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ON A CLASS OF NONCLASSICAL HYPERBOLIC EQUATIONS WITH NONLOCAL CONDITIONS

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This paper proves the existence, uniqueness and continuous dependence of a solution of a class of nonclassical hyperbolic equations with nonlocal boundary and initial conditions. Results are obtained by using a functional analysis method based on an a priori estimate and on the density of the range of the linear operator corresponding to the abstract formulation of the considered problem.

Key words: Nonclassical Hyperbolic Equations, Nonlocal Boundary Conditions, A Priori Estimate, Strong Solutions.

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1. Introduction

In this article, we prove the existence, uniqueness and continuous dependence on a solution of a class of nonclassical hyperbolic partial differential equations with nonlocal boundary and initial conditions. Results are obtained by using a functional analysis method based on an a priori estimate and on the density of the range of the operator corresponding to the abstract formulation of the considered problem. The precise statement of the problem is of the form:

Let $b > 0$, $T_p > 0$ $(p = 1, 2)$ and Q: $= \{(x, t_1, t_2) \in \mathbb{R}^3 : 0 < x < b,\ 0 < t_1 < T_1,\ 0 < t_2 < T_2\}$. Find a function $(x, t_1, t_2) \to v(x, t_1, t_2)$, where $(x, t_1, t_2) \in \overline{Q}$, satisfying the equation

$$\mathcal{L}v: = \frac{\partial^2 v}{\partial t_1 \partial t_2} - \frac{\partial}{\partial x}\Big(a(x, t_1, t_2)\frac{\partial v}{\partial x}\Big) = \mathrm{f}(x, t_1, t_2), \tag{1.1}$$

the initial conditions

$$\begin{aligned} &\ell_1 v: = v(x, 0, t_2) = \Phi(x, t_2), && (x, t_2) \in (0, b) \times (0, T_2), \\ &\ell_2 v: = v(x, t_1, 0) = \Psi(x, t_1), && (x, t_1) \in (0, b) \times (0, T_1), \end{aligned} \tag{1.2}$$

the integral condition

$$\int_0^b v(x, t_1, t_2)dx = E(t_1, t_2) \qquad (t_1, t_2) \in (0, T_1) \times (0, T_2), \tag{1.3}$$

and one of the following boundary conditions:

$$\frac{\partial v(0, t_1, t_2)}{\partial x} = \mu(t_1, t_2), \quad (t_2, t_2) \in (0, T_1) \times (0, T_2), \tag{1.4a}$$

$$\int_0^b xv(x,t_1,t_2)dx = \chi(t_1,t_2), \qquad (t_1,t_2) \in (0,T_1)\times(0,T_2), \tag{1.4b}$$

where a, f, Φ, Ψ, E, μ and χ are given functions.

Mixed problems with integral conditions for parabolic equations are studied by different method in Batten [1], Cannon [14], Kamynin [22], Ionkin [19], Cannon and van der Hoek [17, 18], Yurchuk [25], Benouar and Yurchuk [2], Cahlon, Kulkarni, and Shi [15], Cannon, Esteva, and van der Hoek [16], Byszewski [11], Lin [23], Shi [24], Bouziani [3-5], Bouziani-Benouar [7], Jones, Jumarhon, McKee, and Scott [19], and Jumarhon and McKee [20]. A mixed problem with integral condition for a second order pluriparabolic equation has been investigated by Bouziani [6]. Nonlocal nonlinear hyperbolic problems were studied by Byszewski [9-11], by Byszewski and Lakshmikantham [12], and by Byszewski and Papageorgiou [13]. A mixed problem with integral conditions for a second order classical hyperbolic equation has been treated under growth conditions in Bouziani and Benouar [8]. The results of the paper are generalizations of those given in [8-12]. These findings are also continuations of those obtained by the author in [8].

The paper is organized as follows. In Section 2, we state three assumptions on the functions involved in problem (1.1)-(1.4) and we reduce the posed problem to one with homogeneous boundary conditions. In addition, we present an abstract formulation of the considered problem and we define the strong solution of the problem. In Section 3, the uniqueness and continuous dependence of the solution are established. Finally, in Section 4 we prove the existence of the strong solution and offer remarks on its generalizations.

2. Preliminaries

First, we begin with the following assumptions on function a:

Assumption A1:

$$c_0 \le a \le c_1;\ \tfrac{\partial a}{\partial x} \le c_2;\ \tfrac{\partial a}{\partial t_p} \le c_3\ \ (p=1,2) \text{ in } \overline{Q}\,.$$

Assumption A2:

$$c_4 \le \tfrac{\partial a}{\partial t_p},\ c_5 \le \tfrac{\partial^2 a}{\partial t_p^2} \le c_6,\ \tfrac{\partial^2 a}{\partial t_p \partial x} \le c_7\ (p=1,2);\ \tfrac{\partial^2 a}{\partial t_2 \partial t_1},\ \tfrac{\partial^2 a}{\partial t_1 \partial t_2} \le c_8;$$

$$\tfrac{\partial^3 a}{\partial t_1 \partial t_2^2},\ \tfrac{\partial^3 a}{\partial t_2 \partial t_1^2} \le c_9 \text{ in } \overline{Q}\,.$$

In Assumptions A1-A2, we assume that c_i $(i=0,\ldots,9)$ are positive constants. We also, assume that the functions Φ and Ψ satisfy the following:

Assumption A3: Functions Φ and Ψ satisfy the compatibility conditions:

$$\int_0^b \Phi(x,t_2)dx = E(0,t_2), \qquad \int_0^b \Psi(x,t_1)dx = E(t_1,0),$$

$$\tfrac{\partial \Phi(0,t_2)}{\partial x} = \mu(0,t_2), \qquad \tfrac{\partial \Psi(0,t_1)}{\partial x} = \mu(t_1,0)$$

$$\left[\int_0^b x\Phi(x,t_2)dx = \chi(0,t_2), \int_0^b x\Psi(x,t_1)dx = \chi(t_1,0), \text{ respectively}\right],$$

and

$$\Phi(x,0)=\Psi(x,0)$$

where $x\in(0,b), t_p\in[0,T_p](p=1,2)$.

Let us reduce problem (1.1)-(1.3) and $(1.4a)$ [respectively $(1.4b)$] with nonhomogeneous boundary conditions (1.3), $(1.4a)$ [respectively $(1.4b)$] to an equivalent problem with homogeneous conditions by assuming that we are able to find a function $u=\ u(x,t_1,t_2)$ defined as follows:

$$u(x,t_1,t_2)=v(x,t_1,t_2)-U(x,t_1,t_2),$$

where

$$U(x,t_1,t_2)\colon=\tfrac{x}{2b}(2b-3x)\mu(t_1,t_2)+\tfrac{3x^2}{b^3}E(t_1,t_2)$$

$$\big[U(x,t_1,t_2)\colon=\tfrac{1}{b}(-18bx^2+12b^2x+1)E(t_1,t_2)-\tfrac{12}{b}(3x^2-2bx)\chi(t_1,t_2),\ \text{respectively}\big].$$

Consequently, we have to find a function $(x,t_1,t_2)\to u(x,t_1,t_2)$, which is a solution of the problem

$$\mathcal{L}u=\mathrm{f}(x,t_1,t_2)-\mathcal{L}U=f, \tag{2.1}$$

$$\begin{aligned}\ell_1u=u(x,0,t_2)=\Phi(x,t_2)-\ell_1U=\varphi(x,t_2),\\ \ell_2u=u(x,t_1,0)=\Psi(x,t_1)-\ell_2U=\psi(x,t_1),\end{aligned} \tag{2.2}$$

$$\int_0^b u(x,t_1,t_2)dx=0, \tag{2.3}$$

$$\tfrac{\partial u(0,t_1,t_2)}{\partial x}=0 \tag{2.4a}$$

$$\left[\int_0^b xu(x,t_1,t_2)dx=0,\ \text{respectively}\right]. \tag{2.4b}$$

We assume that the functions φ and ψ satisfy conditions of the form (2.3), $(2.4a)$ [respectively $(2.4b)$], i.e.:

Assumption A3′:

$$\int_0^b\varphi(x,0)dx=0,\ \tfrac{\partial\varphi(0,t_2)}{\partial x}=0\ \left[\int_0^b x\varphi(x,0)dx=0,\ \text{respectively}\right],$$

$$\int_0^b\psi(x,0)dx=0,\ \tfrac{\partial\psi(0,t_1)}{\partial x}=0\ \left[\int_0^b x\psi(x,0)dx=0,\ \text{respectively}\right]$$

and

$$\varphi(x,0)=\psi(x,0).$$

In order to write down an abstract formulation of the problem we will need suitable function spaces. Let $B_2^1(0,b)$ be a Hilbert space defined by the author in general form in [3].

