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OSCILLATION OF SYSTEMS OF CERTAIN NEUTRAL DELAY PARABOLIC DIFFERENTIAL EQUATIONS[1]

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Sufficient conditions are established for the oscillation of systems of neutral delay parabolic differential equations. These results are illustrated by some examples.
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1 Introduction

During the past decade, considerable attention has been given to boundary value problems and initial value problems for partial differential equations with piecewise constant delay by several authors including Wiener and Debnath [9-11]. In all of these papers, the major focus was to investigate the influence of certain piecewise constant time delays, continuous time delays and discontinuous time delays on the solutions of partial differential equations. These results have also been extended to equations with positive definite operators in Hilbert spaces. A class of initial value problems for partial differential equations with piecewise constant argument (EPCA) in partial derivatives. A class of loaded partial differential equations that arise in solving certain inverse problems has been studied within the general framework with piecewise constant delay. An abstract Cauchy problem for partial differential equations with time delays

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in a Banach space has also been examined by Wiener and Debnath [11]. Subsequently, Wiener and Debnath [12] have studied boundary value problems for the diffusion equation with piecewise continuous time delay. This study included boundary value problems for three types of equations: delayed, alternately advanced, and retarded type and most importantly, equations of neutral type. These equation included loaded and impulsive equations as special cases and hence their importance arises in control theory and in certain biomedical models. Recently, Wiener and Heller [13] have made a detailed study of diffusion equations of neutral type with piecewise constant time delay. This study reveals many interesting features including oscillatory and periodic properties of the solutions. On the other hand, Wiener and Debnath [12] have examined the oscillatory properties of the wave equation with discontinuous time delay.

In addition, several authors including Mishev and Bainov [8], Fu and Zhuang [4], Cui et al [2], Bainov et al [1], Li and Cui [6], Debnath and Li [3] have studied the oscillation problems for the partial differential equations of different types. Very recently, Li and Debnath [7] have investigated the theory of oscillations of a system of hyperbolic partial differential equations with continuous distributed deviating arguments. They have obtained sufficient conditions for the oscillation of the system of delay hyperbolic partial differential equations with examples. In spite of the above studies, hardly any attention was given to the problem of oscillation of a system of certain neutral delay parabolic differential equations. The main objective of this paper is to study this problem. Sufficient condition are proved for the oscillation of systems of neutral delay parabolic equations with some examples.

2 Formulation of the Problem

In this paper, we study the oscillation of systems of neutral delay parabolic differential equations of the form

$$\begin{aligned} \frac{\partial}{\partial t}[u_i(x,t) - \sum_{s=1}^{r}\lambda_s(t)u_i(x,t-\rho_s)] - a_i(t)\Delta u_i(x,t) + \sum_{k=1}^{m}\sum_{j=1}^{d}a_{ikj}(t)\Delta u_k(x,t-\tau_j) \\ - p_i(x,t)u_i(x,t) - \sum_{k=1}^{m}\sum_{h=1}^{l}\int_a^b q_{ikh}(x,t,\xi)u_k(x,g_h(t,\xi))d\sigma(\xi), \\ (x,t)\in\Omega\times[0,\infty)\equiv G,\ i=1,2,\ldots,m, \end{aligned} \tag{2.1}$$

where Ω is a bounded domain in R^n with a piecewise smooth boundary $\partial\Omega$, $\Delta u_i(x,t) = \sum_{r=1}^{n}\frac{\partial^2 u_i(x,t)}{\partial x_r^2}$, $i=1,2,\ldots,m$, and the integral in (2.1) is the Riemann-Stieltjes integral.

In this paper, we always suppose that the following conditions hold:

(H1) $a_i \in C([0,\infty);[0,\infty)), a_{ikj}\in C([0,\infty);R), a_{iij}(t)>0$, and

$$A_j(t) = \min_{1\le i\le m}\left\{a_{iij}(t) - \sum_{k=1,k\neq i}^{m} \mid a_{kij}(t) \mid \right\} > 0,\ i=1,2,\ldots,m;\ \ j=1,2,\ldots,d;$$

(H2) $\lambda_s \in C^1([0,\infty);[0,\infty)), s=1,2,\ldots,r$;

(H3) $p_i \in C(\overline{G}\,; [0,\infty))$, $p_i(t) = \inf_{x\in\overline{\Omega}} p_i(x,t)$, $p(t) = \inf_{1\le i\le m}\{p_i(t)\}$, $i = 1,2,\ldots,m$;
(H4) $q_{ikh} \in C(\overline{G} \times [a,b]; R)$, $q_{iih}(x,t,\xi) > 0$, and

$$q_{iih}(t,\xi) = \inf_{x\in\overline{\Omega}} q_{iih}(x,t,\xi), \overline{q}_{\,ikh}(t,\xi) = \sup_{x\in\overline{\Omega}} \mid q_{ikh}(x,t,\xi) \mid,$$

$$Q_h(t,\xi) = \min_{1\le i\le m}\left\{ q_{iih}(t,\xi) - \sum_{k=1,k\ne i}^{m} \overline{q}_{\,kih}(t,\xi)\right\} \ge 0,$$

where $i,k = 1,2,\ldots,m$; $h = 1,2,\ldots,l$;

