become, for the distribution function of the *unitary sojourn time*:

$$\boldsymbol{U_m(t)} = u_m(t, \lambda) = v_0(t) \cdot [v(t, \lambda)]^m, \text{ with } Q_1 = 1 - \rho. \tag{21}$$

4. The Local Sojourn Time in Case of a Renewal Input

In stationary regime with renewal input, the *local sojourn time* $\boldsymbol{s_n^{m+1}}$, at stage $(m+1)$, may be deduced from stochastic equation (19), assuming (5) *of busy periods not broken up* to be able to use the equivalent packet tandem queue [where $\boldsymbol{m}$ is the value of $\boldsymbol{m_0}$ of (9)]:

$$s_n^{m+1} = S_n^m - S_n^{m-1} = \frac{1}{2\pi i} \cdot \int_{+0} \frac{\varphi_0(-u) \cdot \Phi_1(u)}{Q_1} \cdot D(m, u) \cdot \frac{du}{u}, \tag{22}$$

$$\text{with } D(m, u) = m \cdot U(m, u) - (m-1) \cdot U(m-1, u).$$

From (18) we write:

$$\begin{aligned} D(m, u) &= \text{Min}[m \cdot V_m(u), m \cdot U(m-1, u)] - (m-1) \cdot U(m-1, u) \\ &= \text{Min}[m \cdot V_m(u) - (m-1) \cdot U(m-1, u), U(m-1, u)]. \end{aligned} \tag{23}$$

The distribution function $\boldsymbol{d_m(t, u)}$ of $D(m, u)$ corresponds to the inequalities;

$$U(m-1, u) < t, \; m \cdot V_m(u) - (m-1) \cdot U(m-1, u) < t.$$

These inequalities are satisfied by the expression:

$$\boldsymbol{d_m(t, u)} = \int_0^t v\left(\frac{t+w}{m}, u\right) \cdot d_w u_{m-1}\left(\frac{w}{m-1}, u\right), \tag{24}$$

where $\boldsymbol{v(t, u)}$ and $\boldsymbol{u_{m-1}(t, u)}$ are defined by (13) and (16), respectively. Finally, from (22) and (24) we can state:

Theorem 2: (Local sojourn time) *In stationary regime, with* (5) *of* ***busy periods not broken up****, and a number* $(m+1)$ *of equivalent stages* [*as defined by the value of* m_0 *in* (9)], *the distribution function of the* ***local sojourn time*** [*at stage* $(m+1)$] *for an* ***arbitrary*** *customer is*:

$$\boldsymbol{s(t, m+1)} = \frac{1}{2\pi i} \cdot \int_{+0} \frac{\varphi_0(-u) \cdot \Phi_1(u)}{Q_1} d_m(t, u) \cdot \frac{du}{u}, \tag{25}$$

where $\boldsymbol{d_m(t,u)}$ *is given by* (24).

It may be very useful to simplify (24) of $\boldsymbol{d_m(t,u)}$. When $\boldsymbol{m}$ increases, we have for $\boldsymbol{t}$ given

$$v\left(\frac{t+w}{m}, u\right) \rightarrow v(0,u).$$

Equation (24) becomes:

$$\boldsymbol{d_m(t,u)} \rightarrow v(0,u) \cdot u_{m-1}\left(\frac{t}{m-1}, u\right).$$

We can state:

Corollary 1: (Approximation) *When* $\boldsymbol{m}$ *increases, expression* $\boldsymbol{d_m(t,u)}$ *in* (25) *becomes:*

$$\boldsymbol{d_m(t,u)} \cong v(0,u) \cdot u_{m-1}\left(\frac{t}{m-1}, u\right), \tag{26}$$

where $\boldsymbol{v(0,u)}$ *and* $\boldsymbol{u_{m-1}(t,u)}$ *are defined by* (13) *and* (15), *respectively. Thus*

$$u_{m-1}\left(\frac{t}{m-1}, u\right) = v_0(t) \cdot \left[v\left(\frac{t}{m-1}, u\right)\right]^{m-1}, \tag{27}$$

with $\boldsymbol{v_0(t)}$, *as defined by* (13), *being related only to stage* 1.