Here, we reformulate the author's definition. For this purpose, let $C_0(0,b)$ be the vector space of continuous functions with compact support in $(0,b)$. Since such functions are Lebesgue integrable with respect to dx, we can define on $C_0(0,b)$ the bilinear form $((\,\cdot\,,\,\cdot\,))$ given by

$$((u,\omega)) := \int_0^b \mathcal{T}_x u \cdot \mathcal{T}_x \omega dx, \tag{2.5}$$

where

$$\mathcal{T}_x u := \int_0^x u(\xi,t)d\xi.$$

We recall that $((\,\cdot\,,\,\cdot\,))$ is a scalar product on $C_0(0,b)$ for which $C_0(0,b)$ is not complete. We denote by $B_2^1(0,b)$ a completion of $C_0(0,b)$ for a scalar product defined by (2.5). The norm of function u in $B_2^1(0,b)$ is defined by

$$\| u \|_{B_2^1(0,b)} = ((u,u))^{1/2} = \| \mathcal{T}_x u \|_{L^2(0,b)}.$$

Lemma 1: *For $x \in (0,b)$, we have*

$$\| u \|^2_{B_2^1(0,b)} \leq \frac{b^2}{2} \| u \|^2_{L^2(0,b)}.$$

Proof: See Corollary 1 for $m=1$ in Bouziani [3].

The spaces $L^2((0,T_p),L^2(0,b))$, $L^2((0,T_p),B_2^1(0,b))$, $H^1((0,T_p),L^2(0,b))$, $H^1((0,T_p),B_2^1(0,b))$ $(p=1,2)$, $L^2((0,T_1)\times(0,T_2),B_2^1(0,b))$ are defined as usual.

To problems (2.1)-(2.3), (2.4a) [(2.4b), respectively] we assign the operator $L=(\mathcal{L},\ell_1,\ell_2)$ with the domain $D(L)$ consisting of functions u belonging to $L^2((0,T_1)\times(0,T_2),B_2^1(0,b))$ satisfying the condition $\frac{\partial u}{\partial t_1}$, $\frac{\partial u}{\partial t_2}$,$\frac{\partial^2 u}{\partial t_1 \partial t_2}$, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2} \in L^2((0,T_1)\times(0,T_2),B_2^1(0,b))$ and conditions (2.3), (2.4a) [(2.4b), respectively]. The operator L is considered from B to F, where B is the Banach space consisting of functions $u \in L^2((0,T_1)\times(0,T_2),B_2^1(0,b))$ having finite norms

$$\| u \|_B := \left(\sup_{0 \leq \tau_2 \leq T_2} \| u(\,\cdot\,,\,\cdot\,,\tau_2) \|^2_{H^1((0,T_1),B_2^1(0,b))} + \sup_{0 \leq \tau_1 \leq T_1} \| u(\,\cdot\,,\tau_1,\,\cdot\,) \|^2_{H^1((0,T_2),B_2^1(0,b))} \right)^{\frac{1}{2}}$$

and satisfying conditions (2.3), (2.4a) [(2.4b), respectively], and F is a Hilbert space of vector-valued functions (f,φ,ψ) with the finite norms

$$\| (f,\varphi,\psi) \|_F := \Big(\| f \|^2_{L^2((0,T_1)\times(0,T_2),B_2^1(0,b))} + \| \varphi \|^2_{H^1((0,T_2),L^2(0,b))} + \| \Psi \|^2_{H^1((0,T_1),L^2(0,b))} \Big)^{\frac{1}{2}}.$$

Finally, we give a definition of a strong solution. Let $\overline{L}$ be the closure of the operator L with the domain $D(\overline{L}\,)$.

Definition: A solution of the operator equation

$$\overline{L}\, u = (f, \varphi, \psi)$$

is called a *strong solution* of problem (2.1)-(2.3), $(2.4a)$ [$(2.4b)$, respectively].

3. Uniqueness and Continuous Dependence of the Solution

In this section we will prove an a priori estimate. The uniqueness and continuous dependence of the solution upon the data presented here are direct consequences of the a priori estimate.

Theorem 1: *Under Assumption A*1*, the solution of problem* (2.1)-(2.3), $(2.4a)$ [*respectively,* $(2.4b)$] *satisfies the following a priori estimate:*

$$\| u \|_B \leq c \| Lu \|_F, \tag{3.1}$$

where $c > 0$ *is a constant independent of* u.

Proof: Taking the scalar product, in $B_2^1(0,b)$, of equation (2.1) and $Mu := 2\left(\frac{\partial u}{\partial t_1} + \frac{\partial u}{\partial t_2}\right)$, we obtain

$$\begin{aligned} &\tfrac{\partial}{\partial t_2}\left\|\tfrac{\partial u}{\partial t_1}\right\|^2_{B_2^1(0,b)} + \tfrac{\partial}{\partial t_1}\left\|\tfrac{\partial u}{\partial t_2}\right\|^2_{B_2^1(0,b)} + \int_0^b a(x,t_1,t_2)\tfrac{\partial}{\partial t_1}u^2 dx \\ &+ \int_0^b a(x,t_1,t_2)\tfrac{\partial}{\partial t_2}u^2 dx + 2\int_0^b \tfrac{\partial a(x,t_1,t_2)}{\partial x} u\mathcal{T}_x\left(\tfrac{\partial}{\partial t_1} + \tfrac{\partial u}{\partial t_2}\right)dx \\ &= 2\left(\mathcal{L}u, \tfrac{\partial u}{\partial t_1} + \tfrac{\partial u}{\partial t_2}\right)_{B_2^1(0,b)}. \end{aligned} \tag{3.2}$$

Integrating (3.2) over $(0,\tau_1)\times(0,\tau_2)$, where $0 < \tau_1 \leq T_1$ and $0 < \tau_2 \leq T_2$, we have

$$\begin{aligned} &\int_0^{\tau_1}\left\|\tfrac{\partial u(x,t_1,\tau_2)}{\partial t_1}\right\|^2_{B_2^1(0,b)} dt_1 + \int_0^b\int_0^{\tau_1} a(x,t_1,\tau_2)(u(x,t_1,\tau_2))^2 dx dt_1 \\ &+ \int_0^{\tau_2}\left\|\tfrac{\partial u(x,\tau_1,t_2)}{\partial t_2}\right\|^2_{B_2^1(0,b)} dt_2 + \int_0^b\int_0^{\tau_2} a(x,\tau_1,t_2)(u(x,\tau_1,t_2))^2 dx dt_2 \\ &= 2\int_0^{\tau_1}\int_0^{\tau_2}\left(f, \tfrac{\partial u}{\partial t_1} + \tfrac{\partial u}{\partial t_2}\right)_{B_2^1(0,b)} dt_1 t_2 + \int_0^{\tau_1}\left\|\tfrac{\partial \psi(x,t_1)}{\partial t_1}\right\|^2_{B_2^1(0,b)} dt_1 \\ &+ \int_0^b\int_0^{\tau_1} a(x,t_1,0)(\psi(x,t_1))^2 dx dt_1 + \int_0^{\tau_2}\left\|\tfrac{\partial \varphi(x,t_2)}{\partial t_2}\right\|^2_{B_2^1(0,b)} dt_2 \\ &+ \int_0^b\int_0^{\tau_2} a(x,0,t_2)(\varphi(x,t_2))^2 dx dt_2 \end{aligned} \tag{3.3}$$

$$+\int_0^b\int_0^{\tau_1}\int_0^{\tau_2}\Big[\tfrac{\partial a(x,t_1,t_2)}{\partial t_1}+\tfrac{\partial a(x,t_1,t_2)}{\partial t_2}\Big]u^2dxdt_1dt_2$$

$$-2\int_0^b\int_0^{\tau_1}\int_0^{\tau_2}\tfrac{\partial a(x,t_1,t_2)}{\partial x}u\mathcal{T}_x\Big(\tfrac{\partial u}{\partial t_1}+\tfrac{\partial u}{\partial t_2}\Big)dxdt_1dt_2.$$

