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HARMONIC CLOSE-TO-CONVEX MAPPINGS

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Sufficient coefficient conditions for complex functions to be close-to-convex harmonic or convex harmonic are given. Construction of close-to-convex harmonic functions is also studied by looking at transforms of convex analytic functions. Finally, a convolution property for harmonic functions is discussed.

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1. Introduction

Harmonic functions are famous for their use in the study of minimal surfaces and also play important roles in a variety of problems in applied mathematics. Harmonic functions have been studied by differential geometers such as Choquet [2], Kneser [7], Lewy [8], and Rado [9]. Recent interest in harmonic complex functions has been triggered by geometric functions theorists Clunie and Sheil-Small [3].

A continuous function $f = u + iv$ is a *complex-valued harmonic functions* in a domain $\mathcal{D} \subset C$ if both u and v are real harmonic in $\mathcal{D}$. In any simply connected domain, we can write

$$f = h + \overline{g} , \tag{1}$$

where h and g are analytic in $\mathcal{D}$. We call h the *analytic part* and g the *co-analytic part* of f. A necessary and sufficient conditions (see [3] or [8]) for f to be *locally univalent* and *sense-preserving in* $\mathcal{D}$ is that $|\, h'(z) \,| > |\, g'(z) \,|$ in $\mathcal{D}$.

Denote by $\mathcal{S}_{\mathcal{H}}$ the class of functions f of the form (1) that are harmonic univalent and sense-preserving in the unit disk $\Delta = \{z\colon |\, z \,| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Thus we may write

[1]Dedicated to KSU Professor Richard S. Varga on his seventieth birthday.

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{2}$$

Note that $\mathcal{S}_{\mathcal{H}}$ reduces to $\mathcal{S}$, the class of normalized univalent analytic functions if the co-analytic part of f is zero. Since $h'(0) = 1 > |g'(0)| = |b_1|$ for $f \in \mathcal{S}_{\mathcal{H}}$, the function $(f - \overline{b_1 f})/(1 - |b_1|^2)$ is also in $\mathcal{S}_{\mathcal{H}}$. Therefore, we may sometimes restrict ourselves to $\mathcal{S}_{\mathcal{H}}^o$, the subclass of $\mathcal{S}_{\mathcal{H}}$ for which $b_1 = f_{\bar{z}}(0) = 0$. In [3], it was shown that $\mathcal{S}_{\mathcal{H}}$ is normal and $\mathcal{S}_{\mathcal{H}}^o$ is compact with respect to the topology of locally uniform convergence. Some coefficient bounds for convex and starlike harmonic functions have recently been obtained by Avci and Zlotkiewicz [1], Jahangiri [5, 6], and Silverman [14].

In this paper, we give sufficient conditions for functions in $\mathcal{S}_{\mathcal{H}}$ to be close-to-convex harmonic or convex harmonic. We also construct close-to-convex harmonic functions by looking at transforms of convex analytic functions. Finally, we discuss a convolution property for harmonic functions.

In the sequel, unless otherwise stated, we will assume that f is of the form (1) with h and g of the form (2).

2. Convex and Close-to-Convex Mappings

Let $\mathcal{K}$, $\mathcal{K}_{\mathcal{H}}$ and $\mathcal{K}_{\mathcal{H}}^o$ denote the respective subclasses of $\mathcal{S}$, $\mathcal{S}_{\mathcal{H}}$ and $\mathcal{S}_{\mathcal{H}}^o$ where the images of $f(\Delta)$ are convex. Similarly, $\mathcal{C}$, $\mathcal{C}_{\mathcal{H}}$ and $\mathcal{C}_{\mathcal{H}}^o$ denote the subclass of $\mathcal{S}$, $\mathcal{S}_{\mathcal{H}}$ and $\mathcal{S}_{\mathcal{H}}^o$ where the images of $f(\Delta)$ are close-to-convex. Recall that a domain $\mathcal{D}$ is convex if the linear segment joining any two points of $\mathcal{D}$ lies entirely in $\mathcal{D}$. A domain $\mathcal{D}$ is called close-to-convex if the complement of $\mathcal{D}$ can be written as a union of non-crossing half-lines. For other equivalent criteria, see [4].

Clunie and Sheil-Small[3] proved the following results.