Note: In the numerical example provided below in Section 5.3, we will see that the approximation is already excellent for $m = 2$.

5. The Local Sojourn Time in Case of a Poisson Input

5.1 The Distribution Function

In the case of *Poisson input*, as already noted for (21), we have one pole $u = \lambda$ (for $Re(u) > 0$) in the integrand of (25). We deduce

$$\boldsymbol{s(t, m+1)} = d_m(t, \lambda), \tag{28}$$

and for (26)

$$\boldsymbol{s(t, m+1)} \cong v_0(t) \cdot \left[v\left(\frac{t}{m-1}, \lambda\right)\right]^{m-1} \cdot v(0, \lambda). \tag{29}$$

5.2 Case of Packet Traffics with Poisson Input

As an important example, consider (in *stationary regime*) the case of a total traffic stream with $\boldsymbol{N}$-component, partial Poisson traffic streams labeled $\boldsymbol{j}$ ($j = 1 \ldots N$). For traffic stream $\boldsymbol{j}$, packet lengths are constant (i.e., deterministic), and equal to $\boldsymbol{T_j}$ ($T_1 < T_2 < \ldots < T_N$). The partial arrival rate is $\boldsymbol{\lambda_j}$, and the total arrival rate (for an arbitrary customer) is: $\lambda = \sum_{j=1}^{N} \lambda_j$. The partial loads are $\boldsymbol{\rho_j} = \lambda_j \cdot T_j$, with a total load: $\boldsymbol{\rho} = \sum_{j=1}^{N} \rho_j$. With (1), we achieve, for the distribution of an *arbitrary* customer (i.e., packet) and for $Re(z) \geq 0$:

$$\varphi_1(z) = \sum_{j=1}^{N} \frac{\lambda_j}{\lambda} \cdot e^{-zT_j}.$$

In Le Gall [1], formula (B.3), we gave the expression:

$$\boldsymbol{Q_1(t)} = 1 - \lambda \cdot \int_0^t u \cdot dF_1(u), \quad Q_1(\infty) = Q_1(T_N) = 1 - \rho. \tag{30}$$

The distribution function of the service times are as follows:
For $t < T_1$:

$$F_1(t) = 0, \quad Q_1(t) = 1;$$

For $T_k < t < T_{k+1}$:

$$F_1(t) = \frac{\lambda_1 + \ldots + \lambda_k}{\lambda}, \quad Q_1(t) = 1 - (\rho_1 + \ldots + \rho_k). \tag{31}$$

a. ***Exact expression for the mean local sojourn time***

From (21), the distribution function of the *unitary sojourn time* is:

$$\boldsymbol{u_m(t,\lambda)} = v_0(t) \cdot [v(t,\lambda)]^m, \tag{32}$$

where $\boldsymbol{v_0(t)}$ and $\boldsymbol{v(t,\lambda)}$ are defined by (13). We deduce for (32):
For $t < T_1$:

$$u_m(t,\lambda) = 0;$$

For $T_k < t < T_{k+1}$:

$$u_m(t,\lambda) = v_0(t) \cdot [v(t,\lambda)]^m,$$

with

$$v_0(t) = \frac{\lambda_1 + \ldots + \lambda_k}{\lambda} \cdot \frac{1 - \rho}{1 - (\rho_1 + \ldots + \rho_k)},$$

and

$$v(t,\lambda) =$$

$$\mathrm{Exp}\left\{ - \left[\frac{\lambda_{k+1} + \ldots + \lambda_N}{1 - (\rho_1 + \ldots + \rho_k)} \cdot (T_{k+1} - t) + \ldots + \frac{\lambda_N}{1 - (\rho_1 + \ldots + \rho_{N-1})} \cdot (T_N - T_{N-1}) \right] \right\}. \tag{33}$$

