EXISTENCE THEORY FOR SINGLE AND MULTIPLE SOLUTIONS TO SEMIPOSITONE DISCRETE DIRICHLET BOUNDARY VALUE PROBLEMS WITH SINGULAR DEPENDENT NONLINEARITIES

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In this paper we establish the existence of single and multiple solutions to the semipositone discrete Dirichlet boundary value problem

$$\begin{cases} \Delta^2 y(i-1) + \mu f(i, y(i)) = 0, & i \in \{1, 2, ..., T\} \\ y(0) = y(T+1) = 0, \end{cases}$$

where $\mu > 0$ is a constant and our nonlinear term $f(i, u)$ may be singular at $u = 0$. .

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1 Introduction

In this paper we establish the existence of single and multiple solutions to the semipositone discrete Dirichlet boundary value problem

$$\begin{cases} \Delta^2 y(i-1) + \mu f(i, y(i)) = 0, & i \in N = \{1, 2, ..., T\} \\ y(0) = y(T+1) = 0, \end{cases} \tag{1.1}$$

where $\mu > 0$ is a constant and $T \in \{1, 2, ...\}$, $N^+ = \{0, 1, ..., T+1\}$ and $y : N^+ \to \mathbf{R}$. Throughout this paper we will assume $f : N \times (0, \infty) \to \mathbf{R}$ is continuous. As a result our nonlinearity $f(i, u)$ may be singular at $u = 0$.

Remark. Recall a map $f : N \times (0, \infty) \to \mathbf{R}$ is continuous if it is continuous as a map of the topological space $N \times (0, \infty)$ into the topological space $\mathbf{R}$. Throughout this paper the topology on N will be the discrete topology.

We will let $C(N^+, \mathbf{R})$ denote the class of map u continuous on N^+ (discrete topology), with norm $|u|_0 = \max_{k \in N^+} |u(k)|$. By a solution to (1.1) we mean a $u \in C(N^+, \mathbf{R})$ such that u satisfies (1.1) for $i \in N$ and u satisfies the boundary (Dirichlet) conditions.

It is of interest to note here that the existence of single and multiple solutions to positone singular boundary value problems in the continuous case have been studied in great detail in the literature [6, 7, 8, 9, 14], and only recently (see for example [10, 16]) papers have appeared which discuss the semipositone nonsingular problem. Also recently, R. P. Agarwal and D. O'Regan [5] discussed the semipositone singular problem, and a very general existence theory was presented in [5] which showed, for example, that the boundary value problem

$$\begin{cases} y'' + \mu(y^{-\alpha} + y^{\beta} - 1) = 0, & 0 < t < 1, \\ y(0) = y(1) = 0, & \alpha > 0, \quad \beta > 1, \quad \mu > 0 \text{ small}, \end{cases} \tag{1.2}$$

has a positive solution $y \in C[0, 1] \cap C^2(0, 1)$ with $y(t) > 0$ for $t \in (0, 1)$. Existence is established in [5] via a general cone fixed point theorem [6, 11]. However, for the discrete case almost all papers in the literature [2, 4, 12, 13] are devoted to singular positone problems, and only recently in [1] the semipositone nonsingular discrete problem has been discussed.

This paper discusses the existence of single and multiple solutions for semipositone singular discrete problems. Some general existence theorems will be presented in Section 2 and there we will show, for example, that the boundary value problem

$$\begin{cases} \Delta^2 y(i) + \mu([y(i)]^{-\alpha} + [y(i)]^{\beta} - 1) = 0, & i \in N, \\ y(0) = 0, \quad y(T+1) = 0, \quad \alpha > 0, \quad \beta > 1, \quad \mu > 0 \text{ small}, \end{cases} \tag{1.3}$$

has two nonnegative solutions. Existence in this paper will be established using the upper and lower solutions method in [15] and a general cone fixed point theorem in [6, 11].

Theorem 1.1.[6] *Let $E = (E, ||\cdot||)$ be a Banach space and let $K \subset E$ be a cone in E, and let $||\cdot||$ be increasing with respect to K. Also, r, R are constants with $0 < r < R$. Suppose $A : \overline{\Omega}_R \cap K \to K$(here $\Omega_R = \{x \in E, ||x|| < R\}$) is a continuous, compact map and assume the conditions*

$$x \neq \lambda A(x), \quad \text{for } \lambda \in [0, 1) \text{ and } x \in \partial\Omega_r \cap K$$

and

$$||Ax|| > ||x||, \qquad for \ \ x \in \partial\Omega_R \cap K$$

hold. Then A has a fixed point in $K \cap \{x \in E : r \le ||x|| \le R\}$.

In this paper we only consider discrete Dirichlet boundary data. It is worth remarking here that we could consider Sturm Liouville boundary data also; however since the arguments are essentially the same (in fact easier if not Dirichlet data) we will leave the details to the reader.

2   Discrete Equations

In this section we present some new results for the semipositone singular problem(1.1). Before we prove our main results we first state two well known results [1].