Observe that

$$2\int_0^{\tau_1}\int_0^{\tau_2}\Big(f,\tfrac{\partial u}{\partial t_1}+\tfrac{\partial u}{\partial t_2}\Big)_{B_2^1(0,b)}dt_1dt_2\le 2\int_0^{\tau_1}\int_0^{\tau_2}\parallel f\parallel^2_{B_2^1(0,b)}dt_1dt_2$$

$$+\int_0^{\tau_1}\int_0^{\tau_2}\Big\|\tfrac{\partial u}{\partial t_1}\Big\|^2_{B_2^1(0,b)}dt_1dt_2+\int_0^{\tau_1}\int_0^{\tau_2}\Big\|\tfrac{\partial u}{\partial t_2}\Big\|^2_{B_2^1(0,b)}dt_2dt_2. \tag{3.4}$$

It is easy to see that

$$-2\int_0^b\int_0^{\tau_1}\int_0^{\tau_2}\tfrac{\partial a(x,t_1,t_2)}{\partial x}u\mathcal{T}_x\Big(\tfrac{\partial u}{\partial t_1}+\tfrac{\partial u}{\partial t_2}\Big)dxdt_1dt_2$$

$$\le 2\int_0^b\int_0^{\tau_1}\int_0^{\tau_2}\Big(\tfrac{\partial a(x,t_1,t_2)}{\partial x}\Big)^2u^2dxdt_1dt_2 \tag{3.5}$$

$$+\int_0^{\tau_1}\int_0^{\tau_2}\Big[\Big\|\tfrac{\partial u}{\partial t_1}\Big\|^2_{B_2^1(0,b)}+\Big\|\tfrac{\partial u}{\partial t_2}\Big\|^2_{B_2^1(0,b)}\Big]dt_1dt_2.$$

From (3.3)-(3.5) and from Assumption A1, we get

$$\int_0^{\tau_1}\Big[\parallel u(x,t_1,\tau_2)\parallel^2_{L^2(0,b)}+\Big\|\tfrac{\partial u(x,t_1,\tau_2)}{\partial t_1}\Big\|^2_{B_2^1(0,b)}\Big]dt_1$$

$$+\int_0^{\tau_2}\Big[\parallel u(x,\tau_1,t_2)\parallel^2_{L^2(0,b)}+\Big\|\tfrac{\partial u(x,\tau_1,t_2)}{\partial t_2}\Big\|^2_{B_2^1(0,b)}\Big]dt_2$$

$$\le c_{10}\ \Bigg(\int_0^{\tau_1}\int_0^{\tau_2}\parallel f\parallel^2_{B_2^1(0,b)}dt_1dt_2+\int_0^{\tau_1}\Big[\parallel\psi\parallel^2_{L^2(0,b)}+\Big\|\tfrac{\partial\psi}{\partial t_1}\Big\|^2_{B_2^1(0,b)}\Big]dt_1 \tag{3.6}$$

$$+\int_0^{\tau_2}\Big[\parallel\varphi\parallel^2_{L^2(0,b)}+\Big\|\tfrac{\partial\varphi}{\partial t_2}\Big\|^2_{B_2^1(0,b)}\Big]dt_2\Bigg)$$

$$+c_{11}\ \Bigg(\int_0^{\tau_1}\int_0^{\tau_2}\Big[\parallel u\parallel^2_{L^2(0,b)}+\Big\|\tfrac{\partial u}{\partial t_1}\Big\|^2_{B_2^1(0,b)}+\Big\|\tfrac{\partial u}{\partial t_2}\Big\|^2_{B_2^1(0,b)}\Big]dt_1dt_2\Bigg),$$

where

$$c_{10}=\tfrac{\max(4,c_1)}{\min(c_0,1)}$$

and

$$c_{11} = \frac{\max(2c_2,1)}{\min(c_0,1)}.$$

To finish the proof of Theorem 1, we will need the following result:

Lemma 2: *If f_1 and f_2 are nonnegative functions on the rectangle $(0,T_1)\times\ (0,T_2)$, f_1 is integrable on $(0,T_1)\times(0,T_2)$, and f_2 is nondecreasing in $(0,T_1)\times\ (0,T_2)$ with respect to each of its variables separately, then*

$$f_1(\tau_1,\tau_2) \le \exp(2c(\tau_1+\tau_2))f_2(\tau_1,\tau_2)$$

is a consequence of the inequality

$$f_1(\tau_1,\tau_2) \le c\ \left(\int_0^{\tau_1} f_1(t_1,\tau_2)dt_1 + \int_0^{\tau_2} f_1(\tau_1,t_2)dt_2\right) + f_2(\tau_1,\tau_2).$$

The proof of the above lemma is analogous to the proof of Lemma 1 in Bouziani [6].

Continuing the proof of Theorem 1, we apply Lemma 2 to (3.6). For this purpose, we denote the left-hand side of (3.6) by $f_1(\tau_1,\tau_2)$, and the sum of three first integrals on the right-hand side of (3.6) by $f_2(\tau_1,\tau_2)$. This procedure eliminates the last integral of the right-hand side of (3.6) and yields:

$$\begin{aligned}
&\int_0^{\tau_1}\left[\parallel u(x,t_1,\tau_2)\parallel^2_{L^2(0,b)} + \left\|\tfrac{\partial u(x,t_1,\tau_2)}{\partial t_1}\right\|^2_{B_2^1(0,b)}\right]dt_1\\
&+\int_0^{\tau_2}\left[\parallel u(x,\tau_1,t_2)\parallel^2_{L^2(0,b)} + \left\|\tfrac{\partial u(x,\tau_1,t_2)}{\partial t_2}\right\|^2_{B_2^1(0,b)}\right]dt_2 \qquad (3.7)\\
&\le c_{11}\exp(2c_{10}(\tau_1+\tau_2))\ \left(\int_0^{\tau_1}\int_0^{\tau_2}\parallel f\parallel^2_{B_2^1(0,b)}dt_1dt_2\right.\\
&\quad+\int_0^{\tau_1}\left[\parallel \psi\parallel^2_{L^2(0,b)} + \left\|\tfrac{\partial \psi}{\partial t_1}\right\|^2_{B_2^1(0,b)}\right]dt_1\\
&\quad\left.+\int_0^{\tau_2}\left[\parallel \varphi\parallel^2_{L^2(0,b)} + \left\|\tfrac{\partial \varphi}{\partial t_2}\right\|^2_{B_2^1(0,b)}\right]dt_2\right).
\end{aligned}$$

According to Lemma 1, we bound below the first and the third terms on the left-hand side of (3.7) and we bound above the third and the fifth terms on the right-hand side of (3.7). Consequently, we obtain

$$\begin{aligned}
&\parallel u(x,t_1,\tau_2)\parallel^2_{H^1((0,T_1),B_2^1(0,b))} + \parallel u(x,\tau_1,t_2)\parallel^2_{H^1((0,T_2),B_2^1(0,b))}\\
&\le c_{12}\Big(\parallel f\parallel^2_{L^2((0,T_1)\times(0,T_2),B_2^1(0,b))} + \parallel \psi\parallel^2_{H^1((0,T_1),L^2(0,b))} \qquad (3.8)\\
&\quad+\parallel \varphi\parallel^2_{H^1((0,T_2),L^2(0,b))}\Big),
\end{aligned}$$

where

$$c_{12} = \frac{c_{11}\exp(2c_{10}(T_1+T_2))\max\left(1,\frac{b^2}{2}\right)}{\min\left(1,\frac{2}{b^2}\right)}.$$

Since the right-hand side of inequality (3.8) does not depend on τ_1 and τ_2, then, in the left-hand side of (3.8) we can take the supremum with respect to τ_p from 0 to T_p $(p = 1, 2)$. Therefore, we obtain (3.1) with $c = c_{12}^{1/2}$. The proof of Theorem 1 is complete.