We deduce the *mean overall sojourn time*, from stage 2 to stage $(m+1)$:

$$E(S_n^m) = m \cdot \int_0^{T_N} [1 - u_m(t,\lambda)] \cdot dt, \tag{34}$$

and we deduce the *mean local sojourn time* at stage $(m+1)$, for an *arbitrary* packet:

$$s(m+1) = m \cdot \int_0^{T_N} [1 - u_m(t,\lambda)] \cdot dt - (m-1) \cdot \int_0^{T_N} [1 - u_{m-1}(t,\lambda)] \cdot dt. \tag{35}$$

b. *Approximated expression for the local sojourn time distribution*

Approximated equation (29) gives for the distribution function of the *local sojourn time* [at stage $(m+1)$], for an *arbitrary* packet:

For $t < T_1$:

$$s(t, m+1) = 0; \tag{36}$$

For $T_k < t < T_{k+1}$:

$$s(t, m+1) = v(0,\lambda) \cdot v_0(t) \cdot \left[v\left(\frac{t}{m-1}, \lambda\right)\right]^{m-1} = v_0(t) \cdot \left[v\left(\frac{t}{m}, \lambda\right)\right]^m,$$

where $\boldsymbol{v_0(t)}$ and $\boldsymbol{v(t,\lambda)}$ are given by (33); and

For $t > T_N$:

$$s(t, m+1) = 1.$$

From (36), we get an *approximated value* for the **mean** local sojourn time at stage $(m+1)$:

$$\boldsymbol{s_1(m+1)} = \int_0^{T_N} [1 - s(t, m+1)] \cdot dt. \tag{37}$$

In the same way, we get the second moment:

$$m_2 m + 1) = 2 \cdot \int_0^{T_N} t \cdot [1 - s(t, m+1)] \cdot dt, \tag{38}$$

the variance

$$v_1(m+1) = m_2(m+1) - [s_1(m+1)]^2,$$

and the ***standard deviation***

$$\sigma_1(m+1) = \sqrt{v_1(m+1)}. \tag{39}$$

To evaluate the accuracy of (36), we compare (35) for $\boldsymbol{s(m+1)}$, and (37) for $\boldsymbol{s_1(m+1)}$ with numerical values.

5.3 An Example of Three Packet Traffic Streams

For numerical calculations, we will consider the case of three Poisson packet traffic streams. Here $\boldsymbol{v_0(t)}$ and $\boldsymbol{v(t,\lambda)}$, used in (36), become:

For $T_1 < t < T_2$:

$$v_0(t) = \frac{\lambda_1}{\lambda} \cdot \frac{1-\rho}{1-\rho_1},$$

$$\boldsymbol{v(t,\lambda)} = \mathrm{Exp}\left\{ -\left[\frac{\lambda_2+\lambda_3}{1-\rho_1}\cdot(T_2-t)+\frac{\lambda_3}{1.-(\rho_1+\rho_2)}\cdot(T_3-T_2)\right]\right\};$$

For $T_2 < t < T_3$:

$$v_0(t) = \frac{\lambda_1+\lambda_2}{\lambda} \cdot \frac{1-\rho}{1-(\rho_1+\rho_2)},$$
$$\boldsymbol{v(t,\lambda)} = \mathrm{Exp}\left\{ -\frac{\lambda_3}{1-(\rho_1+\rho_2)}\cdot(T_3-t)\right\}. \tag{40}$$

Table 1 gives $\boldsymbol{s(m+1)}$, $\boldsymbol{s_1(m+1)}$ and $\boldsymbol{\sigma_1(m+1)}$, related to (35), (37), and (39) respectively, for $m = 1,2,3,5,10$ and 15. We observe that $\boldsymbol{s_1(m+1)}$ is approximately equal to the exact value $\boldsymbol{s(m+1)}$ for $m > 1$, and equal for $m = 1$. When $\boldsymbol{m}$ increases, $\boldsymbol{\sigma_1(m+1) \to 0}$, and consequently $\boldsymbol{s(m+1)} \to T_3$, as explained by the agglutination phenomenon (see Section 3.3).

	exact	approximated	
m	$s(m+1)$	$s_1(m+1)$	$\sigma_1(m+1)$
1	14.7	14.7	12.8
2	18.9	19.2	12.6
3	21.8	22.4	11.6
5	25.5	26.1	9.0
10	28.9	29.3	4.1
15	29.7	29.9	1.8

Table 1: Local Queue Distribution [at stage $(m+1)$] for an Arbitrary Packet
1st approximation: formulae (40).