Lemma 2.1.   *Let* $u \in (N^+, \mathbf{R})$ *satisfy* $u(i) \ge 0$ *for* $i \in N^+$. *If* $y \in C(N^+, \mathbf{R})$ *satisfies*

$$\begin{cases} \Delta^2 y(i-1) + u(i) = 0, & i \in N = \{1, 2, ..., T\} \\ y(0) = y(T+1) = 0, \end{cases}$$

then

$$y(i) \ge q(i)|y|_0 \ \ for \ \ i \in N^+;$$

here

$$q(i) = min\{\frac{T+1-i}{T+1}, \frac{i}{T}\}.$$

Lemma 2.2. *Let* ω *be the solution of*

$$\begin{cases} \Delta^2 y(i-1) + 1 = 0, & i \in N \\ y(0) = y(T+1) = 0, \end{cases}$$

then

$$\omega(i) \le C_0 q(i) \ \ for \ \ i \in N^+;$$

here

$$C_0 = \frac{(T+1)^2}{2} \ \ and \ \ q(i) = min\{\frac{T+1-i}{T+1}, \frac{i}{T}\}.$$

Remark.   In fact $\omega(i) = \frac{i(T+1-i)}{2}, \ \ i \in N^+$.

Theorem 2.1.   *Suppose the following condition are satisfied:*

$$\begin{cases} f : N \times (0, \infty) \to \mathbf{R} \ is \ continuous \ and \ there \ exists \\ a \ constant \ M > 0 \ with \ f(i, u) + M \ge 0 \\ for \ all \ i \in N \ and \ u \in (0, \infty); \end{cases} \tag{2.1}$$

$$\begin{cases} f^*(i, u) = f(i, u) + M \le g(u) + h(u) \ for \ (i, u) \in N \times (0, \infty) \\ with \ g > 0 \ continuous \ and \ nonincreasing \ on \ (0, \infty), \ h \ge 0 \\ continuous \ on \ [0, \infty) \ and \ \frac{h}{g} \ nondecreasing \ on \ (0, \infty); \end{cases} \tag{2.2}$$

$$\exists \ K_0 \ge 0 \ with \ g(ab) \le K_0 g(a)g(b) \quad \forall a > 0, b > 0; \tag{2.3}$$

$$\exists \ r > \mu M C_0 \ with \ \frac{1}{g(1 - \frac{\mu M C_0}{r})\{1 + \frac{h(r)}{g(r)}\}} \int_0^r \frac{du}{g(u)} > \frac{1}{2}\mu K_0 T(T+1); \tag{2.4}$$

and

$$\begin{cases} \text{there exists contants } L > M \text{ and } \epsilon_0 > 0 \text{ such that } g(u) > L \\ \text{for } u \in (0, \epsilon_0), \text{ and } f^*(i, u) > L \text{ for all } (i, u) \in N \times (0, \epsilon_0). \end{cases} \tag{2.5}$$

Then (1.1) has a solution $y \in C(N^+, \mathbf{R})$ *with* $y(i) > 0$ *for* $i \in N$, $|y + \phi|_0 < r$, *here* $\phi(i) = \mu M \omega(i)(\omega$ *is as in Lemma 2.2).*

Proof. Choose $\delta > 0$ and $\delta < r$ with

$$\frac{1}{g(1 - \frac{\mu M C_0}{r})\{1 + \frac{h(r)}{g(r)}\}} \int_\delta^r \frac{du}{g(u)} > \frac{1}{2}\mu K_0 T(T + 1). \tag{2.6}$$

Let $m_0 \in \{1, 2, ...\}$ be chosen so that $\frac{1}{m_0} < \min\{\frac{\delta}{2}, \epsilon_0\}$ and let $N_0 = \{m_0, m_0 + 1, ...\}$.

To show (1.1) has a nonnegative solution $y \in C(N^+, \mathbf{R})$ with $|y + \phi|_0 < r$, we will show

$$\begin{cases} \Delta^2 y(i - 1) + \mu f^*(i, y(i) - \phi(i)) = 0, & i \in N \\ y(0) = y(1) = 0 \end{cases} \tag{2.7}$$

has a solution $y_1 \in C(N^+, \mathbf{R})$ with $y_1(i) > \phi(i)$ for $i \in N$ and $|y_1|_0 < r$, here $f^*(i, u)$ is as in (2.2).

If this is true then $u(i) = y_1(i) - \phi(i)(i \in N^+)$ is a nonnegative solution (positive on N) of (1.1), and $|u + \phi|_0 < r$, since

$$\begin{aligned} \Delta^2 u(i - 1) &= \Delta^2 y_1(i - 1) - \Delta^2 \phi(i - 1) \\ &= -\mu f^*(i, y_1(i) - \phi(i)) + \mu M \\ &= -\mu(f(i, y_1(i) - \phi(i)) + M) + \mu M \\ &= -\mu f(i, y_1(i) - \phi(i)) \\ &= -\mu f(i, u(i)), \quad i \in N. \end{aligned}$$

As a result we will concentrate our study on (2.7).

The idea is to first show that

$$\begin{cases} \Delta^2 y(i - 1) + \mu f^*(i, y(i) - \phi(i)) = 0, & i \in N \\ y(0) = \frac{1}{m}, \quad y(T + 1) = \frac{1}{m}, \quad m \in N_0 \end{cases} \tag{2.8}^m$$

has a solution y_m for each $m \in N_0$ with $y_m(i) \geq \frac{1}{m}$ and $y_m(i) \geq \phi(i)$ for $i \in N^+$, and $|y_m|_0 < r$.