(H5) $g_h \in C([0,\infty)\times[a,b]; R)$, $g_h(t,\xi) \le t$, $\xi \in [a,b]$ and $g_h(t,\xi)$ is a nondecreasing function with respect to t and ξ, respectively,

$$\lim_{t\to\infty} \min_{\xi\in[a,b]}\{g_h(t,\xi)\} = \infty, h = 1,2,\ldots,l;$$

(H6) $\sigma \in ([a,b]; R)$ and $\sigma(\xi)$ is nondecreasing in ξ;
(H7) ρ_s, τ_j are positive constants, $s = 1,2,\ldots,r$; $j = 1,2,\ldots,d$.
We consider two kinds of boundary conditions:

$$\tfrac{\partial u_i(x,t)}{\partial N} + f_i(x,t)u_i(x,t) = 0, (x,t) \in \partial\Omega\times[0,\infty),\ i = 1,2,\ldots,m, \tag{2.2}$$

where N is the unit exterior normal vector to $\partial\Omega$ and $f_i(x,t)$ is a nonnegative continuous function on $\partial\Omega\times[0,\infty)$, $i = 1,2,\ldots,m$, and

$$u_i(x,t) = 0, (x,t) \in \partial\Omega\times[0,\infty), i = 1,2,\ldots,m. \tag{2.3}$$

Definition 2.1: The vector function $u(x,t) = \{u_1(x,t), u_2(x,t),\ldots,u_m(x,t)\}^T$ is said to be a solution of the problem (2.1), (2.2) (or (2.1), (2.3)) if it satisfies (2.1) in $G = \Omega\times[0,\infty)$ and boundary condition (2.2) (or (2.3)).

Definition 2.2: The vector solution $u(x,t) = \{u_1(x,t), u_2(x,t),\ldots,u_m(x,t)\}^T$ of the problem (2.1), (2.2) (or (2.1), (2.3)) is said to be oscillatory in the domain $G = \Omega\times[0,\infty)$ if at least one of its nontrivial component is oscillatory in G. Otherwise, the vector solution $u(x,t)$ is said to be nonoscillatory.

3 Oscillation of the Problem (2.1), (2.2)

Theorem 3.1: *If the neutral differential inequality*

$$[V(t) - \sum_{s=1}^{r}\lambda_s(t)V(t-\rho_s)]' + p(t)V(t) + \sum_{h=1}^{l}\int_a^b Q_h(t,\xi)V(g_h(t,\xi))d\sigma(\xi) \le 0, \tag{3.1}$$

has no eventually positive solution, then every solution of the problem (2.1)*,* (2.2) *is oscillatory in G.*

Proof: Suppose to the contrary that there is a nonoscillatory solution $u(x,t) = \{u_1(x,t), u_2(x,t),\ldots,u_m(x,t)\}^T$ of the problem (2.1), (2.2). We assume that $\mid u_i(x,t)\mid > 0$

for $t \geq t_0 \geq 0,\ i = 1, 2, \ldots, m$. Let $\delta_i = \operatorname{sgn} u_i(x,t),\ Z_i(x,t) = \delta_i u_i(x,t)$, then $Z_i(x,t) > 0$, $(x,t) \in \Omega \times [t_0, \infty)$, $i = 1, 2, \ldots, m$. From (H5) and (H7) there exists a number $t_1 \geq t_0$ such that $Z_i(x,t) > 0,\ Z_i(x, t-\rho_s) > 0,\ Z_i(x, t-\tau_j) > 0$ and $Z_i(x, g_h(t,\xi)) > 0$ in $\Omega \times [t_1, \infty)$, $i = 1, 2, \ldots, m;\ s = 1, 2, \ldots, r;\ j = 1, 2, \ldots, d;\ h = 1, 2, \ldots, l$.

Integrating (2.1) with respect to x over the domain Ω, we have

$$\begin{aligned}
&\frac{d}{dt}\left[\int_\Omega u_i(x,t)dx - \sum_{s=1}^{r} \lambda_s(t) \int_\Omega u_i(x, t-\rho_s)dx\right] = a_i(t)\int_\Omega \Delta u_i(x,t)dx \\
&+ \sum_{k=1}^{m} \sum_{j=1}^{d} a_{ikj}(t) \int_\Omega \Delta u_k(x, t-\tau_j)dx - \int_\Omega p_i(x,t)u_i(x,t)dx \qquad (3.2) \\
&- \sum_{k=1}^{m} \sum_{h=1}^{l} \int_\Omega \int_a^b q_{ikh}(x,t,\xi) u_k(x, g_h(t,\xi)) d\sigma(\xi) dx,\ t \geq t_1,\ i = 1, 2, \ldots, m,
\end{aligned}$$

where $t \geq t_1$, $i = 1, 2, \ldots, m$.