Theorem A: *If h, g are analytic in Δ with $|h'(0)| > |g'(0)|$ and $h + \epsilon g$ is close-to-convex for each ϵ, $|\epsilon| = 1$, then $f = h + \overline{g}$ is harmonic close-to-convex.*

Theorem B: *If $f = h + \overline{g}$ is locally univalent in Δ and $h + \epsilon g$ is convex for some ϵ, $|\epsilon| \leq 1$, then f is univalent close-to-convex.*

A domain $\mathcal{D}$ is called convex in the direction ϕ $(0 \leq \phi < \pi)$ if every line parallel to the line through 0 and $e^{i\phi}$ has a connected intersection with $\mathcal{D}$. Such a domain is close-to-convex. The convex domains are those convex in every direction. We will also make use of the following result, which may be found in [3].

Theorem C: *A function $f = h + \overline{g}$ is harmonic convex if and only if the analytic functions $h(z) - e^{i\phi} g(z)$, $0 \leq \phi < 2\pi$, are convex in the direction $\phi/2$ and f is suitably normalized.*

The harmonic Koebe function $k_0 = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}^o$ is defined by $h(z) - g(z) = z/(1-z)^2$, $g'(z) = z h'(z)$, which leads to

$$h(z) = \tfrac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}, \; g(z) = \tfrac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}.$$

The function k_o maps Δ onto the complex plane minus the real slit from $-1/6$ to $-\infty$. The coefficients of k_o are $a_n = (2n+1)(n+1)/6$ and $b_n = (2n-1)(n-1)/6$. These coefficient bounds are known to be extremal for the subclass of $\mathcal{S}_{\mathcal{H}}^o$ consisting of typically real functions (e.g., see [3]) and functions that are either starlike or convex in one direction (e.g., see [12]). It is not known if the coefficients of k_o are extremal for all of $\mathcal{S}_{\mathcal{H}}^o$.

Necessary coefficient conditions were found in [3] for functions to be in $\mathcal{C}_{\mathcal{H}}$ and $\mathcal{K}_{\mathcal{H}}$. We now give some sufficient condition for functions to be in these classes. But first we need the following results. See, for example, [13].

Lemma 1: *If $q(z) = z + \sum_{n=2}^{\infty} c_n z^n$ is analytic in Δ, then q maps onto a starlike domain if $\sum_{n=2}^{\infty} n \mid c_n \mid \ \leq 1$ and onto a convex domains if $\sum_{n=2}^{\infty} n^2 \mid c_n \mid \ \leq 1$.*

3. Main Results

Theorem 1: *If $f = h + \overline{g}$ with*

$$\sum_{n=2}^{\infty} n \mid a_n \mid \ + \sum_{n=1}^{\infty} n \mid b_n \mid \ \leq 1, \tag{3}$$

then $f \in \mathcal{C}_{\mathcal{H}}$. The result is sharp.

Proof: In view of Theorem A, we need only prove that $h + \epsilon g$, $\mid \epsilon \mid \ = 1$, is close-to-convex. It suffices to show that

$$t(z) = \tfrac{h+\epsilon g}{1+\epsilon b_1} = z + \sum_{n=2}^{\infty} \left(\tfrac{a_n + \epsilon b_n}{1+\epsilon b_1} \right) z^n \in \mathcal{C}.$$

Since

$$\sum_{n=2}^{\infty} n \left| \tfrac{a_n + \epsilon b_n}{1+\epsilon b_1} \right| \leq \sum_{n=2}^{\infty} \tfrac{n(|a_n| + |b_n|)}{1-|b_1|} \leq 1$$

if and only if (3) holds, $t(z)$ maps Δ onto a starlike domain and consequently $t(z) \in \mathcal{C}$.

To see that the upper bound in (3) cannot be extended to $1 + \delta$, $\delta > 0$, we note that the function $f(z) = z + \frac{1+\delta}{n} z^n$ is not univalent in Δ.

Theorem 2: *If f is locally univalent with $\sum_{n=2}^{\infty} n^2 \mid a_n \mid \ \leq 1$, then $f \in \mathcal{C}_{\mathcal{H}}$.*

Proof: Take $\epsilon = 0$ in Theorem B and apply Lemma 1.