We have the following claim.

Claim 1 $\alpha_m(i) = \frac{1}{m} + lw(i) + \phi(i) = \frac{1}{m} + (l + \mu M)w(i)$, $i \in N^+$ *is a (strict) lower solution for problem* $(2.8)^m$, *here* $0 < l < \min\{\mu(L - M), \frac{\varepsilon_0 - \frac{1}{m_0}}{|w|_0}\}$, $m \in N_0$.

Proof. Notice $\alpha_m(0) = \alpha_m(T + 1) = \frac{1}{m}$, and

$$\begin{aligned} & \Delta^2 \alpha_m(i - 1) + \mu f_m^*(i, \alpha_m(i) - \phi(i)) \\ = & (l + \mu M)\Delta^2 w(i) + \mu f^*(i, \frac{1}{m} + lw(i)) \\ = & -(l + \mu M) + \mu f^*(i, \frac{1}{m} + lw(i)) \\ \geq & -(l + \mu M) + \mu L \\ > & 0, \quad i \in N, \end{aligned}$$

since $lw(i) + \frac{1}{m} \leq l|w|_0 + \frac{1}{m_0} < \varepsilon_0$, and $l < \mu(L - M)$.

In order to seek upper solutions of $(2.8)^m$, we consider the following problem

$$\begin{cases} \Delta^2 y(i - 1) + \mu g(y(i) - \phi(i))(1 + \frac{h(r)}{g(r)}) = 0, & i \in N, \\ y(0) = \frac{1}{m}, \quad y(T + 1) = \frac{1}{m}, \quad m \in N_0. \end{cases} \tag{2.9}^m$$

In the same way as Claim 1, we can easily prove $\alpha_m(i) = \frac{1}{m} + lw(i) + \phi(i) = \frac{1}{m} + (l + \mu M)w(i)$ is also a (strict) lower solution of $(2.9)^m$.

Let $\beta_m^0 \in C(N^+, \mathbf{R})$ be the unique solution of the discrete boundary value problem

$$\begin{cases} \Delta^2 y(i-1) + \mu g(\alpha_m(i) - \phi(i))(1 + \frac{h(r)}{g(r)}) = 0, & i \in N, \\ y(0) = \frac{1}{m}, \quad y(T+1) = \frac{1}{m}, \quad m \in N_0. \end{cases}$$

Then we have

$$\beta_m^0(i) = \frac{1}{m} + \mu \sum_{j=1}^{T} G(i,j)g(\alpha_m(j) - \phi(j))(1 + \frac{h(r)}{g(r)}), \quad i \in N^+;$$

here $G(i,j)$ is the Green's function to the problem

$$\begin{cases} -\Delta^2 u(i-1) = 0, & i \in N \\ u(0) = u(T+1) = 0, \end{cases}$$

so

$$G(i,j) = \begin{cases} \frac{j(T+1-i)}{T+1}, & 0 \le j \le i-1, \\ \frac{i(T+1-j)}{T+1}, & i \le j \le T+1. \end{cases} \tag{2.10}$$

Then we have

$$\begin{aligned} \beta_m^0(i) &= \frac{1}{m} + \mu \sum_{j=1}^{T} G(i,j)g(\frac{1}{m} + lw(j))(1 + \frac{h(r)}{g(r)}) \\ &\ge \frac{1}{m} + \mu(1 + \frac{h(r)}{g(r)})g(\frac{1}{m_0} + l|w|_0) \sum_{j=1}^{T} G(i,j) \\ &= \frac{1}{m} + \mu(1 + \frac{h(r)}{g(r)})g(\frac{1}{m_0} + l|w|_0)w(i) \\ &\ge \frac{1}{m} + \mu(1 + \frac{h(r)}{g(r)})Lw(i) \\ &\ge \frac{1}{m} + \mu Lw(i) \\ &\ge \frac{1}{m} + (l + \mu M)w(i) \\ &= \alpha_m(i), \quad i \in N^+. \end{aligned}$$

Thus we obtain

$$\begin{aligned} &\Delta^2\beta_m^0(i-1) + \mu g(\beta_m^0(i) - \phi(i))(1 + \frac{h(r)}{g(r)}) \\ =\ &\mu(1 + \frac{h(r)}{g(r)})[g(\beta_m^0(i) - \phi(i)) - g(\alpha_m(i) - \phi(i))] \le 0, \quad i \in N, \end{aligned}$$

so that β_m^0 is an upper solution for problem $(2.9)^m$.