Proposition 1: *The operator L from B to F has a closure.*

The proof of the above proposition is similar to the proof of Proposition 1 in Bouziani [4].

Theorem 1 can be extended to cover strong solutions by passing to the limit.

Corollary 1: *Under the assumptions of Theorem* 1*, there exists a positive constant c such that*

$$\| u \|_B \leq c \| \overline{L}\, u \|_F,$$

where c does not depend on u.

Corollary 1 implies the following:

Corollary 2: *A strong solution of* (2.1)-(2.3), $(2.4a)$ [*respectively,* $(2.4b)$] *is unique, if it exists, and it depends continuously on* (f, φ, ψ).

Corollary 3: *The range $R(\overline{L}\,)$ of solutions for the closure $\overline{L}$ of L is closed in F, $R(\overline{L}\,) = \overline{R(L)}$ and $\overline{L}^{\,-1} = \overline{L^{-1}}$, where $\overline{L^{-1}}$ is the unique extension of L^{-1} (by continuity from $R(L)$ to $\overline{R(L)}$).*

4. Existence of the Solution

In this section we concentrate on the existence of the strong solution of problem (2.1)-(2.3), $(2.4a)$ [respectively, $(2.4b)$]. The main idea is to demonstrate that the range $R(L)$ is dense in F.

Theorem 2: *Suppose that Assumptions A*1 *and A*2 *are satisfied. Then, for arbitrary $f \in L^2((0,T_1)\times(0,T_2), B_2^1(0,b))$, $\varphi \in H^1((0,T_2), L^2(0,b))$ and $\psi \in H^1((0,T_1),$ $L^2(0,b))$, problem* (2.1)-(2.3), $(2.4a)$ [*respectively,* $(2.4b)$] *admits a strong solution* $u = \overline{L}^{\,-1}(f,\varphi,\psi) = \overline{L^{-1}}(f,\varphi,\psi)$.

Proof: To prove the existence of a strong solution of problem (2.1)-(2.3), $(2.4a)$ [respectively $(2.4b)$] for all $(f,\varphi,\psi) \in F$, it remains to prove that $\overline{R(L)} = F$. To this end, we establish the density in the special case, where u belongs to $D_0(L)$, which is equal to the set of all $u \in D(L)$ vanishing in a neighborhood of $t_1 = 0$ and $t_2 = 0$. Then we proceed to the general case.

Proposition 2: *Let the assumptions of Theorem* 2 *be satisfied. If for $g \in L^2((0,T_1)\times(0,T_2); B_2^1(0,b))$ and for all $u \in D_0(L)$, we have*

$$\int_0^{T_1}\int_0^{T_2} ((\mathcal{L}u, g))dt_1dt_2 = 0 \ \ \forall u \in D_0(L). \tag{4.1}$$

Then g vanishes almost everywhere in Q.

Proof of Proposition 2: By the fact that relation (4.1) holds for any function $u \in D_0(L)$, we express it in a special form. Let

$$u=\begin{cases} 0, & 0\le t_p\le s_p,\\ \int\limits_{s_1}^{t_1}\int\limits_{s_2}^{t_2}\frac{\partial^2 u}{\partial\tau_1\partial\tau_2}d\tau_1 d\tau_2, & s_p\le t_p\le T_p \end{cases} \tag{4.2}$$

for $p=1,2$ and let $\frac{\partial^2 u}{\partial t_1\partial t_2}$ be a solution of

$$a(\sigma,t_1,t_2)\tfrac{\partial^2 u}{\partial t_1\partial t_2}=\mathcal{T}_{t_p}^{*}g_p=\int\limits_{t_p}^{T_p}g_p d\tau_p, \tag{4.3}$$

where σ is a fixed number belonging to $[0,T]$. Formulas (4.2) and (4.3) imply that u belongs to $D_{s_1,s_2}(L)$, which denotes the set of all $u\in D(L)$ vanishing in the neighborhood of $t_p=s_p$ and $t_p<s_p$ $(p=1,2)$. Putting $s_p=0$ $(p=1,2)$, we have that $u\in D_0(L)$. It follows from (4.2) and (4.3) that

$$g=\sum_{p=1}^{2}g_p=-\left(\tfrac{\partial}{\partial t_1}+\tfrac{\partial}{\partial t_2}\right)\left(a(\sigma,t_1,t_2)\tfrac{\partial^2 u}{\partial t_1\partial t_2}\right). \tag{4.4}$$

To prove that the function g, defined by (4.4), belongs to $L^2((s_1,T_1)\times(s_2,T_2),B_2^1(0,b))$, we apply the following result:

Lemma 3: *If the assumptions of Proposition* 2 *are satisfied, then the function* u*, defined by* (4.2)*, has derivatives of the form* $\frac{\partial^3 u}{\partial t_1\partial t_2^2}$ *and* $\frac{\partial^3 u}{\partial t_1^2\partial t_2}$ *belonging to* $L^2((s_1,T_1)\times(s_2,T_2),B_2^1(0,b))$.

Proof: See Lemma 2 in Bouziani [3].

Relation (4.4) implies that (4.1) can be written in the form

$$\begin{aligned}
&-\int_0^{T_1}\int_0^{T_2}\left(\left(\tfrac{\partial^2 u}{\partial t_1\partial t_2},\tfrac{\partial}{\partial t_1}\left(a(\sigma,t_1,t_2)\tfrac{\partial^2 u}{\partial t_1\partial t_2}\right)\right)\right)dt_1dt_2\\
&-\int_0^{T_1}\int_0^{T_2}\left(\left(\tfrac{\partial^2 u}{\partial t_1\partial t_2},\tfrac{\partial}{\partial t_2}\left(a(\sigma,t_1,t_2)\tfrac{\partial^2 u}{\partial t_1\partial t_2}\right)\right)\right)dt_1dt_2\\
&+\int_0^{T_1}\int_0^{T_2}\left(\left(\tfrac{\partial}{\partial x}\left(a(x,t_1,t_2)\tfrac{\partial u}{\partial x}\right),\tfrac{\partial}{\partial t_1}\left(a(\sigma,t_1,t_2)\tfrac{\partial^2 u}{\partial t_1\partial t_2}\right)\right)\right)dt_1dt_2\\
&+\int_0^{T_1}\int_0^{T_2}\left(\left(\tfrac{\partial}{\partial x}\left(a(x,t_1,t_2)\tfrac{\partial u}{\partial x}\right),\tfrac{\partial}{\partial t_2}\left(a(\sigma,t_1,t_2)\tfrac{\partial^2 u}{\partial t_1\partial t_2}\right)\right)\right)dt_1dt_2=0.
\end{aligned} \tag{4.5}$$