Example: (3 packet traffic streams): $N = 3; T_1 = 1, T_2 = 5$, $\boxed{\boldsymbol{T_3 = 30}}$; $\rho_1 = \rho_2 = \rho_3 = 0.2$ ($\rho = 0.6$).

Mean local queue: Exact value: $\boldsymbol{s(m+1)}$, Equation (35);
Approximated value: $\boldsymbol{s_1(m+1)}$, Equation (37).

Standard deviation: Approximated value: $\boldsymbol{\sigma_1(m+1)}$, Equation (39).

6. A General Approximation in Case of Poisson Input

In (36) we may write, for the long service durations:
For $T_{n-1} < t < T_N$:

$$\boldsymbol{s_2(t,m+1)} = \left(1 - \frac{\lambda_N}{\lambda}\right) \cdot \frac{1-\rho}{1-(\rho-\rho_N)} \cdot \mathrm{Exp}\left\{ - \frac{m\rho_N}{1-(\rho-\rho_N)} \cdot \left(1 - \frac{t}{mT_N}\right)\right\}, \quad (41)$$

For $t > T_N$:

$$s_2(t,m+1) = 1,$$

where the set (λ_N, ρ_N, T_N) relates to the longest packets.

In Table 2, we use this approximated distribution function for $0 < t < T_N$, independent of T_1 and T_2. We observe the accuracy of this approximation by comparing the exact value $\boldsymbol{s(m+1)}$, and the new approximated value $\boldsymbol{s_2(m+1)}$ deduced from (37) by using (41):

$$\boldsymbol{s_2(m+1)} = \int_0^{T_N} [1 - s_2(t,m+1)] \cdot dt. \quad (42)$$

	exact	approximated	
m	$s(m+1)$	$s_2(m+1)$	$\sigma_2(m+1)$
3	21.8	21.5	13.0
5	25.5	25.6	10.2
10	28.9	29.2	4.8
15	29.7	29.8	2.1

Table 2: Local Queue Distribution [at stage $(m+1)$] for an Arbitrary Packet
2nd approximation: (41), using Example of Table 1.

Mean local queue: Exact value: $\boldsymbol{s(m+1)}$, Equation (35);
Approximated value: $\boldsymbol{s_2(m+1)}$, Equation (42).
Standard deviation: Approximated value: $\boldsymbol{\sigma_2(m+1)}$.

In the same way, an analogous expression to (39) gives the *standard deviation* $\boldsymbol{\sigma_2(m+1)}$. In Table 2, the accuracy of $\boldsymbol{s_2(m+1)}$ is good for $m > 2$, but the relative error of $\boldsymbol{\sigma_2(m+1)}$ stands between 10% and 15%. For a first use, it is sufficient. The agglutination phenomenon, presented in Section 3.3, explains that the value T_N alone appears: we have seen that busy periods initiated by T_N tend to amalgamate busy periods initiated by T_j $(j < N)$. We state for any distribution function of the service durations, in case of a finite support:

Property 4: (Impact of longest service times) In stationary regime and in case of **Poisson input** with (5) of busy periods not broken up, and a number $\boldsymbol{(m+1)}$ of equivalent stages as defined by (9), the distribution function of the **local sojourn time**

[at stage $(m+1)$] for an **arbitrary** customer can be approximated (for $0 < t < T_N$) by (41), when the support of this distribution is finite.

7. Conclusion

Finally, Equation (5) *of busy periods not broken up* leads us to a general simplified (41) to easily evaluate the distribution of the local sojourn time in tandem queues, in the case of a finite support for the distribution of the successive service times of a customer (services which are different and which may or may not be mutually dependent).

References

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