If we now take $\alpha_m^0 \equiv \alpha_m$, we have that α_m^0 and β_m^0 are, respectively, a lower and an upper solution of $(2.9)^m$ with $\alpha_m^0(i) \le \beta_m^0(i)$, for all $i \in N^+$. So by the upper and lower solutions method in [15], we know that there exists a solution $\beta_m \in C(N^+, \mathbf{R})$ of $(2.9)^m$ such that

$$\alpha_m(i) = \alpha_m^0(i) \le \beta_m(i) \le \beta_m^0(i), \quad \forall i \in N^+.$$

Now we claim that $|\beta_m|_0 < r$. Suppose this is false, i.e., suppose $|\beta_m|_0 \ge r$. Since $\Delta^2\beta_m(i-1) \le 0$ on N and $\beta_m(0) - \frac{1}{m} = \beta_m(T+1) - \frac{1}{m} = 0$, then $\beta_m(i) - \frac{1}{m} \ge q(i)|\beta_m - \frac{1}{m}|_0$, $i \in N^+$(from Lemma 2.1). Thus, $\beta_m(i) \ge \frac{1}{m} + q(i)(|\beta_m|_0 - \frac{1}{m}) \ge q(i)|\beta_m|_0 \ge q(i)r$, $i \in N^+$. Also, there exists $\sigma_m \in N$ with $\Delta\beta_m(i) \ge 0$ on $[0, \sigma_m) = \{0, ..., \sigma_m - 1\}$, $\Delta\beta_m(i) \le 0$ on $[\sigma_m, T+1) = \{\sigma_m, ..., T\}$ (note $\Delta^2\beta_m(i-1) \le 0$ on N), and $\beta_m(\sigma_m) = |\beta_m|_0 \ge r$. Notice also for $i \in N^+$ that

$$\beta_m(i) - \phi(i) = \beta_m(i)[1 - \frac{\mu Mw(i)}{\beta_m(i)}] \ge \beta_m(i)[1 - \frac{\mu MC_0}{r}],$$

since $\beta_m(i) \ge q(i)|\beta_m|_0 \ge q(i)r$, and $w(i) \le C_0 q(i)$ for $i \in N^+$. Thus

$$\beta_m(i) - \phi(i) \ge \beta_m(i)[1 - \frac{\mu M C_0}{r}] > 0, \quad for \quad i \in N, \tag{2.11}$$

since $r > \mu M C_0$. Thus for $i \in N$, we have

$$-\Delta^2 \beta_m(i-1) = \mu g(\beta_m(i) - \phi(i))\{1 + \frac{h(r)}{g(r)}\},$$

and this together with (2.11) yields

$$-\Delta^2 \beta_m(i-1) \le \mu K_0 g(1 - \frac{\mu M C_0}{r})\{1 + \frac{h(r)}{g(r)}\}g(\beta_m(i)), \quad i \in N. \tag{2.12}$$

Sum the inequality (2.12) from $j(1 \le j < \sigma_m)$ to σ_m to obtain

$$\Delta\beta_m(j-1) \le \Delta\beta_m(\sigma_m) + \mu K_0 g(1 - \frac{\mu M C_0}{r})\{1 + \frac{h(r)}{g(r)}\}\sum_{s=j}^{\sigma_m} g(\beta_m(s)),$$

i.e.,

$$\Delta\beta_m(j) \le \Delta\beta_m(\sigma_m) + \mu K_0 g(1 - \frac{\mu M C_0}{r})\{1 + \frac{h(r)}{g(r)}\}\sum_{s=j}^{\sigma_m-1} g(\beta_m(s+1)).$$

Since $\Delta\beta_m(\sigma_m) \le 0$, and $\beta_m(s+1) \ge \beta_m(j+1)$ when $j \le s < \sigma_m$, then we have

$$\Delta\beta_m(j) \le g(\beta_m(j+1))\mu K_0 g(1 - \frac{\mu M C_0}{r})\{1 + \frac{h(r)}{g(r)}\}(\sigma_m - j), \quad j < \sigma_m,$$

i.e.,
$$\frac{\Delta\beta_m(j)}{g(\beta_m(j+1))} \le \mu K_0 g(1 - \frac{\mu M C_0}{r})\{1 + \frac{h(r)}{g(r)}\}(\sigma_m - j), \quad j < \sigma_m.$$

Since $g(\beta_m(j+1)) \le g(u) \le g(\beta_m(j))$ for $\beta_m(j) \le u \le \beta_m(j+1)$, when $j < \sigma_m$, then we have

$$\int_{u(j)}^{u(j+1)} \frac{du}{g(u)} \le \frac{\Delta\beta_m(j)}{g(\beta_m(j+1))} \le \mu K_0 g(1 - \frac{\mu M C_0}{r})\{1 + \frac{h(r)}{g(r)}\}(\sigma_m - j), \quad j < \sigma_m,$$

and then sum the above from 0 to $\sigma_m - 1$ to obtain

$$\int_{\frac{1}{m}}^{\beta_m(\sigma_m)} \frac{du}{g(u)} \le \mu K_0 g(1 - \frac{\mu M C_0}{r})\{1 + \frac{h(r)}{g(r)}\}\frac{\sigma_m(\sigma_m+1)}{2}. \tag{2.13}$$

Similarly, sum the inequality (2.12) from σ_m to $j(\sigma_m < j < T+1)$ to obtain

$$-\Delta\beta_m(j) \le -\Delta\beta_m(\sigma_m - 1) + \mu K_0 g(1 - \frac{\mu M C_0}{r})\{1 + \frac{h(r)}{g(r)}\}\sum_{s=\sigma_m}^{j} g(\beta_m(s)), \quad j > \sigma_m.$$

Since $\Delta\beta_m(\sigma_m - 1) \ge 0$, $\beta_m(s) \ge \beta_m(j)$ for $\sigma_m \le s \le j$, then we have

$$-\Delta\beta_m(j) \le \mu K_0 g(1 - \frac{\mu M C_0}{r})\{1 + \frac{h(r)}{g(r)}\}(j - \sigma_m + 1), \quad j > \sigma_m.$$