Integrating by parts, each term of (4.5), we obtain

$$\begin{aligned}
&\int_0^{b}\int_{s_2}^{T_2}a(\sigma,s_1,t_2)\left(\mathcal{T}_x\tfrac{\partial^2 u(x,s_1,t_2)}{\partial t_1\partial t_2}\right)^2dxdt_2\\
&=\int_0^{b}\int_{s_1}^{T_1}\int_{s_2}^{T_2}\tfrac{\partial a(\sigma,t_1,t_2)}{\partial t_1}\left(\mathcal{T}_x\tfrac{\partial^2 u}{\partial t_1\partial t_2}\right)^2dxdt_1dt_2\\
&-2\int_0^{T_1}\int_0^{T_2}\left(\left(\tfrac{\partial^2 u}{\partial t_1\partial t_2},\tfrac{\partial}{\partial t_1}\left(a(\sigma,t_1,t_2)\tfrac{\partial^2 u}{\partial t_1\partial t_2}\right)\right)\right)dt_1dt_2,
\end{aligned} \tag{4.6}$$

$$\int_0^b \int_{s_1}^{T_1} a(\sigma,t_1,s_2)\Big(\mathcal{T}_x \tfrac{\partial^2 u(x,t_1,s_2)}{\partial t_1 \partial t_2}\Big)^2 dx dt_1$$

$$= \int_0^b \int_{s_1}^{T_1} \int_{s_2}^{T_2} \tfrac{\partial a(\sigma,t_1,t_2)}{\partial t_2} \Big(\mathcal{T}_x \tfrac{\partial^2 u}{\partial t_1 \partial t_2}\Big)^2 dx dt_1 dt_2 \tag{4.7}$$

$$- 2\int_0^{T_1} \int_0^{T_2} \Big(\Big(\tfrac{\partial^2 u}{\partial t_1 \partial t_2}, \tfrac{\partial}{\partial t_2}\Big(a(\sigma,t_1,t_2)\tfrac{\partial^2 u}{\partial t_1 \partial t_2}\Big)\Big)\Big) dt_1 dt_2,$$

$$\int_0^b \int_{s_1}^{T_1} \Big[\tfrac{\partial^2 a(x,t_1,T_2)}{\partial t_1^2} a(\sigma,t_1,T_2) + \tfrac{\partial a(x,t_1,T_2)}{\partial t_1}\ \tfrac{\partial a(\sigma,t_1,T_2)}{\partial t_1}\Big](u(x,t_1,T_2))^2 dx dt_1$$

$$+ \int_0^b \int_{s_1}^{T_1} a(x,t_1,T_2)a(\sigma,t_1,T_2)\Big(\tfrac{\partial u(x,t_1,T_2)}{\partial t_1}\Big)^2 dx dt_1 \tag{4.8}$$

$$= \int_0^b \int_{s_1}^{T_1} \int_{s_2}^{T_2} \Big[\tfrac{\partial^3 a(x,t_1,t_2)}{\partial t_2 \partial t_1^2} a(\sigma,t_1,t_2) + \tfrac{\partial^2 a(x,t_1,t_2)}{\partial t_1^2}\ \tfrac{\partial a(\sigma,t_1,t_2)}{\partial t_2}$$

$$+ \tfrac{\partial a(x,t_1,t_2)}{\partial t_2 \partial t_1}\ \tfrac{\partial a(\sigma,t_1,t_2)}{\partial t_1} + \tfrac{\partial a(x,t_1,t_2)}{\partial t_1}\ \tfrac{\partial^2 a(\sigma,t_1,t_2)}{\partial t_2 \partial t_1}\Big] u^2 dx dt_1 dt_2$$

$$- 2\int_0^b \int_{s_2}^{T_2} \tfrac{\partial a(x,t_1,t_2)}{\partial t_1} a(\sigma,T_1,t_2) u(x,T_1,t_2) \tfrac{\partial u(x,T_1,t_2)}{\partial t_2} dx dt_2$$

$$+ \int_0^b \int_{s_1}^{T_1} \int_{s_2}^{T_2} \Big[\tfrac{\partial a(x,t_1,t_2)}{\partial t_2} a(\sigma,t_1,t_2) + a(x,t_1,t_2)\tfrac{\partial a(\sigma,t_1,t_2)}{\partial t_2}\Big]\Big(\tfrac{\partial u}{\partial t_1}\Big)^2 dx dt_1 dt_2$$

$$- 2\int_0^b \int_{s_1}^{T_1} \int_{s_2}^{T_2} \tfrac{\partial a(x,T_1,t_2)}{\partial t_1} a(\sigma,t_1,t_2) \tfrac{\partial u}{\partial t_1}\ \tfrac{\partial u}{\partial t_2} dx dt_1 dt_2$$

$$- 2\int_0^b \int_{s_1}^{T_1} \int_{s_2}^{T_2} \Big[\tfrac{\partial^2 a(x,t_1,t_2)}{\partial t_1 \partial x} u + \tfrac{\partial a(x,t_1,t_2)}{\partial x}\ \tfrac{\partial u}{\partial t_1}\Big] a(\sigma,t_1,t_2) \mathcal{T}_x \tfrac{\partial^2 u}{\partial t_1 \partial t_2} dx dt_1 dt_2$$

$$+ 2\int_0^{T_1} \int_0^{T_2} \Big(\Big(\tfrac{\partial}{\partial x}\big(a(x,t_1,t_2)\tfrac{\partial u}{\partial x}\big), \tfrac{\partial}{\partial t_1}\Big(a(\sigma,t_1,t_2)\tfrac{\partial^2 u}{\partial t_1 \partial t_2}\Big)\Big)\Big) dt_1 dt_2,$$

$$\int_0^b \int_{s_2}^{T_2} \Big[\tfrac{\partial a(x,T_1,t_2)}{\partial t_2^2} a(\sigma,T_1,t_2) + \tfrac{\partial a(x,T_1,t_2)}{\partial t_2}\ \tfrac{\partial a(\sigma,T_1,t_2}{\partial t_2}\Big](u(x,T_1,t_2))^2 dx dt_2$$

$$+ \int_0^b \int_{s_2}^{T_2} a(x,T_1,t_2)a(\sigma,T_1,t_2)\Big(\tfrac{\partial u(x,T_1,t_2)}{\partial t_2}\Big)^2 dx dt_2 \tag{4.9}$$

$$= \int_0^b \int_{s_1}^{T_1} \int_{s_2}^{T_2} \Big[\tfrac{\partial^3 a(x,t_1,t_2)}{\partial t_1 \partial t_2^2} a(\sigma,t_1,t_2) + \tfrac{\partial^2 a(x,t_1,t_2)}{\partial t_2^2}\ \tfrac{\partial a(\sigma,t_1,t_2)}{\partial t_1}$$

$$+ \tfrac{\partial^2 a(x,t_1,t_2)}{\partial t_1 \partial t_2}\ \tfrac{\partial a(\sigma,t_1,t_2)}{\partial t_2} + \tfrac{\partial a(x,t_1,t_2)}{\partial t_2}\ \tfrac{\partial^2 a(\sigma,t_1,t_2)}{\partial t_1 \partial t_2}\Big] u^2 dx dt_1 dt_2$$

$$+\int_0^b\int_{s_1}^{T_1}\int_{s_2}^{T_2}\Big[\tfrac{\partial a(x,t_1,t_2)}{\partial t_1}a(\sigma,t_1,t_2)+a(x,t_1,t_2)\tfrac{\partial a(\sigma,t_1,t_2)}{\partial t_1}\Big]\Big(\tfrac{\partial u}{\partial t_2}\Big)^2dxdt_1dt_2$$

$$-2\int_0^b\int_{s_1}^{T_1}\tfrac{\partial a(x,t_1,T_2)}{\partial t_2}a(\sigma,t_1,T_2)u(x,t_1,T_2)\tfrac{\partial u(x,t_1,T_2)}{\partial t_1}dxdt_1$$

$$-2\int_0^b\int_{s_1}^{T_1}\int_{s_2}^{T_2}\tfrac{\partial a(x,t_1,t_2)}{\partial t_2}a(\sigma,t_1,t_2)\tfrac{\partial u}{\partial t_1}\ \tfrac{\partial u}{\partial t_2}dxdt_1dt_2$$

$$-2\int_0^b\int_{s_1}^{T_1}\int_{s_2}^{T_2}\Big[\tfrac{\partial^2 a(x,t_1,t_2)}{\partial t_2\partial x}u+\tfrac{\partial a(x,t_1,t_2)}{\partial x}\ \tfrac{\partial u}{\partial t_2}\Big]a(\sigma,t_1,t_2)\mathcal{T}_x\tfrac{\partial^2 u}{\partial t_1\partial t_2}dxdt_1dt_2$$

$$+2\int_0^{T_1}\int_0^{T_2}\Big(\Big(\tfrac{\partial}{\partial x}\big(a(x,t_1,t_2)\tfrac{\partial u}{\partial x}\big),\tfrac{\partial}{\partial t_2}\Big(a(\sigma,t_1,t_2)\tfrac{\partial^2 u}{\partial t_1\partial t_2}\Big)\Big)\Big)dt_1dt_2.$$