It is easy to see that

$$\begin{aligned}
&\int_\Omega \int_a^b q_{ikh}(x,t,\xi) u_k(x, g_h(t,\xi)) d\sigma(\xi) dx \qquad (3.3) \\
&= \int_a^b \int_\Omega q_{ikh}(x,t,\xi) u_k(x, g_h(t,\xi)) dx d\sigma(\xi).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{d}{dt}\left[\int_\Omega Z_i(x,t)dx - \sum_{s=1}^{r} \lambda_s(t) \int_\Omega Z_i(x, t-\rho_s)dx\right] = a_i(t)\int_\Omega \Delta Z_i(x,t)dx \\
&+ \sum_{j=1}^{d} a_{iij}(t) \int_\Omega \Delta Z_i(x, t-\tau_j)dx + \sum_{k=1,k\neq i}^{m} \sum_{j=1}^{d} a_{ikj}(t) \frac{\delta_k}{\delta_i} \int_\Omega \Delta Z_k(x, t-\tau_j)dx \\
&- \int_\Omega p_i(x,t) Z_i(x,t)dx - \sum_{h=1}^{l} \int_a^b \int_\Omega q_{iih}(x,t,\xi) Z_i(x, g_h(t,\xi)) dx d\sigma(\xi) \qquad (3.4) \\
&- \sum_{h=1}^{l} \sum_{k=1,k\neq i}^{m} \frac{\delta_k}{\delta_i} \int_a^b \int_\Omega q_{ikh}(x,t,\xi) Z_k(x, g_h(t,\xi)) dx d\sigma(\xi),
\end{aligned}$$

where $t \geq t_1$, $i = 1, 2, \ldots, m$.

The Green's formula and (2.2) yield

$$\int\limits_{\Omega} \Delta Z_i(x,t)dx = \int\limits_{\partial\Omega} \frac{\partial Z_i(x,t)}{\partial N} dS = -\int\limits_{\partial\Omega} f_i(x,t) Z_i(x,t) dS \le 0, \tag{3.5}$$

and

$$\begin{aligned}\int\limits_{\Omega} \Delta Z_k(x,t-\tau_j)dx &= \int\limits_{\partial\Omega} \frac{\partial Z_k(x,t-\tau_j)}{\partial N} dS \\ &= -\int\limits_{\partial\Omega} f_k(x,t-\tau_j) Z_k(x,t-\tau_j) dS,\end{aligned} \tag{3.6}$$

where $t \ge t_1$, $i,k = 1,2,\ldots,m$; $j = 1,2,\ldots,d$, and dS is the surface element on $\partial\Omega$.

Combining (3.4)-(3.6), we get

$$\begin{aligned}&\frac{d}{dt}\left[\int\limits_{\Omega} Z_i(x,t)dx - \sum_{s=1}^{r} \lambda_s(t) \int\limits_{\Omega} Z_i(x,t-\rho_s)dx\right] \\ &\le -\sum_{j=1}^{d} a_{iij}(t) \int\limits_{\partial\Omega} f_i(x,t-\tau_j) Z_i(x,t-\tau_j) dS \\ &+ \sum_{j=1}^{d} \sum_{k=1,k\ne i}^{m} \mid a_{ikj}(t) \mid \int\limits_{\partial\Omega} g_k(x,t-\tau_j) Z_k(x,t-\tau_j) dS - p_i(t) \int\limits_{\Omega} Z_i(x,t)dx \\ &- \sum_{h=1}^{l} \int_a^b q_{iih}(t,\xi) \int\limits_{\Omega} Z_i(x,g_h(t,\xi))dx d\sigma(\xi) \\ &+ \sum_{h=1}^{l} \sum_{k=1,k\ne i}^{m} \int_a^b \overline{q}_{ikh}(t,\xi) \int\limits_{\Omega} Z_k(x,g_h(t,\xi))dx d\sigma(\xi),\end{aligned} \tag{3.7}$$

where $t \ge t_1$, $i = 1,2,\ldots,m$.

Setting

$$V_i(t) = \int\limits_{\Omega} Z_i(x,t)dx, W_i(t) = \int\limits_{\partial\Omega} f_i(x,t) Z_i(x,t) dS, t \ge t_1, i = 1,2,\ldots,m,$$

we obtain

$$\begin{aligned}&\left[V_i(t) - \sum_{s=1}^{r} \lambda_s(t) V_i(t-\rho_s)\right]' + \sum_{j=1}^{d} a_{iij}(t) W_i(t-\tau_j) \\ &- \sum_{j=1}^{d} \sum_{k=1,k\ne i}^{m} \mid a_{ikj}(t) \mid W_k(t-\tau_j) + p_i(t) V_i(t)\end{aligned} \tag{3.8}$$

$$+\sum_{h=1}^{l}\int_a^b q_{iih}(t,\xi)V_i(g_h(t,\xi))d\sigma(\xi)$$

$$-\sum_{h=1}^{l}\quad\sum_{k=1,k\neq 1}^{m}\int_a^b \overline{q}_{\ ikh}(t,\xi)V_k(g_h(t,\xi))d\sigma(\xi)\leq 0,$$

where $t\geq t_1$, $i=1,2,\ldots m$.