So we have

$$\int_{\beta_m(j+1)}^{\beta_m(j)} \frac{du}{g(u)} \leq \frac{-\Delta\beta_m(j)}{g(\beta_m(j))} \leq \mu K_0 g(1 - \frac{\mu M C_0}{r})\{1 + \frac{h(r)}{g(r)}\}(j - \sigma_m + 1), \quad j > \sigma_m,$$

and then we sum the above from σ_m to T to obtain

$$\int_{\frac{1}{m}}^{\beta_m(\sigma_m)} \frac{du}{g(u)} \leq \mu K_0 g(1 - \frac{\mu M C_0}{r})\{1 + \frac{h(r)}{g(r)}\}\frac{(T - \sigma_m + 1)(T - \sigma_m + 2)}{2}. \quad (2.14)$$

Now, (2.13) and (2.14) imply

$$\int_{\delta}^{r} \frac{du}{g(u)} \leq \int_{\frac{1}{m}}^{\beta_m(\sigma_m)} \frac{du}{g(u)} \leq \frac{1}{2}\mu K_0 g(1 - \frac{\mu M C_0}{r})\{1 + \frac{h(r)}{g(r)}\}T(T+1).$$

This contradicts (2.6) and consequently $|\beta_m|_0 < r$.

Observe that

$$\begin{aligned} f^*(i, \beta_m(i) - \phi(i)) &= g(\beta_m(i) - \phi(i))(1 + \frac{h(\beta_m(i) - \phi(i))}{g(\beta_m(i) - \phi(i))}) \\ &\leq g(\beta_m(i) - \phi(i))\{1 + \frac{h(r)}{g(r)}\}, \quad i \in N. \end{aligned}$$

Thus we have

$$\begin{aligned} &\Delta^2\beta_m(i-1) + \mu f^*(i, \beta_m(i) - \phi(i)) \\ &= -\mu g(\beta_m(i) - \phi(i))\{1 + \frac{h(r)}{g(r)}\} + \mu f^*(i, \beta_m(i) - \phi(i)) \\ &\leq 0, \quad i \in N, \end{aligned}$$

so that β_m is an upper solution for $(2.8)^m$. This together with Claim 1, implies that α_m and β_m are, respectively, a lower and an upper solution for problem $(2.8)^m$ with $\alpha_m(i) \leq \beta_m(i)$, for all $i \in N^+$. So we conclude that $(2.8)^m$ has a solution $y_m \in C(N^+, \mathbf{R})$ such that

$$\alpha_m(i) \leq y_m(i) \leq \beta_m(i), \quad i \in N^+.$$

Thus we have

$$|y_m|_0 < r, \quad y_m(i) - \phi(i) \geq \frac{1}{m} + lw(i) > lw(i), \quad i \in N^+. \quad (2.15)$$

The Arzela-Ascoli Theorem guarantees the existence of a subsequence N_1 of N_0 and a function $y \in C(N^+, \mathbf{R})$ with $y_m \to y$ in $C(N^+, \mathbf{R})$ as $m \to \infty$ through N_1. Also $y(0) = y(T+1) = 0$ and

$$|y|_0 < r, \quad y(i) - \phi(i) \geq lw(i), \quad i \in N^+.$$

Now fix $i \in N$. Then $y_m, \ m \in N_1$, satisfies

$$y_m(i) = \frac{1}{m} + \sum_{j=1}^{T} G(i,j)f^*(j, y_m(j) - \phi(j)). \quad (2.16)$$

Also

$$0 < l\min_{k \in N} w(k) \leq y_m(j) < r, \ for \ j \in N \ and \ m \in N_1.$$

Let $m \to \infty$ (through N_1) in (2.16) to obtain

$$y(i) = \sum_{j=1}^{T} G(i,j) f^*(j, y(j) - \phi(j))$$

We can do this for each $i \in N$ and so

$$\Delta^2 y(i-1) + f^*(i, y(i) - \phi(i)) = 0, \quad i \in N.$$

Theorem 2.2. *Suppose (2.1)-(2.4) hold. In addition assume the following hold:*

$$\begin{cases} f^*(i, u) = f(i, u) + M \geq g_1(u) + h_1(u) \text{ with} \\ g_1 > 0 \text{ continuous and nonincreasing on } (0, \infty), \; h_1 \geq 0 \\ \text{continuous on } [0, \infty) \text{ and } \frac{h_1}{g_1} \text{ nondecreasing on } (0, \infty) \end{cases} \qquad (2.17)$$

and

$$\begin{cases} \exists \; R > r \text{ with} \\ \frac{R g_1(\frac{\epsilon}{T+1} R)}{g_1(R) g_1(\frac{\epsilon}{T+1} R) + g_1(R) h_1(\frac{\epsilon}{T+1} R)} \leq \mu \sum_{j=1}^{T} G(\sigma, j); \end{cases} \qquad (2.18)$$

here $\epsilon > 0$ is any constant (choose and fix it) so that $1 - \frac{\mu M C_0}{R} \geq \epsilon$ (note ϵ exists since $R > r > \mu M C_0$) and $G(i,j)$ is the Green's function for

$$\begin{cases} -\Delta^2 y(i-1) = 0, \quad i \in N \\ y(0) = y(T+1) = 0, \end{cases}$$

and $\sigma \in N^+$ is such that

$$\sum_{j=1}^{T} G(\sigma, j) = \max_{i \in N^+} \sum_{j=1}^{T} G(i,j).$$

Then (1.1) has a solution $y \in C(N^+, \mathbf{R})$ with $y(i) > 0$ for $i \in N, r < |y + \phi|_0 \leq R$, here $\phi(i) = \mu M \omega(i)$ (ω is as in Lemma 2.2).