According to Assumptions A1 and A2, we get

$$c_0\int_{s_2}^{T_2}\Big\|\tfrac{\partial^2 u(x,s_1,t_2)}{\partial t_1\partial t_2}\Big\|^2_{B_2^1(0,b)}dt_2\le c_3\int_{s_1}^{T_1}\int_{s_2}^{T_2}\Big\|\tfrac{\partial^2 u}{\partial t_1\partial t_2}\Big\|^2_{B_2^1(0,b)}dt_1dt_2$$

$$-2\int_0^{T_1}\int_0^{T_2}\Big(\Big(\tfrac{\partial^2 u}{\partial t_1\partial t_2},\tfrac{\partial}{\partial t_1}\Big(a(\sigma,t_1,t_2)\tfrac{\partial^2 u}{\partial t_1\partial t_2}\Big)\Big)\Big)dt_1dt_2, \tag{4.10}$$

$$c_0\int_{s_1}^{T_1}\Big\|\tfrac{\partial^2 u(x,t_1,s_2)}{\partial t_1\partial t_2}\Big\|^2_{B_2^1(0,b)}dt_1\le c_3\int_{s_1}^{T_1}\int_{s_2}^{T_2}\Big\|\tfrac{\partial^2 u}{\partial t_1\partial t_2}\Big\|^2_{B_2^1(0,b)}dt_1dt_2$$

$$-2\int_0^{T_1}\int_0^{T_2}\Big(\Big(\tfrac{\partial^2 u}{\partial t_1\partial t_2},\tfrac{\partial}{\partial t_2}\Big(a(\sigma,t_1,t_2)\tfrac{\partial^2 u}{\partial t_1\partial t_2}\Big)\Big)\Big)dt_1dt_2, \tag{4.11}$$

$$(c_5c_0+c_4^2)\int_{s_1}^{T_1}\parallel u(x,t_1,T_2)\parallel^2_{L^2(0,b)}dt_1+c_0^2\int_{s_1}^{T_1}\Big\|\tfrac{\partial u(x,t_1,T_2)}{\partial t_1}\Big\|^2_{L^2(0,b)}dt_1$$

$$\le(c_9c_1+c_6c_3+2c_8c_3+2c_7^2)\int_{s_1}^{T_1}\int_{s_2}^{T_2}\parallel u\parallel^2_{L^2(0,b)}dt_1dt_2 \tag{4.12}$$

$$+(2c_1c_3+c_3^2+2c_2^2)\int_{s_1}^{T_1}\int_{s_2}^{T_2}\Big\|\tfrac{\partial u}{\partial t_1}\Big\|^2_{L^2(0,b)}dt_1dt_2$$

$$+\tfrac{2c_3^2c_1^2}{c_0^2}\int_{s_2}^{T_2}\parallel u(x,T_1,t_2)\parallel^2_{L^2(0,b)}dt_2+\tfrac{c_0^2}{2}\int_{s_2}^{T_2}\Big\|\tfrac{\partial u(x,T_1,t_2)}{\partial t_2}\Big\|^2_{L^2(0,b)}dt_2$$

$$
\begin{aligned}
&+c_1^2\int_{s_1}^{T_1}\int_{s_2}^{T_2}\Big\|\tfrac{\partial^2 u}{\partial t_1\partial t_2}\Big\|^2_{B_2^1(0,b)}dt_1dt_2+c_1^2\int_{s_1}^{T_1}\int_{s_2}^{T_2}\Big\|\tfrac{\partial u}{\partial t_2}\Big\|^2_{L^2(0,b)}dt_1dt_2\\
&+2\int_0^{T_1}\int_0^{T_2}\Big(\Big(\tfrac{\partial}{\partial x}\big(a(x,v,t_1,t_2)\tfrac{\partial u}{\partial x}\big),\tfrac{\partial}{\partial t_1}\Big(a(\sigma,t_1,t_2)\tfrac{\partial^2 u}{\partial t_1\partial t_2}\Big)\Big)\Big)dt_1dt_2,\\
&(c_5c_0+c_4^2)\int_{s_2}^{T_2}\parallel u(x,T_1,t_2)\parallel^2_{L^2(0,b)}dt_2+c_0^2\int_{s_2}^{T_1}\Big\|\tfrac{\partial u(x,T_1,t_2)}{\partial t_2}\Big\|^2_{L^2(0,b)}dt_2\\
&\le 2\tfrac{c_1^2c_3^2}{c_0^2}\int_{s_1}^{T_1}\parallel u(x,t_1,T_2)\parallel^2_{L^2(0,b)}dt_1+\tfrac{c_0^2}{2}\int_{s_1}^{T_1}\Big\|\tfrac{\partial u(x,t_1,T_2)}{\partial t_1}\Big\|^2dxdt_1\\
&+(c_9c_1+c_6c_3+2c_8c_3+2c_7^2)\int_{s_1}^{T_1}\int_{s_2}^{T_2}\parallel u\parallel^2_{L^2(0,b)}dt_1dt_2\\
&+(2c_1c_3+c_3^2+2c_2^2)\int_{s_1}^{T_1}\int_{s_2}^{T_2}\Big\|\tfrac{\partial u}{\partial t_2}\Big\|^2_{L^2(0,b)}dt_1dt_2\\
&+c_1^2\int_{s_1}^{T_1}\int_{s_2}^{T_2}\Big\|\tfrac{\partial u}{\partial t_1}\Big\|^2_{L^2(0,b)}dt_1dt_2+c_1^2\int_{s_1}^{T_1}\int_{s_2}^{T_2}\Big\|\tfrac{\partial^2 u}{\partial t_1\partial t_2}\Big\|^2_{B_2^1(0,b)}dt_1dt_2\\
&+2\int_0^{T_1}\int_0^{T_0}\Big(\Big(\tfrac{\partial}{\partial x}\big(a(x,t_1,t_2)\tfrac{\partial u}{\partial x}\big),\tfrac{\partial}{\partial t_2}\Big(a(\sigma,t_1,t_2)\tfrac{\partial^2 u}{\partial t_1\partial t_2}\Big)\Big)\Big)dt_1dt_2.
\end{aligned}
\tag{4.13}
$$

Observe that

$$
\begin{aligned}
&\tfrac{2c_1^2c_3^2}{c_0^2}\int_{s_1}^{T_1}\parallel u(x,t_1,T_2)\parallel^2_{L^2(0,b)}dt_1\\
&\le\tfrac{2c_1^2c_3^2}{c_0^2}\int_{s_1}^{T_1}\int_{s_2}^{T_2}\Big(\parallel u\parallel^2_{L^2(0,b)}+\Big\|\tfrac{\partial u}{\partial t_2}\Big\|^2_{L^2(0,b)}\Big)dt_1dt_2,
\end{aligned}
\tag{4.14}
$$

$$
\begin{aligned}
&\tfrac{2c_1^2c_3^2}{c_0^2}\int_{s_2}^{T_2}\parallel u(x,T_1,t_2)\parallel^2_{L^2(0,b)}dt_2\\
&\le\tfrac{2c_1^2c_3^2}{c_0^2}\int_{s_1}^{T_1}\int_{s_2}^{T_2}\Big(\parallel u\parallel^2_{L^2(0,b)}+\Big\|\tfrac{\partial u}{\partial t_1}\Big\|^2_{L^2(0,b)}\Big)dt_1dt_2.
\end{aligned}
\tag{4.15}
$$

