Journal of Applied Mathematics and Stochastic Analysis, **16:3** (2003), 201-207.

ON THE TIME OF THE MAXIMUM OF BROWNIAN MOTION WITH DRIFT

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(Received October, 2002; Revised March, 2003)

The distribution of the time at which Brownian motion with drift attains its maximum on a given interval is obtained by elementary methods. The proof depends on a remarkable integral identity involving Gaussian distribution functions.
Keywords: Brownian Motion with Drift, Girsanov's Theorem, Integral Identity.
AMS (MOS) Subject Classification: 60J65, 60G17.

1 Introduction

The properties of the maximum value attained by Brownian motion on a given interval are well understood. Indeed, in the absence of drift, its distribution is easily obtained from the reflection principle; moreover the case of Brownian motion with drift can be reduced to the above through Girsanov's theorem, using an appropriate change of probability measure, see Karatzas and Shreve [4], p. 196.

The time at which the maximum is attained is a less familiar and somewhat more subtle object. First one needs to prove that such a time is almost surely unique. So, if B_t, $t \geq 0$ is standard Brownian motion on a suitable probability space and if we denote its running maximum by

$$\bar{B}_t = \max_{0 \leq s \leq t} B_s,$$

then

$$\theta_t = \sup\{s \leq t \ : B_s = \bar{B}_t\} = \inf\{s \leq t \ : B_s = \bar{B}_t\},$$

where the second equality holds almost surely, see Karatzas and Shreve [4], p. 102.

Next one can obtain the joint probability density of B_t, $\bar{B}_t$, θ_t:

$$f_{B_t,\bar{B}_t,\theta_t}(a,b,s) = \begin{cases} \frac{b(b-a)}{\pi(s(t-s))^{3/2}} e^{-(b-a)^2/2(t-s)} e^{-b^2/2s} & \text{if } b \geq a,\ b \geq 0,\ s \leq t, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

This can be established (with some effort) by completely elementary means using nothing more than the defining properties of the increments of Brownian motion, see Karatzas and Shreve [4], p. 101.

As a consequence of (1.1), the time of the maximum of standard Brownian motion on [0,t] follows an arcsine law, see Karatzas and Shreve [4], p. 102.

$$f_{\theta_t}(s) = \frac{1}{\pi\sqrt{s(t-s)}}, \quad 0 < s < t. \tag{1.2}$$

The formula that replaces (1.2) when a drift term is added to B_t is known, and is considerably more complex; it reads

$$f_{\theta_t}^{(\mu)}(s) = 2\left[\frac{1}{\sqrt{s}}\varphi(\mu\sqrt{s}) + \mu\Phi(\mu\sqrt{s})\right] \times \left[\frac{1}{\sqrt{t-s}}\varphi(\mu\sqrt{t-s}) - \mu\Phi(-\mu\sqrt{t-s})\right], \quad 0 < s < t \tag{1.3}$$

where φ and Φ denote the standard Gaussian density and distribution function respectively. The route through which (1.3) is identified as being the density of the time of the maximum of Brownian motion with drift is somewhat circuitous: that formula was in fact derived in Akahori [1] as the density of A_t, the time spent by Brownian motion with drift in $(0, \infty)$ up to instant t. This in turn is known to coincide with the density of θ_t in the presence of a drift because $(B_t,\ A_t)$ and (B_t,θ_t) have identical laws when $\mu = 0$. This fact is offered as an observation in Karatzas and Shreve [4], p. 425 after both laws have been obtained separately; the derivation of the joint law of $(B_t,\ A_t)$ involves excursion theory.

A probabilistic explanation for the above identity in law is given in Karatzas and Shreve [3] by means of path decomposition methods. Two alternative explanations are offered in Embrechts et al. [2]. One of these relies on the observation that both θ_t and A_t can be expressed in terms of hitting times of appropriate Brownian bridges; the other one depends on the properties of Brownian meanders.

To be sure, the articles described above provide a fascinating insight into fundamental questions of stochastic analysis; but to someone primarily interested in (1.3), none of these approaches can be described as direct. Moreover, the level of sophistication required is considerable. By contrast, this article offers a direct and straightforward derivation of (1.3).

Remark 1.1: When comparing formula (1.3) to Theorem 1.1 in Akahori [1] the reader should note that what is actually calculated there is the distribution of the time spent in $(-\infty, 0)$; also the author uses the notation Φ for the tail of the Gaussian distribution. Finally, (1.3) has the advantage of showing explicitly the invariance of $f_{\theta_t}^{(\mu)}$ under the combined transformation, $\mu \to -\mu$, $s \to t - s$.