Proof. Choose $\delta > 0$ and $\delta < r$ with

$$\frac{1}{g(1 - \frac{\mu M C_0}{r})\{1 + \frac{h(r)}{g(r)}\}} \int_{\delta}^{r} \frac{du}{g(u)} > \frac{1}{2} \mu K_0 T(T+1). \qquad (2.19)$$

Let $m_0 \in \{1, 2, ...\}$ be chosen so that $\frac{1}{m_0} < \frac{\delta}{2}, \; \frac{1}{m_0} < \frac{\epsilon R}{T+1}, \; \frac{1}{m_0} < \frac{1}{T+1}(r - \mu M C_0)$, and $N_0 = \{m_0, m_0 + 1, ...\}$.

To show (1.1) has a nonnegative solution $y \in C(N^+, \mathbf{R})$ with $r < |y + \phi|_0 \leq R$, we will show

$$\begin{cases} \Delta^2 y(i-1) + \mu f^*(i, y(i) - \phi(i)) = 0, \quad i \in N \\ y(0) = y(1) = 0 \end{cases} \qquad (2.20)$$

has a solution $y_2 \in C(N^+, \mathbf{R})$ with $y_2(i) > \phi(i)$ for $i \in N$ and $r < |y_2|_0 \leq R$, here $f^*(i, u)$ is as in (2.2). If this is true, then $u(i) = y_2(i) - \phi(i)$ is a nonnegative solution (positive on N) of (1.1). As a result we will concentrate our study on (2.20).

The idea is to first show that

$$\begin{cases} \Delta^2 y(i-1) + \mu f_m(i, y(i) - \phi(i)) = 0, \quad i \in N, \\ y(0) = \frac{1}{m}, \; y(T+1) = \frac{1}{m}, \quad m \in N_0 \end{cases} \qquad (2.21)^m$$

has a solution y_m for each $m \in N_0$ with $y_m(i) \geq \frac{1}{m}$ and $y_m(i) \geq \phi(i)$ for $i \in N^+$, and $r < |y_m|_0 \leq R$, here

$$f_m(i, u) = \begin{cases} f^*(i, u), & u \geq \frac{1}{m}, \\ f^*(i, \frac{1}{m}), & u \leq \frac{1}{m}. \end{cases}$$

Note $f_m(i, u) \geq 0$ for $u \in (-\infty, \infty)$.

Fix $m \in N_0$. Let $E = (C(N^+, \mathbf{R}), \ | \bullet |_0)$ and

$$K = \{ \ u \in C(N^+, \mathbf{R}) : \quad u(k) \geq q(k)|u|_0 \ \text{ for } \ k \in N^+ \ \},$$

here $q(k)$ is as in Lemma 2.1.

Let $A : \ K \to C(N^+, \mathbf{R})$ be defined by

$$(Ay)(i) = \frac{1}{m} + \mu \sum_{j=1}^{T} G(i, j) f_m(j, y(j) - \phi(j)), \quad i \in \ N^+.$$

A standard argument [3] implies $A : \ K \to C(N^+, \mathbf{R})$ is continuous and completely continuous. Next we show $A : \ K \to \ K$. If $u \in K$, then clearly $(Au)(i) \geq 0$ for $i \in N^+$. Also notice

$$\begin{cases} \Delta^2(Au)(i) \leq 0 \quad \text{on } N \\ (Au)(0) = (Au)(T + 1) = \frac{1}{m} \end{cases}$$

then by Lemma 2.1, $(Au)(i) - \frac{1}{m} \geq q(i)|Au - \frac{1}{m}|_0$ for $i \in N^+$, and so we have $(Au)(i) \geq \frac{1}{m} + q(i)(|Au|_0 - \frac{1}{m}) \geq q(i)|Au|_0$. Consequently $Au \in K$ so $A : \ K \to K$. Let

$$\Omega_1 = \{ \ u \in C(N^+, \mathbf{R}) : \ |u_0| < r\} \ \text{and} \ \Omega_2 = \{ \ u \in C(N^+, \mathbf{R}) : \ |u_0| < R\}$$

We first show

$$y \neq \lambda Ay \text{ for } \lambda \in [0, 1) \text{ and } y \in K \cap \partial \Omega_1. \tag{2.22}$$