Adding inequalities (4.10)-(4.15) and applying (4.5), we get

$$\int_{s_2}^{T_2}\left[\left\|\frac{\partial^2 u(x,s_1,t_2)}{\partial t_1\partial t_2}\right\|^2_{B_2^1(0,b)} + \parallel u(x,T_1,t_2)\parallel^2_{L^2(0,b)} + \left\|\frac{\partial u(x,T_1,t_2)}{\partial t_2}\right\|^2_{L^2(0,b)}\right]dt_2$$

$$+\int_{s_1}^{T_1}\left[\left\|\frac{\partial^2 u(x,t_1,s_2)}{\partial t_1\partial t_2}\right\|^2_{B_2^1(0,b)} + \parallel u(x,t_1,T_2)\parallel^2_{L^2(0,b)} + \left\|\frac{\partial u(x,t_1,T_2)}{\partial t_1}\right\|^2_{L^2(0,b)}\right]dt_1$$

$$\le c_{13}\int_{s_1}^{T_1}\int_{s_2}^{T_2}\left[\left\|\frac{\partial^2 u}{\partial t_1\partial t_2}\right\|^2_{B_2^1(0,b)} + \parallel u\parallel^2_{L^2(0,b)}\right.$$

$$\left.+\left\|\frac{\partial u}{\partial t_1}\right\|^2_{L^2(0,b)} + \left\|\frac{\partial u}{\partial t_2}\right\|^2_{L^2(0,b)}\right]dt_1dt_2, \tag{4.16}$$

where

$$c_{13}\colon = \frac{\max\left(c_3, c_1c_9+c_3c_6+c_2c_8+2c_7^2+\frac{2c_1^2c_3^2}{c_0^2}, 2c_1c_3+c_3^2+2c_2^2+c_1^2+\frac{2c_1^2c_3^2}{c_0^2}\right)}{\min\left(c_0, \frac{c_0^2}{2}, \frac{1}{2}(c_0c_5+c_4^2)\right)}.$$

Inequality (4.16) is basic in the proof of Proposition 2. In order to apply inequality (4.16), we introduce a new function θ by the formula

$$\theta(x,t_1,t_2)\colon = \int_{t_1}^{T_1} u_{\tau_1}d\tau_1 + \int_{t_2}^{T_2} u_{\tau_2}d\tau_2.$$

Consequently,

$$u(x,T_1,t_2) = \theta(x,s_1,t_2), \qquad u(x,t_1,T_2) = \theta(x,t_1,s_2),$$

$$\frac{\partial u(x,T_1,t_2)}{\partial t_2} = \frac{\partial\theta(x,s_1,t_2)}{\partial t_2}, \qquad \frac{\partial u(x,t_1,T_2)}{\partial t_1} = \frac{\partial\theta(x,t_1,s_2)}{\partial t_1}.$$

Therefore (4.16) yields

$$\int_{s_2}^{T_2}\left[\left\|\frac{\partial^2 u(x,s_1,t_2)}{\partial t_1\partial t_2}\right\|^2_{B_2^1(0,b)}\right.$$

$$\left.+\left(1-\tfrac{3}{4}c_{13}(T_2-s_2)\right)\left(\parallel\theta(x,s_1,t_2)\parallel^2_{L^2(0,b)} + \left\|\frac{\partial\theta(x,s_1,t_2)}{\partial t_2}\right\|^2_{L^2(0,b)}\right)\right]dt_2$$

$$+\int_{s_1}^{T_1}\left[\left\|\frac{\partial^2 u(x,t_1,s_2)}{\partial t_1\partial t_2}\right\|^2_{B_2^1(0,b)} + \left(1-\tfrac{3}{4}c_{13}(T_1-s_1)\right)\right.$$

$$\left.\times\left(\parallel\theta(x,t_1,s_2)\parallel^2_{L^2(0,b)} + \left\|\frac{\partial\theta(x,t_1,s_2)}{\partial t_1}\right\|^2_{L^2(0,b)}\right)\right]dt_1$$

$$\le \frac{3c_{13}}{4}\int_{s_1}^{T_1}\int_{s_2}^{T_2}\left[\left\|\frac{\partial^2 u}{\partial t_1\partial t_2}\right\|^2_{B_2^1(0,b)} + \parallel\theta\parallel^2_{L^2(0,b)} + \left\|\frac{\partial\theta}{\partial t_2}\right\|^2_{L^2(0,b)} + \left\|\frac{\partial\theta}{\partial t_1}\right\|^2_{L^2(0,b)}\right]dt_1dt_2. \tag{4.17}$$

Hence, if $s_{p_0} > 0$ satisfies $1 - \frac{3}{4}c_{13}(T_p - s_{p_0}) = 1/2$ $(p = 1, 2)$, then (4.17) becomes

$$\int_{s_2}^{T_2}\left[\left\|\frac{\partial^2 u(x,s_1,t_2)}{\partial t_1 \partial t_2}\right\|^2_{B_2^1(0,b)} + \parallel \theta(x,s_1,t_2) \parallel^2_{L^2(0,b)} + \left\|\frac{\partial \theta(x,s_1,t_2)}{\partial t_2}\right\|^2_{L^2(0,b)}\right]dt_2$$

$$+\int_{s_1}^{T_1}\left[\left\|\frac{\partial^2 u(x,t_1,s_2)}{\partial t_1 \partial t_2}\right\|^2_{B_2^1(0,b)} + \parallel \theta(x,t_1,s_2) \parallel^2_{L^2(0,b)} + \left\|\frac{\partial \theta(x,t_1,s_2)}{\partial t_1}\right\|^2_{L^2(0,b)}\right]dt_1$$

$$\leq \frac{3c_{13}}{2}\int_{s_1}^{T_1}\int_{s_2}^{T_2}\left[\left\|\frac{\partial^2 u}{\partial t_1 \partial t_2}\right\|^2_{B_2^1(0,b)} + \parallel \theta \parallel^2_{L^2(0,b)}\right.$$

$$\left. + \left\|\frac{\partial \theta}{\partial t_2}\right\|^2_{L^2(0,b)} + \left\|\frac{\partial \theta}{\partial t_1}\right\|^2_{L^2(0,b)}\right]dt_1 dt_2. \tag{4.18}$$

Applying a variant of Lemma 2 for the case of the reversed intervals to (4.18), to eliminate the double integrals on the right-hand side in (4.18), we conclude that $g = 0$ almost everywhere in $(0,b) \times (s_1, T_1) \times (s_2, T_2)$. Proceeding in this way, step by step, we prove that $g = 0$ almost everywhere in Q. The proof of Proposition 2 is complete.

To finish the proof of Theorem 2, consider the general case. Let the element $W = (f, \varphi, \psi) \in F$ be orthogonal to $R(L)$, so that

$$(\mathcal{L}u, f)_{L^2((0,T_1)\times(0,T_2),B_2^1(0,b))} + (\ell_1 u, \varphi)_{H^1((0,T_2),L^2(0,b))}$$

$$+ (\ell_2 u, \psi)_{H^1((0,T_1),L^2(0,b))} = 0, \ \forall u \in D(L). \tag{4.19}$$

Putting u in (4.19) equal to any element of $D_0(L)$, we get

$$(\mathcal{L}u, f)_{L^2((0,T_1)\times(0,T_2),B_2^1(0,b))} = 0, \ \forall u \in D_0(L).$$

Proposition 2 implies that $f \equiv 0$. Hence, (4.19) implies that

$$(\ell_1 u, \varphi)_{H^1((0,T_2),L^2(0,b))} + (\ell_2 u, \psi)_{H^1((0,T_1),L^2(0,b))} = 0, \forall u \in D(L).$$

Since $\ell_1 u$ and $\ell_2 u$ are independent and the ranges of values ℓ_1 and ℓ_2 are everywhere dense in $H^1((0,T_2),L^2(0,b))$, $H^1((0,T_1),L^2(0,b))$, the above equality implies that $\varphi \equiv 0$, and $\psi \equiv 0$ (recall that φ and ψ satisfy Assumption A3′). Hence $\overline{R(L)} = F$. Therefore, the proof of Theorem 2 is complete.