Let $V(t)=\sum_{i=1}^{m}V_i(t), W(t)=\sum_{i=1}^{m}W_i(t)$ for $t\geq t_1$. It follows from (3.8) that

$$\left[V(t)-\sum_{s=1}^{r}\lambda_s(t)V(t-\rho_s)\right]'+\sum_{j=1}^{d}\left\{\sum_{i=1}^{m}\left[a_{iij}(t)W_i(t-\tau_j)-\sum_{k=1,k\neq i}^{m}\mid a_{ikj}(t)\mid W_k(t-\tau_j)\right]\right\}$$

$$+p(t)V(t)+\sum_{h=1}^{l}\left\{\sum_{i=1}^{m}\int_a^b[q_{iih}(t,\xi)V_i(t_h(t,\xi))d\sigma(\xi)\right. \tag{3.9}$$

$$\left.-\sum_{k=1,k\neq i}^{m}\int_a^b \overline{q}_{\ ikh}(t,\xi)V_k(g_h(t,\xi))d\sigma(\xi)]\right\}\leq 0,\ t\geq t_1.$$

Noting that

$$\sum_{i=1}^{m}\int_a^b\left[q_{iih}(t,\xi)V_i(g_h(t,\xi))-\sum_{k=1,k\neq i}^{m}\overline{q}_{\ ikh}(t,\xi)V_k(g_h(t,\xi))\right]d\sigma(\xi)$$

$$=\int_a^b\left[q_{11h}(t,\xi)V_1(g_h(t,\xi))-\sum_{k=1,k\neq 1}^{m}\overline{q}_{\ 1kh}(t,\xi)V_k(g_h(t,\xi))\right]d\sigma(\xi)$$

$$+\int_a^b\left[q_{22h}(t,\xi)V_2(g_h(t,\xi))-\sum_{k=1,k\neq 2}^{m}\overline{q}_{\ 2kh}(t,\xi)V_k(g_h(t,\xi))\right]d\sigma(\xi)$$

$$+\ldots$$

$$+\int_a^b\left[q_{mmh}(t,\xi)V_m(g_h(t,\xi))-\sum_{k=1,k\neq m}^{m}\overline{q}_{\ mkh}(t,\xi)V_k(g_h(t,\xi))\right]d\sigma(\xi)$$

$$=\int_a^b\left[q_{11h}(t,\xi)-\sum_{k=1,k\neq 1}^{m}\overline{q}_{\ k1h}(t,\xi)\right]V_1(g_h(t,\xi))d\sigma(\xi)$$

$$+\int_a^b \left[q_{22h}(t,\xi) - \sum_{k=1,k\neq 2}^{m} \overline{q}_{\ k2h}(t,\xi)\right] V_2(g_h(t,\xi))d\sigma(\xi)$$

$$+\dots$$

$$+\int_a^b \left[q_{mmh}(t,\xi)V_m(g_h(t,\xi)) - \sum_{k=1,k\neq m}^{m} \overline{q}_{\ kmh}(t,\xi)V_m(g_h(t,\xi))\right] d\sigma(\xi)$$

$$\geq \int_a^b \min_{1\leq i\leq m}\left\{q_{iih}(t,\xi) - \sum_{k=1,k\neq i}^{m} \overline{q}_{\ kih}(t,\xi)\right\}\sum_{i=1}^{m} V_i(g_h(t,\xi))d\sigma(\xi)$$

$$= \int_a^b Q_h(t,\xi)V(g_h(t,\xi))d\sigma(\xi), t\geq t_1, h=1,2,\dots,l.$$

Similarly,

$$\sum_{i=1}^{m}\left[a_{iij}(t)W_i(t-\tau_j) - \sum_{k=1,ki}^{m} \mid a_{ikj}(t)\mid W_k(t-\tau_j)\right]$$

$$\geq \min_{1\leq i\leq m}\left[a_{iij}(t) - \sum_{k=1,ki}^{m} \mid a_{kij}\mid\right]\sum_{i=1}^{m} W_i(t-\tau_j)$$

$$= A_j(t)W(t-\tau_j), t\geq t_1, j=1,2,\dots,d.$$

Then, from (3.9), we have

$$\left[V(t) - \sum_{s=1}^{r}\lambda_s(t)V(t-\rho_s)\right]' + \sum_{j=1}^{d} A_j(t)W(t-\tau_j) + p(t)V(t)$$
$$+\sum_{h=1}^{l}\int_a^b Q_h(t,\xi)V(g_h(t,\xi))d\sigma(\xi)\leq 0, t\geq t_1. \tag{3.10}$$