Corollary: *If $\sum_{n=2}^{\infty} n^2 \mid a_n \mid \ \leq 1$ and $\mid g'(z) \mid \ \leq 1/2, z \in \Delta$, then $f \in \mathcal{C}_{\mathcal{H}}$.*

Proof: The function f is locally univalent if $\mid h'(z) \mid \ > \ \mid g'(z) \mid$ for $z \in \Delta$. Since $2\sum_{n=2}^{\infty} n \mid a_n \mid \ \leq \sum_{n=2}^{\infty} n^2 \mid a_n \mid \ \leq 1$, we have $h'(z) > 1 - \sum_{n=2}^{\infty} n \mid a_n \mid \ \geq 1/2$.

We next give a sufficient coefficient condition for f to be convex harmonic.

Theorem 3: *If*

$$\sum_{n=2}^{\infty} n^2 \mid a_n \mid \ + \sum_{n=1}^{\infty} n^2 \mid b_n \mid \ \leq 1, \tag{4}$$

then $f \in \mathcal{K}_{\mathcal{H}}$. The result is sharp.

Proof: By Theorem C, it suffices to show that $h - e^{i\phi} g$ is convex in Δ. Set

$$s(z) = \tfrac{h - e^{i\phi} g}{1 - e^{i\phi} b_1} = z + \sum_{n=2}^{\infty} \left(\tfrac{a_n - e^{i\phi} b_n}{1 - e^{i\phi} b_1} \right) z^n.$$

Since

$$\sum_{n=2}^{\infty} n^2 \left| \tfrac{a_n - e^{i\phi} b_n}{1 - e^{i\phi} b_1} \right| \leq \sum_{n=2}^{\infty} \tfrac{n^2(|a_n| + |b_n|)}{1-|b_1|} \leq 1$$

if and only if (4) holds, we see from Lemma 1 that $s(z) \in \mathcal{K}$ and consequently $f \in \mathcal{K}_{\mathcal{H}}$.

The function $f(z) = z + \frac{1+\delta}{n^2} z^n$, $\delta > 0$, shows that the upper bound in (4) cannot be improved.

Remark: The coefficient bound given in Theorem 3 can also be found in [5] and [14]. However, our approach in this paper is different from those given in [5] and [14].

Remark: The well-known results for univalent functions that f is convex if and only if zf' is starlike does not carry over to harmonic univalent functions. See [12]. Hence, we cannot conclude from Theorem 3 that (3) is a sufficient condition for f to map Δ onto a starlike domain. Nevertheless, we believe this to be the case. See [5, 6, 14].

We now introduce a class of harmonic close-to-convex functions that are constructed from convex analytic functions.

Theorem 4: *If $h(z) \in \mathcal{K}$ and $w(z)$ is a Schwartz function, then*

$$f(z) = h(z) + \overline{\int_0^z w(t)h'(t)dt} \in \mathcal{C}_{\mathcal{H}}^o.$$

Proof: Write $g'(z) = w(z)h'(z)$. Now for each ϵ, $|\epsilon| = 1$, we observe that

$$Re\frac{h'(z)=\epsilon g'(z)}{h'(z)} = Re(1+\epsilon w(z)) \geq 1 - |z| > 0,\, z \in \Delta.$$

Consequently, $h + \epsilon g$ is close-to-convex and the result follows from Theorem A.

Remark: If we only require that w in Theorem 4 be analytic with $|w(z)| < 1$, $z \in \Delta$, then we may conclude that $f \in \mathcal{C}_{\mathcal{H}}$.

Corollary: *If $h \in \mathcal{K}$ and n is a positive integer, then*

$$f_n(z) = \int_0^z \left(\frac{h(t)}{t}\right)^2 dt + \overline{\int_0^z t^{n-2}h^2(t)dt} \in \mathcal{C}_{\mathcal{H}}^o.$$

Proof: A result of Sheil-Small [10] shows that $\int_0^z (h(t)/t)^2 dt \in \mathcal{K}$ whenever $h \in \mathcal{K}$. Set $w(z) = z^n$ in Theorem 4, and the result follows.

We now give some examples from Theorem 4.