2 Changing measure

The most natural method for establishing formula (1.3) consists in reducing the problem to a driftless one through a change of measure, exactly as is done when studying the distribution of the maximum value of Brownian motion. By Girsanov's theorem

(Karatzas and Shreve [4], p. 196) if B_s, $0 \le s \le t$ is standard Brownian motion on $(\Omega, \mathcal{F}, \mathrm{P})$ then $W_s = B_s + \mu s$ is itself standard Brownian motion on $(\Omega, \mathcal{F}, \mathrm{P}_\mu)$ where

$$dP_\mu = e^{-\mu B_t - \mu^2 t/2}\, dP,$$

or equivalently

$$dP = e^{\mu W_t - \mu^2 t/2} dP_\mu.$$

This results in the following identity for the probability density $f_{\theta_t}^{(\mu)}$ of the time of the maximum of $B_s + \mu s$, $0 \le s \le t$:

$$f_{\theta_t}^{(\mu)}(s) = e^{-\mu^2 t/2} \int_{-\infty}^{\infty} e^{\mu a} f_{B_t,\theta_t}(a, s) da. \tag{2.1}$$

Hence our first task must be to extract f_{B_t,θ_t} from the trivariate density (1.1). Although the calculation is straighforward, the result does not seem to appear in the standard treatises; neither is it particularly simple.

Proposition 2.1: *The joint density of standard Brownian motion and the time of its maximum up to instant t is given for $0 < s < t$ by*

$$f_{B_t,\theta_t}(a, s) = \frac{a}{\pi t^2}\sqrt{\frac{s}{t-s}} e^{-a^2/2s} + \sqrt{\frac{2}{\pi}} \frac{1}{t^{3/2}} e^{-a^2/2t} (1 - \frac{a^2}{t}) \Phi(-a\sqrt{\frac{t-s}{st}})$$

if $a \ge 0$, and

$$f_{B_t,\theta_t}(a, s) = \frac{-a}{\pi t^2}\sqrt{\frac{t-s}{s}} e^{-a^2/2(t-s)} + \sqrt{\frac{2}{\pi}} \frac{1}{t^{3/2}} e^{-a^2/2t} (1 - \frac{a^2}{t}) \Phi(a\sqrt{\frac{s}{t(t-s)}})$$

if $a \le 0$.

Proof: Clearly

$$f_{B_t,\theta_t}(a, s) = \int_{a^+}^{\infty} f_{B_t,\bar{B}_t,\theta_t}(a, b, s) db,$$

where $a^+ = \max(a, 0)$ denotes the positive part of a. The integration becomes easy if the exponent in (1.1) is written in the form

$$-\frac{1}{2}\{\frac{a^2}{t} + \frac{(b - as/t)^2}{\sigma^2}\},$$

with $\sigma^2 = s(t-s)/t$. Standard manipulations lead to

$$\begin{aligned} f_{B_t,\theta_t}(a, s) = {} & \frac{e^{-a^2/2t}}{\pi t\sqrt{s(t-s)}} (a^+ - a + \frac{as}{t}) e^{-(a^+ - as/t)^2/2\sigma^2} \\ & + \sqrt{\frac{2}{\pi}} \frac{1}{t^{3/2}} e^{-a^2/2t} (1 - a^2/t) \Phi(-\frac{a^+ - as/t}{\sigma}) \end{aligned}$$

which is equivalent to the stated result. □

The next step is to combine (2.1) and Proposition 2.1. This yields

$$\begin{aligned} e^{\mu^2 t/2} f_{\theta_t}^{(\mu)}(s) = & \frac{s}{\pi t^2 \sqrt{s(t-s)}} \int_0^\infty e^{\mu a} a e^{-a^2/2s}\, da \\ & + \sqrt{\frac{2}{\pi}} \frac{1}{t^{3/2}} \int_0^\infty e^{\mu a} e^{-a^2/2t} (1 - \frac{a^2}{t}) \Phi(-a\sqrt{\frac{t-s}{st}})\, da \\ & - \frac{1}{\pi t^2} \sqrt{\frac{t-s}{s}} \int_{-\infty}^0 e^{\mu a} a e^{-a^2/2(t-s)}\, da \\ & + \sqrt{\frac{2}{\pi}} \frac{1}{t^{3/2}} \int_{-\infty}^0 e^{\mu a} e^{-a^2/2t} (1 - \frac{a^2}{t}) \Phi(a\sqrt{\frac{s}{t(t-s)}})\, da. \end{aligned} \tag{2.2}$$

The first and third terms are easily integrated to give

$$\begin{aligned} \frac{1}{\pi t^2 \sqrt{s(t-s)}} \Big[& s^2 + (t-s)^2 + \mu\sqrt{2\pi} s^{5/2} e^{\mu^2 s/2} \Phi(\mu\sqrt{s}) \\ & - \mu\sqrt{2\pi}(t-s)^{5/2} e^{\mu^2 (t-s)/2} \Phi(-\mu\sqrt{t-s}) \Big]. \end{aligned} \tag{2.3}$$

However, the other two terms do not appear to lend themselves to an explicit evaluation, so that we are left with a frustratingly untidy formula for $f_{\theta_t}^{(\mu)}$, a far cry from the compact (1.3). We develop in the next section an integral identity which resolves this conundrum.