It is easy to see that

$$W(t-\tau_j) = \sum_{i=1}^{m} W_i(t-\tau_j)\geq 0, t\geq t_1, j=1,2,\dots,d.$$

Therefore,

$$\left[V(t)-\sum_{s=1}^{r}\lambda_s(t)V(t-\rho_s)\right]'+p(t)V(t)$$

$$+\sum_{h=1}^{l}\int_a^b Q_h(t,\xi)V(g_h(t,\xi))d\sigma(\xi)\le 0, \quad t\ge t_1,$$

which contradicts the assumption that (3.1) has no eventually positive solution. This completes the proof. □

Theorem 3.2: *Suppose there exists some* $h_0\in\{1,2,\dots,l\}$ *such that*

(B1) *there exists a function* $\eta_{h_0}\in C([0,\infty)\times[a,b];(0,\infty))$ *such that*

$$\eta_{h_0}(\eta_{h_0}(t,\xi),\xi)=g_{h_0}(t,\xi),$$

$\eta_{h_0}(t,\xi)$ *is a nondecreasing function with respect to* t *and* ξ*, and*

$$t\ge\eta_{h_0}(t,\xi)\ge g_{h_0}(t,\xi), \lim_{t\to\infty}\min_{\xi\in[a,b]}\eta_{h_0}(t,\xi)=\infty;$$

(B2) $\liminf_{t\to\infty}\int_{g_{h_0}(t,b)}^{t}\int_a^b Q_{h_0}(s,\xi)d\sigma(\xi)ds>\frac{1}{e}$;

(B3) $\liminf_{t\to\infty}\int_{\eta_{h_0}(t,b)}^{t}\int_a^b Q_{h_0}(s,\xi)d\sigma(\xi)ds>0$;

(B4) $\lim_{t\to\infty}\sum_{s=1}^{r}\lambda_s(t)=\lambda$ *and* $0<\lambda<1$;

(B5) *there exists a constant* $m>0$ *such that* $Q_{h_0}(t,\xi)\ge m$*,* $(t,\xi)\in[0,\infty)\times[a,b]$.

Then every solution of the problem (2.1), (2.2) *is oscillatory in* G.

Proof: We prove that inequality (3.1) has no eventually positive solution if the conditions of Theorem 3.2 hold. Suppose $V(T)$ is an eventually positive solution of inequality (3.1). Then there exists a number $t_1\ge 0$ such that $V(g_h(t,\xi))>0$, $h=1,2,\dots,l$, for $t\ge t_1$. Thus we have

$$\left[V(t)-\sum_{s=1}^{r}\lambda_s(t)V(t-\rho_s)\right]'+p(t)V(t)$$
$$+\int_a^b Q_{h_0}(t,\xi)V(g_{h_0}(t,\xi))d\sigma(\xi)\le 0, \quad t\ge t_1. \tag{3.11}$$

By Theorem 2 of Fu and Zhang [4], we obtain that inequality (3.11) has no eventually solution, which contradicts the fact that $V(t)>0$ is a solution of inequality (3.11). □

The proof of the following theorem is similar to that of Theorem 3.2 by using Theorem 3 of Fu and Zhang [4].

Theorem 3.3: *Suppose that conditions* (B1), (B2), (B3) *and* (B5) *hold. If*

(B4)$'$ $\sum_{s=1}^{r}\lambda_s(t)\le 1$,

then every solution of the problem (2.1), (2.2) *is oscillatory in* G.

Example 3.1: Consider the system of parabolic differential equations

$$\begin{cases} \frac{\partial}{\partial t}[u_1(x,t)-u_1(x,t-\pi)] = \Delta u_1(x,t)+2\Delta u_1(x,t-\frac{3\pi}{2}) \\ \quad +\frac{1}{2}\Delta u_2(x,t-\frac{3\pi}{2})-\frac{3}{2}u_1(x,t) \\ \quad -\int_{-\pi}^{-\frac{\pi}{2}} 3u_1(x,t+\xi)d\xi-\int_{-\pi}^{-\frac{\pi}{2}} u_2(x,t+\xi)d\xi, \\ \frac{\partial}{\partial t}[u_2(x,t)-u_2(x,t-\pi)] = 2\Delta u_2(x,t)-\Delta u_1(x,t-\frac{3\pi}{2}) \\ \quad +\Delta u_2(x,t-\frac{3\pi}{2})-4u_2(x,t) \\ \quad -\int_{-\pi}^{-\frac{\pi}{2}} u_1(x,t+\xi)d\xi-\int_{-\pi}^{-\frac{\pi}{2}} 4u_2(x,t+\xi)d\xi, \\ \quad (x,t)\in(0,\pi)\times[0,\infty), \end{cases} \tag{3.12}$$

with the boundary condition

$$\frac{\partial}{\partial x}u_i(0,t)=\frac{\partial}{\partial x}u_i(\pi,t)=0, t\geq 0, i=1,2. \tag{3.13}$$