Suppose this is false i.e. suppose there exists $y \in K \cap \partial \Omega_1$ and $\lambda \in [0, 1)$ with $y = \lambda Ay$. We can assume $\lambda \neq 0$. Now since $y = \lambda Ay$ we have

$$\begin{cases} \Delta^2 y(i - 1) + \lambda \mu q(t) f_m(i, y(i) - \phi(i)) = 0, \quad i \in N, \\ y(0) = y(T + 1) = \frac{\lambda}{m} \end{cases}$$

with $y(i) \geq q(i)r$ for $i \in N^+$. Since $\Delta^2 y(i-1) \leq 0$ on N and $y \geq \frac{\lambda}{m}$ on N^+, there exists $i_0 \in N$ with $\Delta y(i) \geq 0$ on $[0, i_0] = \{0, ..., i_0 - 1\}$, $\Delta y(i) \leq 0$ on $[i_0, \ T+1) = \{i_0, ..., T\}$ and $y(i_0) = |y|_0 = r$. Notice also for $i \in N^+$ that

$$y(i) - \phi(i) = y(i)[1 - \frac{\mu M w(i)}{y(i)}] \geq y(i)[1 - \frac{\mu M C_0}{r}],$$

since $y(i) \geq q(i)r$ and $w(i) \leq C_0 q(i)$ for $i \in N^+$. Thus

$$y(i) - \phi(i) = y(i)[1 - \frac{\mu M C_0}{r}] > 0 \quad \text{for} \ \ i \in N, \tag{2.23}$$

since $r > \mu M C_0$. Also notice

$$\begin{aligned} y(i) - \phi(i) \ & \geq y(i)[1 - \frac{\mu M C_0}{r}] \\ & \geq q(i)r[1 - \frac{\mu M C_0}{r}] \\ & \geq \frac{1}{T+1}r[1 - \frac{\mu M C_0}{r}] \\ & \geq \frac{1}{m_0}, \quad i \in N. \end{aligned}$$

Thus we have

$$f_m(y(i) - \phi(i)) = f^*(y(i) - \phi(i)) \leq g(y(i) - \phi(i)) + h(y(i) - \phi(i)), \quad i \in N.$$

Now for $i \in N$ we have

$$-\Delta^2 y(i-1) \leq \mu g(y(i) - \phi(i))\{1 + \frac{h(y(i) - \phi(i))}{g(y(i) - \phi(i))}\}$$

and this together with (2.23) yields

$$-\Delta^2 y(i-1) \leq \mu K_0 g(1 - \frac{\mu M C_0}{r})\{1 + \frac{h(r)}{g(r)}\}g(y(i)), \quad i \in N. \tag{2.24}$$

The reasoning used to obtain (2.13) and (2.14) in Theorem 2.1, yields

$$\int_{\frac{\lambda}{m}}^{y(i_0)} \frac{du}{g(u)} \leq \mu K_0 g(1 - \frac{\mu M C_0}{r})\{1 + \frac{h(r)}{g(r)}\}\frac{i_0(i_0 + 1)}{2}, \tag{2.25}$$

and

$$\int_{\frac{\lambda}{m}}^{y(i_0)} \frac{du}{g(u)} \leq \mu K_0 g(1 - \frac{\mu M C_0}{r})\{1 + \frac{h(r)}{g(r)}\}\frac{(T - i_0 + 1)(T - i_0 + 2)}{2}. \tag{2.26}$$

Now (2.25) and (2.26) imply

$$\int_{\delta}^{r} \frac{du}{g(u)} = \int_{\delta}^{y(i_0)} \frac{du}{g(u)} \leq \frac{1}{2}\mu K_0 g(1 - \frac{\mu M C_0}{r})\{1 + \frac{h(r)}{g(r)}\}T(T + 1).$$

This contradicts (2.19) and consequently (2.22) is true.

Next we show

$$|Ay|_0 \geq |y|_0 \quad \text{for} \quad y \in K \cap \partial\Omega_2. \tag{2.27}$$

To see this let $y \in K \cap \partial\Omega_2$ so $y(i) \geq q(i)R$ for $i \in N^+$. Also for $i \in N^+$ we have

$$y(i) - \phi(i) \geq y(i) - \mu M C_0 q(i) \geq y(i)[1 - \frac{\mu M C_0}{R}] \geq \epsilon y(i) \geq \epsilon q(i)R.$$

Thus for $i \in N$ we have

$$y(i) - \phi(i) \geq \epsilon q(i)R \geq \frac{\epsilon}{T + 1}R,$$

and so

$$f_m(y(i) - \phi(i)) = f^*(y(i) - \phi(i)) \geq g_1(y(i) - \phi(i)) + h_1(y(i) - \phi(i)), \quad i \in N,$$

since $y(i) - \phi(i) \geq \frac{\epsilon}{T+1}R > \frac{1}{m_0}$ for $i \in N$. As a result we have

$$
\begin{aligned}
(Ay)(\sigma) &= \frac{1}{m} + \mu\Sigma_{j=1}^T G(\sigma, j)f_m(y(j) - \phi(j)) \\
&\geq \mu\sum_{j=1}^T G(\sigma, j)g_1(y(j) - \phi(j))\{1 + \frac{h_1(y(j) - \phi(j))}{g(y(i) - \phi(i))}\} \\
&\geq \mu g_1(R)\{1 + \frac{h_1(\frac{\epsilon}{T+1}R)}{g_1(\frac{\epsilon}{T+1}R)}\}\sum_{j=1}^T G(\sigma, j) \\
&\geq R = |y|_0,
\end{aligned}
$$

using(2.18). Thus $|Ay|_0 \geq |y|_0$, so (2.27) is true.