Example 1: Suffridge [15] showed for the partial sums $p_n(z)$ of $e^{1+z} = \sum_{k=0}^{\infty}(1+z)^k/k!$ that

$$C_n(z) = \frac{p_n(z)-p_n(0)}{p_n'(0)} = \sum_{k=1}^{n}\left(\frac{\sum_{l=0}^{n-k}\frac{1}{l!}}{\sum_{l=0}^{n-1}\frac{1}{l!}}\right)\frac{1}{k!}z^k \in \mathcal{K}.$$

Setting $w(z) = z$ in Theorem 4, we see that

$$f(z) = \sum_{k=1}^{n}\left(\frac{\sum_{l=0}^{n-k}\frac{1}{l!}}{\sum_{l=0}^{n-1}\frac{1}{l1}}\right)\left(\frac{(k+1)z^k+k\overline{z}^{k+1}}{(k+1)!}\right) \in \mathcal{C}_{\mathcal{H}}^o.$$

Example 2: Since $h_k(z) = z + z^k/k^2 \in \mathcal{K}$, we get from the Corollary that

$$f_{k,n}(z) = z + \frac{2}{k^3}z^k + \frac{1}{k^4(2k-1)}z^{2k-1} + \overline{\frac{z^{n+1}}{n+1} + \frac{2z^{n+k}}{k^2(n+k)} + \frac{z^{2k+n-1}}{k^4(2k+n-1)}} \in \mathcal{C}_{\mathcal{H}}^o$$

for $k = 2, 3, \ldots,$ and $n = 1, 2, \ldots.$

Example 3: Set $h(z) = z/(1-z)$ and $w(z) = z$ in Theorem 4. Then

$$f(z) = \frac{z}{1-z} + \overline{\int_0^z \frac{t}{(1-t)^2}dt} = 2Re\frac{z}{1-z} + \log(1-\overline{z}\,) \in \mathcal{C}_{\mathcal{H}}^o.$$

We can actually state a more general result for which Example 3 is a special case.

Theorem 5: *If* $b(z)$ *is analytic with* $|b(z)| < 1/|1-z|^2$, $z \in \Delta$, *then*

$$f(z) = \tfrac{z}{1-z} + \overline{\int_0^z b(t)dt} \in \mathcal{C}_{\mathcal{H}}.$$

Proof: Set $h(z) = z/(1-z)$ and $g(z) = \int_o^z b(t)dt$. Then $|h'(z)| = (1/|1-z|^2) > |g'(z)| = |b(z)|$, so that f is locally univalent. Set $\epsilon = 0$ in Theorem B, and the result follows.

Corollary: *If* $b(z)$ *is analytic with* $|b(z)| \leq 1/4$, $z \in \Delta$, *then*

$$\tfrac{z}{1-z} + \overline{\int_0^z b(t)dt} \in \mathcal{C}_{\mathcal{H}}.$$

4. Convolution Condition

The convolution of two harmonic functions $f_1(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^{\,n}$ and $f_2(z) = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} B_n \overline{z}^{\,n}$ is defined by

$$f_1(z) * f_2(z) = (f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} b_n B_n \overline{z}^{\,n}.$$

In [3], it was shown for $\phi \in \mathcal{K}$ and $f \in \mathcal{K}_{\mathcal{H}}$ that $(\phi + \epsilon\overline{\phi}\,) * f \in \mathcal{C}_{\mathcal{H}}(\,|\,\epsilon\,| \leq 1)$. We given an example to show that $\mathcal{K}$ cannot be replaced by $S^*(\alpha)$, $0 \leq \alpha < 1$, the family of functions starlike of order α.

Set

$$\phi(z) = z + \tfrac{1-\alpha}{n-\alpha} z^n \in S^*(\alpha),\ h(z) = \tfrac{z - z^2/2}{(1-z)^2},\ g(z) = \tfrac{-z^2/2}{(1-z)^2}.$$

Then $f = h + \overline{g} \in \mathcal{K}_{\mathcal{H}}$, see [3]. Setting $\epsilon = 0$ in $(\phi + \epsilon\overline{\phi}\,) * f$ we obtain

$$\phi * f = \phi * (h + \overline{g}\,) = \phi * h = \left(z + \tfrac{1-\alpha}{n-\alpha} z^n\right) * \left(z + \sum_{n=2}^{\infty} \tfrac{n+1}{2} z^n\right)$$

$$= z + \tfrac{(1-\alpha)(n+1)}{2(n-\alpha)} z^n,$$

which is not even univalent for $n > 2\alpha/(1-\alpha)$.

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