Here $n=1,\ m=2,\ r=1,\ d=1,\ l=1,\ \lambda_1(t)=1,\ \rho_1=\pi,\ a_1(t)=1,\ a_{111}(t)=2,$ $a_{121}(t)=\frac{1}{2}, \tau_1=\frac{3\pi}{2},\ p_1(x,t)=\frac{3}{2}, q_{111}(x,t,\xi)=3, q_{121}(x,t,\xi)=1, g_1(t,\xi)=t+\xi,$ $a_2(t)=2, a_{211}(t)=\frac{1}{2}, a_{221}(t)=1, p_2(x,t)=4, q_{211}(x,t,\xi)=1, q_{221}(x,t,\xi)=4,$ $a=-\pi, b=-\frac{\pi}{2}$. It is easy to see that $Q_1(t,\xi)=2, \eta_1(t,\xi)=t+\frac{\xi}{2}, t>\frac{3\pi}{2}$,

$$\lim_{t\to\infty}\ \inf\int_{g_1(t,b)}^{t}\int_a^b Q_1(s,\xi)d\xi ds = \lim_{t\to\infty}\ \inf\int_{t-\frac{\pi}{2}}^{t}\int_{-\pi}^{-\frac{\pi}{2}} 2d\xi ds=\frac{\pi^2}{2}>\frac{1}{e},$$

and

$$\lim_{t\to\infty}\ \inf\int_{\eta_1(t,b)}^{t}\int_a^b Q_1(s,\xi)d\xi ds = \lim_{t\to\infty}\ \inf\int_{t-\frac{\pi}{4}}^{t}\int_{-\pi}^{-\frac{\pi}{2}} 2d\xi ds=\frac{\pi^2}{4}>0.$$

Hence all the conditions of Theorem 3.3 are fulfilled. Then every solution of problem (3.12), (3.13) is oscillatory in $(0,\pi)\times[0,\infty)$. In fact, such a solution is $u_1(x,t)=\cos x\sin t$, $u_2(x,t)=\cos x\cos t$.

4 Oscillation of the Problem (2.1), (2.3)

In the domain Ω, we consider the following Dirichlet problem

$$\begin{cases} \Delta\omega(x)+\alpha\omega(x)=0 \text{ in } \Omega, \\ \omega(x)=0 \text{ on } \partial\Omega, \end{cases} \tag{4.1}$$

where α is a constant. It is well known that the least eigenvalue α_0 of problem (4.1) is positive and the corresponding eigenfunction $\varphi(x)$ is positive on Ω.

Theorem 4.1: *If the differential inequality*

$$\left[V(t)-\sum_{s=1}^{r}\lambda_s(t)V(t-\rho_s)\right]'+\alpha_0\sum_{j=1}^{d}A_j(t)V(t-\tau_j)+p(t)V(t)$$
$$+\sum_{h=1}^{l}\int_a^b Q_h(t,\xi)V(g_h(t,\xi))d\sigma(\xi)\le 0, \tag{4.2}$$

has no eventually positive solution, then every solution of problem (2.1)*,* (2.3) *is oscillatory in* G*.*

Proof: Suppose to the contrary that there is a nonoscillatory solution $u(x,t)=\{u_1(x,t),\ u_2(x,t),\dots,u_m(x,t)\}^T$ of problem (2.1), (2.3). We assume that $|\,u_i(x,t)\,|>0$ for $t\ge t_0\ge 0$, $i=1,2,\dots,m$. Let $\delta_i=\mathrm{sgn}u_i(x,t)$, $Z_i(x,t)=\delta_i u_i(x,t)$, then $Z_i(x,t)>0$, $(x,t)\in\Omega\times[t_0,\infty)$, $i=1,2,\dots,m$. From (H5) and (H7), there exists a number $t_1\ge t_0$ such that $Z_i(x,t)>0$, $Z_i(x,t-\rho_s)>0$, $Z_i(x,t-\tau_j)>0$ and $Z_i(x,g_h(t,\xi))>0$ in $\Omega\times[t_1,\infty)$, $i=1,2,\dots,m$; $s=1,2,\dots,r$; $j=1,2,\dots,d$; $h=1,2,\dots,l$.

Multiplying both sides of (2.1) by $\varphi(x)$ and integrating with respect to x over the domain Ω, we obtain

$$\frac{d}{dt}\left[\int_\Omega u_i(x,t)\varphi(x)dx-\sum_{s=1}^{r}\lambda_s(t)\int_\Omega u_i(x,t-\rho_s)\varphi(x)dx\right]=a_i(t)\int_\Omega \Delta u_i(x,t)\varphi(x)dx$$
$$+\sum_{k=1}^{m}\ \sum_{j=1}^{d}a_{ikj}(t)\int_\Omega \Delta u_k(x,t-\tau_j)\varphi(x)dx-\int_\Omega p_i(x,t)u_i(x,t)\varphi(x)dx \tag{4.3}$$
$$-\sum_{k=1}^{m}\ \sum_{h=1}^{l}\int_\Omega\int_a^b q_{ikh}(x,t,\xi)u_k(x,g_h(t,\xi))\varphi(x)d\sigma(\xi)dx,$$

where $t\ge t_1$, $i=1,2,\dots,m$.