Now Theorem 1.1 implies A has a fixed point $y_m \in K \bigcap (\bar{\Omega}_2 \setminus \Omega_1)$ i.e., $r \leq |y_m|_0 \leq R$ and $y_m(i) \geq rq(i)$ for $i \in N^+$ In fact $|y_m|_0 > r$ (note if $|y_m|_0 = r$ then essentially the same argument from (2.24)-(2.26) will yield a contradiction). In addition $y_m(i) > \phi(i)$ for $i \in N$ since

$$y_m(i) \geq rq(i) > \mu M C_0 q(i) \geq \mu M w(i) = \phi(i).$$

Thus y_m is a solution of $(2.21)^m$.

Essentially the same argument as in the proof of Theorem 2.1, guarantees the existence of a subsequence N_1 of N_0 and a function $y \in C(N^+, \mathbf{R})$ with $y_m \to y$ in $C(N^+, \mathbf{R})$ as $m \to \infty$ through N_1. Also, $y \in C(N^+, \mathbf{R})$ is a nonnegative solution of (2.20) such that $r \leq |y|_0 \leq R$, $y(i) \geq q(i)r$ for $i \in N^+$. In addition $y(i) > \phi(i)$ for $i \in N$. Finally it is easy to see that $|y|_0 > r$(note if $|y_0| = r$ then essentially the argument from (2.24)-(2.26) will yield a contradiction).

Theorem 2.3. *Suppose (2.1)-(2.5) and (2.17)-(2.18) hold. Then (1.1) has two solutions $y_n \in C(N^+, \mathbf{R})$ ($n = 1, 2$) with $y_n(i) > 0$ for $i \in N$, $0 < |y_1 + \phi|_0 < r < |y_2 + \phi|_0 \leq R$, here $\phi(i) = \mu M w(i)$ (w is as in Lemma 2.2).*

Proof. The existence of y_1 follows from Theorem 2.1 and the existence of y_2 follows from Theorem 2.2.

Example. Consider the boundary value problem

$$\begin{cases} \Delta^2 y(i) + \mu([y(i)]^{-\alpha} + [y(i)]^\beta - 1) = 0, & i \in N, \\ y(0) = 0, \quad y(T+1) = 0, \quad \alpha > 0, \quad \beta > 1 \end{cases} \quad (2.28)$$

with $\mu \in (0, \mu_0)$ is such that

$$[\mu_0 T(T+1)(\alpha+1)]^{\frac{1}{\alpha}} + \frac{\mu_0(T+1)^2}{2} \leq 1. \quad (2.29)$$

Then (2.28) have two solutions y_n with $y_n(i) > 0$, for $i \in N$, $n = 1, 2$, and $0 < |y_1 + \phi|_0 < 1 < |y_2 + \phi|_0$, here $\phi(i) = \mu w(i) = \frac{\mu i(T+1-i)}{2}$, $i \in N^+$.

To see this we will apply Theorem 2.3 with (here $R > 1$ will be chosen below; in fact here we choose R so that $\epsilon = \frac{1}{2}$ works i.e., we choose R so that $1 - \frac{\mu M C_0}{R} \geq \frac{1}{2}$)

$$M = 1, \; w(i) = \frac{i(T+1-i)}{2}, \; \phi(i) = \frac{\mu i(T+1-i)}{2},$$

and

$$g(y) = g_1(y) = y^{-\alpha}, \; h(y) = h_1(y) = y^\beta, \; \epsilon = \frac{1}{2}, \; K_0 = 1, \; C_0 = \frac{(T+1)^2}{2}.$$

Clearly (2.1),(2.2), (2.3), (2.4), (2.5),(2.17) hold. Also note

$$\frac{1}{g(1 - \frac{\mu M C_0}{r})\{1 + \frac{h(r)}{g(r)}\}} \int_0^r \frac{du}{g(u)} = (1 - \frac{\mu(T+1)^2}{2r})^\alpha \frac{1}{1 + r^{\alpha+\beta}} \frac{r^{\alpha+1}}{\alpha+1}.$$

Now (2.6) holds with $r = 1$ since

$$\frac{1}{2}\mu K_0 T(T+1) = \frac{\mu T(T+1)}{2} < \frac{\mu_0 T(T+1)}{2} \leq \frac{(1 - \frac{\mu_0(T+1)^2}{2})^\alpha}{2(\alpha+1)}$$

$$\leq \frac{(1 - \frac{\mu(T+1)^2}{2})^\alpha}{2(\alpha+1)} = \frac{1}{g(1 - \frac{\mu M C_0}{r})\{1 + \frac{h(r)}{g(r)}\}} \int_0^r \frac{du}{g(u)}$$

from (2.29). Finally notice (2.18) is satisfied for R large since

$$\frac{Rg_1(\frac{\epsilon}{T+1}R)}{g_1(R)g_1(\frac{\epsilon}{T+1}R) + g_1(R)h_1(\frac{\epsilon}{T+1}R)} = \frac{R^{1+\alpha}}{1 + (\frac{\epsilon}{T+1})^{\alpha+\beta}R^{\alpha+\beta}} \to 0$$

as $R \to \infty$, since $\beta > 1$. Thus all the conditions of Theorem 2.3 are satisfied so existence of two positive solutions is guaranteed.

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