Remark 1: If equation (1.1) is replaced by

$$\mathcal{L}v: = \frac{\partial^2 v}{\partial t_1 \partial t_2} - \frac{\partial}{\partial x}\left(a(x,t_1,t_2)\frac{\partial v}{\partial x}\right) = f(x,t_1,t_2,v)$$

then the existence, uniqueness and continuous dependence upon the data of a solution can be proved by the same method, provided that f satisfies a growth condition of the form

$$\mid f(x,t_1,t_2,v) \mid \ \leq \widetilde{f}\ (x,t_1,t_2) + C \mid v \mid .$$

Remark 2: Our results can be extended to the following nonlocal hyperbolic problem:

$$\begin{cases} \mathcal{L}v: = \frac{\partial^2 v}{\partial t_1 \partial t_2} + \text{sign}[(1 - \mid \lambda_1 \mid^2)(1 - \mid \lambda_2 \mid^2)] & \\ \qquad\qquad \frac{\partial}{\partial x}(a(x,t_1,t_2)\frac{\partial v}{\partial x}) = f,\ (x,t_1,t_2) \in Q, \lambda_1, \lambda_2 \in C, & \\ \ell_1 v: = v \mid_{t_1=0} - \lambda_1 v \mid_{t_1=T_1} = \Phi(x,t_2), & (x,t_2) \in (0,b) \times (0,T_2), \\ \ell_2 v: = v \mid_{t_2=0} - \lambda_2 v \mid_{t_2=T_2} = \Psi(x,t_1), & (x,t_1) \in (0,b) \times (0,T_1), \\ \frac{\partial v(0,t_1,t_2)}{\partial x} = \mu_1(t_1,t_2), & (t_1,t_2) \in (0,T_1) \times (0,T_2), \\ \int\limits_0^b v(x,t_1,t_2)dx = E(t_1,t_2), & (t_1,t_2) \in (0,T_1) \times (0,T_2) \\ [\int\limits_0^b xv(x,t_1,t_2)dx = \chi(t_1,t_2), & (t_1,t_2) \in (0,T_1) \in (0,T_2), \text{ respectively}]. \end{cases}$$

Remark 3: Our results can also be extended to the following problem:

$$\begin{cases} \mathcal{L}v: = \frac{\partial^2 v}{\partial t_1 \partial t_2} - \frac{\partial^{2m}}{\partial x^{2m}}(a(x,t_1,t_2)v) = f, & (x,t_1,t_2) \in Q, \\ \ell_1 v: = v(x,0,t_2) = \Phi(x,t_2), & (x,t_2) \in (0,b) \times (0,T_2), \\ \ell_2 v: = v(x,t_1,0) = \Psi(x,t_1), & (x,t_1) \in (0,b) \times (0,T_1), \\ \frac{\partial^p v(0,t_1,t_2)}{\partial x^p} = 0, \quad 0 < p \le m, & (t_1,t_2) \in (0,T_1) \times (0,T_2), \\ \int\limits_0^b x^q v(x,t)dx = 0, \quad p \le q \le 2m-1, & (t_1,t_2) \in (0,T_1) \times (0,T_2), \end{cases}$$

where m, p, q are integers.

References

[1] Batten, Jr., G.W., Second order correct boundary condition for the numerical solution of the mixed boundary problem for parabolic equations, *Math. Comput.* **17** (1963), 405-413.

[2] Benouar, N.E. and Yurchuk, N.I., Mixed problem with an integral condition for parabolic equations with the Bessel operator, *Differ. Urav.* **27**:12 (1991), 2094-2098.

[3] Bouziani, A., Mixed problem with integral conditions for a certain parabolic equation, *J. of Appl. Math. and Stoch. Anal.* **9**:3 (1996), 323-330.

[4] Bouziani, A., On a third order parabolic equation with a nonlocal boundary condition, *J. of Appl. Math. and Stoch. Anal.*, (in press).

[5] Bouziani, A., Solution forte d'un probléme mixte avec condition non locale pour une classe d'équations paraboliques, *Maghreb Math. Rev.* **6**:1 (1997), 1-17.

[6] Bouziani, A., Mixed problem with nonlocal condition for certain pluriparabolic equations, *Hiroshima Math. J.* **27**:3 (1997), 373-390.

[7] Bouziani, A. and Benouar, N.E., Probléme mixte avec conditions intégrales pour une classe d'équations paraboliques, *Comptes Rendus de l'Académie des Sciences Paris* **321** (1995), 1177-1182.

[8] Bouziani, A. and Benouar, N.E., Probléme mixte avec conditions intégrales pour une classe d'équations hyperboliques, *Bull. Belgian Math. Soc. Simon Stevan* **31** (1996), 125-133.

[9] Byszewski, L., Existence and uniqueness of solutions of nonlocal problems for hyperbolic equation $u_{xt} = F(x,t,u,u_x)$, *J. of Appl. Math. and Stoch. Anal.* **3**:3 (1990), 163-168.

[10] Byszewski, L., Theorem about existence and uniqueness of continuous solutions of nonlocal problem for nonlinear hyperbolic equation, *Appl. Anal.* **40** (1991), 173-180.
[11] Byszewski, L., *Differential and Functional-Differential Problems with Nonlocal Conditions*, Cracow Univ. of Tech. Monograph **184**, Cracow 1995.
[12] Byszewski, L. and Lakshmikantham, V., Monotone iterative technique for nonlocal hyperbolic differential problem, *J. of Math. and Phys. Sci.* **26**:4 (1992), 345-359.
[13] Byszewski, L. and Papageorgiou, N.S., An application of noncompactness technique to an investigation of the existence of solutions to a nonlocal multivalued Darboux problem, *J. of Appl. Math. and Stoch. Anal.* (1998), (in press).
[14] Cannon, J.R., The solution of the heat equation subject to the specification of energy, *Quart. Appl. Math.* **21**:2 (1963), 155-160.
[15] Cannon, J.R., The one-dimensional heat equation, In: *Encyclopedia of Mathematics and its Applications* **23**, Addison-Wesley, Reading, MA, Menlo Park, CA (1984).
[16] Cannon, J.R., Esteva, S.P. and Van der Hoek, J., A Galerkin procedure for the diffusion equation subject to the specification of mass, *SIAM J. Numer. Anal.* **24** (1987), 499-515.
[17] Cannon, J.R. and Van der Hoek, J., The existence and the continuous dependence for the solution of the heat equation subject to the specification of energy, *Boll. Uni. Math. Ital. Suppl.* **1** (1981), 253-282.
[18] Cannon, J.R. and Van der Hoek, J., An implicit finite difference scheme for the diffusion of mass in a portion of the domain, In: *Numerical Solutions of Partial Differential Equations* (ed. by J. Noye), North-Holland, Amsterdam (1982), 527-539.
[19] Ionkin, N.I., Solution of boundary value problem in heat conduction theorem with nonlocal boundary conditions, *Diff. Urav.* **13**:2 (1977), 294-304.
[20] Jones, S., Jumarhorn, B., McKee, S. and Scott, J.A., A mathematical model of biosensor, *J. Eng. Math.* **30** (1996), 312-337.
[21] Jumarhorn, B. and McKee, S., On the heat equation with nonlinear and nonlocal boundary conditions, *J. Math. Anal. Appl.* **190** (1995), 806-820.
[22] Kamynin, N.I., A boundary value problem in the theory of the heat conduction with nonclassical boundary condition, *Th. Vychisl. Mat. Mat. Fiz.* **43**:6 (1964), 1006-1024.
[23] Lin, Y., *Parabolic Partial Differential Equations Subject to Nonlocal Boundary Conditions*, Ph.D. Thesis, Washington State University, Pullman, WA 1988.
[24] Shi, P., Weak solution to an evolution problem with a nonlocal constraint, *SIAM J. Math. Anal.* **24**:1 (1993), 46-58.
[25] Yurchuk, N.I., Mixed problem with an integral condition for certain parabolic equations, *Diff. Urav.* **22**:19 (1986), 2117-2126.