Therefore, we have

$$\frac{d}{dt}\left[\int_\Omega Z_i(x,t)\varphi(x)dx-\sum_{s=1}^{r}\lambda_s(t)\int_\Omega Z_i(x,t-\rho_s)\varphi(x)dx\right]=a_i(t)\int_\Omega \Delta Z_i(x,t)\varphi(x)dx$$
$$+\sum_{j=1}^{d}a_{iij}(t)\int_\Omega \Delta Z_i(x,t-\tau_j)\varphi(x)dx+\sum_{k=1,k\ne i}^{m}\ \sum_{j=1}^{d}a_{ikj}(t)\frac{\delta_k}{\delta_i}\int_\Omega \Delta Z_k(x,t-\tau_j)\varphi(x)dx$$
$$-\int_\Omega p_i(x,t)Z_i(x,t)\varphi(x)dx-\sum_{h=1}^{l}\int_a^b\int_\Omega q_{iih}(x,t,\xi)Z_i(x,g_h(t,\xi))\varphi(x)dxd\sigma(\xi) \tag{4.4}$$
$$-\sum_{h=1}^{l}\ \sum_{k=1,k\ne i}^{m}\frac{\delta_k}{\delta_i}\int_a^b\int_\Omega q_{ikh}(x,t,\xi)Z_k(x,g_h(t,\xi))\varphi(x)dxd\sigma(\xi),$$

where $t \geq t_1$, $i = 1, 2, \ldots, m$.

Using Green's formula and (2.3), we have

$$\int_\Omega \Delta Z_i(x,t)\varphi(x)dx = \int_\Omega Z_i(x,t)\Delta\varphi(x)dx = -\alpha_0 \int_\Omega Z_i(x,t)\varphi(x)dx \leq 0, \tag{4.5}$$

and

$$\begin{aligned}\int_\Omega \Delta Z_k(x,t-\tau_j)\varphi(x)dx &= \int_\Omega Z_k(x,t-\tau_j)\Delta\varphi(x)dx \\ &= -\alpha_0 \int_\Omega Z_k(x,t-\tau_j)\varphi(x)dx,\end{aligned} \tag{4.6}$$

where $t \geq t_1$, $k = 1, 2, \ldots, m$; $j = 1, 2, \ldots, d$.

It follows from (4.4)-(4.6) that

$$\begin{aligned}&\frac{d}{dt}\left[\int_\Omega Z_i(x,t)\varphi(x)dx - \sum_{s=1}^{r}\lambda_s(t)\int_\Omega Z_i(x,t-\rho_s)\varphi(x)dx\right] \\ &\leq -\alpha_0 \sum_{j=1}^{d} a_{iij}(t)\int_\Omega Z_i(x,t-\tau_j)\varphi(x)dx \\ &+ \alpha_0 \sum_{j=1}^{d} \sum_{k=1,k\neq i}^{m} |\, a_{ikj}(t)\,| \int_\Omega Z_k(x,t-\tau_j)\varphi(x)dx - p_i(t)\int_\Omega Z_i(x,t)\varphi(x)dx \\ &- \sum_{h=1}^{l}\int_a^b q_{iih}(t,\xi)\int_\Omega Z_i(x,g_h(t,\xi))\varphi(x)dxd\sigma(\xi) \\ &+ \sum_{h=1}^{l} \sum_{k=1,k\neq i}^{m} \int_a^b \overline{q}_{ikh}(t,\xi)\int_\Omega Z_k(x,g_h(t,\xi))\varphi(x)dxd\sigma(\xi),\end{aligned} \tag{4.7}$$

where $t \geq t_1$, $i = 1, 2, \ldots, m$.

Setting

$$V_i(t) = \int_\Omega Z_i(x,t)\varphi(x)dx,\ t \geq t_1, i = 1, 2, \ldots, m,$$

we have

$$\left[V_i(t) - \sum_{s=1}^{r}\lambda_s(t)V_i(t-\rho_s)\right]' + \alpha_0 \sum_{j=1}^{d} a_{iij}(t)V_i(t-\tau_j)$$

$$
\begin{aligned}
&- \alpha_0 \sum_{j=1}^{d} \sum_{k=1,k\neq i}^{m} \mid a_{ikj}(t) \mid V_k(t-\tau_j) \\
&+ p_i(t)V_i(t) + \sum_{h=1}^{l} \int_a^b q_{iih}(t,\xi)V_i(g_h(t,\xi))d\sigma(x) \\
&- \sum_{h=1}^{l} \sum_{k=1,k\neq i}^{m} \int_a^b \overline{q}_{\ ikh}(t,\xi)V_k(g_h(t,\xi))d\sigma(\xi) \le 0,
\end{aligned}
\tag{4.8}
$$

where $t \ge t_1$, $i = 1, 2, \ldots, m$.

Let $V(t) = \sum_{i=1}^{m} V_i(t)$ for $t \ge t_1$. It follows from (4.8) that

$$
\begin{aligned}
&\left[V(t) - \sum_{s=1}^{r} \lambda_s(t)V(t-\rho_s)\right]' \\
&+ \alpha_0 \sum_{j=1}^{d} \left\{ \sum_{i=1}^{m} \left[a_{iij}(t)V_i(t-\tau_j) - \sum_{k=1,k\neq i}^{m} \mid a_{ikj}(t) \mid V_k(t-\tau_j) \right] \right\} \\
&+ p(t)V(t) + \sum_{h=1}^{l} \left\{ \sum_{i=1}^{m} \int_a^b [q_{ih}(t,\xi)V_i(g_h(t,\xi))d\sigma(\xi) \right. \\
&\left. - \sum_{k=1,k\neq i}^{m} \int_a^b \overline{q}_{\ ikh}(t,\xi)V_k(g_h(t,\xi))d\sigma(x) \Big] \right\} \le 0, \ t \ge t_1.
\end{aligned}
\tag{4.9}
$$

As in the proof of Theorem 3.1, with (4.9), we obtain

$$
\begin{aligned}
&\left[V(t) - \sum_{s=1}^{r} \lambda_s(t)V(t-\rho_s)\right]' + \alpha_0 \sum_{j=1}^{d} A_j(t)V(t-\tau_j) + p(t)V(t) \\
&+ \sum_{h=1}^{l} \int_a^b Q_h(t,\xi)V(g_h(t,\xi))d\sigma(\xi) \le 0, \ \ t \ge t_1,
\end{aligned}
$$

which shows that $V(t) = \sum_{i=1}^{m} V_i(t) > 0$ is a positive solution of the inequality (4.2). This is a contradiction.

By using Theorem 4.1, we have the following theorems.

Theorem 4.2: *If all conditions of Theorem* 3.1 *hold, then every solution of problem* (2.1), (2.3) *is oscillatory in G.*

Theorem 4.3: *If all conditions of Theorem* 3.2 *hold, then every solution of problem* (2.1), (2.3) *is oscillatory in G.*

Theorem 4.4: *If all conditions of Theorem* 3.3 *hold, then every solution of problem* (2.1), (2.3) *is oscillatory in* G.

Example 4.1: Consider the system of parabolic differential equations

$$\begin{cases} \frac{\partial}{\partial t}[u_1(x,t)-u_1(x,t-\pi)] = \Delta u_1(x,t)+\Delta u_1(x,t-\frac{3\pi}{2}) \\ \qquad -\Delta u_2(x,t-\frac{3\pi}{2})-7u_1(x,t) \\ \qquad -\int_{-\pi}^{-\frac{\pi}{2}} 5u_1(x,t+\xi)d\xi-\int_{-\pi}^{-\frac{\pi}{2}} 2u_2(x,t+\xi)d\xi, \\ \frac{\partial}{\partial t}[u_2(x,t)-u_2(x,t-\pi)] = \frac{3}{2}\Delta u_2(x,t)+\frac{1}{2}\Delta u_1(x,t-\frac{3\pi}{2}) \\ \qquad +2\Delta u_2(x,t-\frac{3\pi}{2})-u_2(x,t) \\ \qquad -\int_{-\pi}^{-\frac{\pi}{2}} u_1(x,t+\xi)d\xi-\int_{-\pi}^{-\frac{\pi}{2}} 3u_2(x,t+\xi)d\xi, \\ \qquad (x,t)\in(0,\pi)\times[0,\infty), \end{cases} \tag{4.10}$$

with boundary condition

$$u_i(0,t)=u_i(\pi,t)=0,\ t\geq 0,\ i=1,2. \tag{4.11}$$

Here $n=1$, $m=2$, $r=1$, $d=1$, $l=1$, $\lambda_1(t)=1$, $\rho_1=\pi$, $a_1(t)=1$, $a_{111}(t)=1$, $a_{121}(t)=-1$, $\tau_1=\frac{3\pi}{2}$, $p_1(x,t)=7$, $q_{111}(x,t,\xi)=5$, $q_{121}(x,t,\xi)=2$, $g_1(t,\xi)=t+\xi$, $a_2(t)=\frac{3}{2}$, $a_{211}(t)=\frac{1}{2}$, $a_{221}(t)=2$, $p_2(x,t)=1$, $q_{211}(x,t,\xi)=1$, $q_{221}(x,t,\xi)=3$, $a=-\pi$, $b=-\frac{\pi}{2}$. It is easy to see that all the conditions of Theorem 4.4 are met. Thus, all the solutions of the problem (4.10), (4.11) are oscillatory in $(0,\pi)\times[0,\infty)$. For instance, $u_1(x,t)=\sin x\cos t$, $u_2(x,t)=\sin x\sin t$ is such a solution.

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