3 The Key Integral Identity

Theorem 3.1: *The following holds whenever $\alpha, \beta, \mu \in \mathbf{R}$ and $\alpha\beta > 0$:*

$$\begin{aligned} & \int_0^\infty e^{-\alpha\beta x^2/2} \left\{ e^{-\mu x} \Phi(-\alpha x) + e^{\mu x} \Phi(-\beta x) \right\} dx \\ & \qquad = \sqrt{\frac{2\pi}{\alpha\beta}} e^{\mu^2/2\alpha\beta} \Phi\left(\frac{\mu}{\sqrt{\beta(\alpha+\beta)}} \right) \Phi\left(\frac{-\mu}{\sqrt{\alpha(\alpha+\beta)}} \right). \end{aligned}$$

Proof: Rewrite the integral as

$$\begin{aligned} & e^{\mu^2/2\alpha\beta} \left\{ \int_0^\infty e^{-\alpha\beta(x+\mu/\alpha\beta)^2/2} \Phi(-\alpha x) dx + \int_0^\infty e^{-\alpha\beta(x-\mu/\alpha\beta)^2/2} \Phi(-\beta x) dx \right\} \\ & = e^{\mu^2/2\alpha\beta} \left\{ \int_{\frac{\mu}{\sqrt{\alpha\beta}}}^\infty e^{-\frac{u^2}{2}} \Phi(-u\sqrt{\frac{\alpha}{\beta}} + \frac{\mu}{\beta}) \frac{du}{\sqrt{\alpha\beta}} + \int_{-\frac{\mu}{\sqrt{\alpha\beta}}}^\infty e^{-\frac{u^2}{2}} \Phi(-u\sqrt{\frac{\beta}{\alpha}} - \frac{\mu}{\alpha}) du \right\} \\ & = \frac{e^{\mu^2/2\alpha\beta}}{\sqrt{2\pi\alpha\beta}} \left\{ \int_{\frac{\mu}{\sqrt{\alpha\beta}}}^\infty du \int_{-\infty}^{-u\sqrt{\frac{\alpha}{\beta}} + \frac{\mu}{\beta}} e^{-(u^2+v^2)/2} dv \right. \\ & \qquad \left. + \int_{-\frac{\mu}{\sqrt{\alpha\beta}}}^\infty du \int_{-\infty}^{-u\sqrt{\frac{\beta}{\alpha}} - \frac{\mu}{\alpha}} e^{-(u^2+v^2)/2} dv \right\}. \end{aligned} \tag{3.1}$$

The two integrals in (3.1) are over the regions A and B represented in Figure 1.

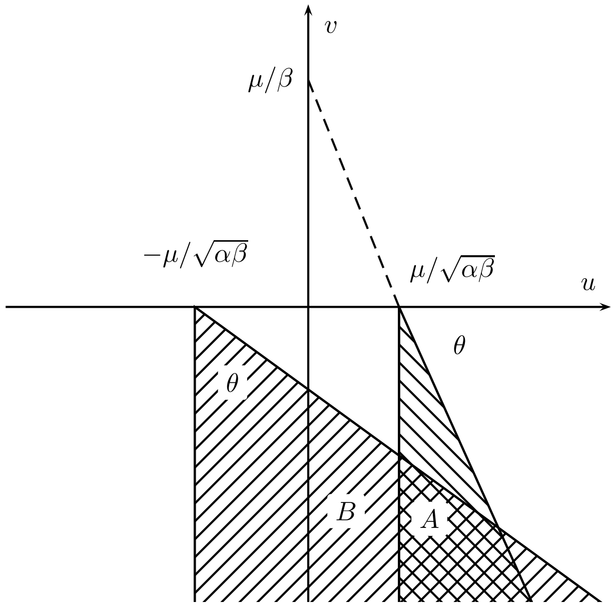

Figure 1: The regions A and B.

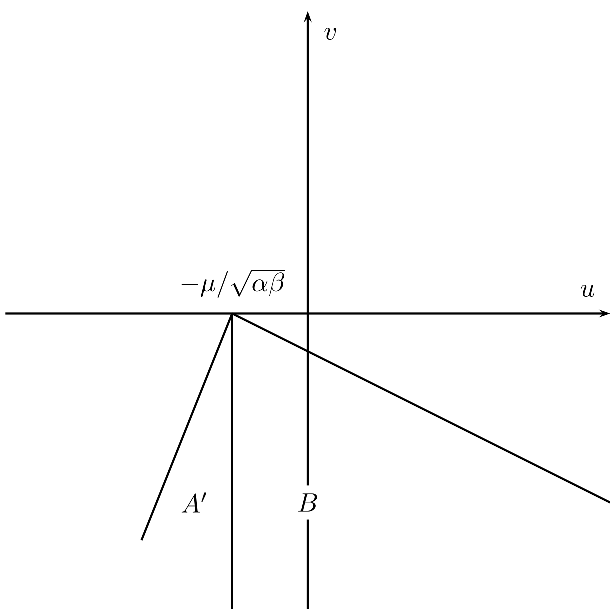

Figure 2: The regions A' and B.

In the above the angle θ is determined by

$$\cos\theta = \sqrt{\frac{\beta}{\alpha+\beta}}, \quad \sin\theta = \sqrt{\frac{\alpha}{\alpha+\beta}}. \tag{3.2}$$

In the view of the symmetry of the integrand, the region A can be replaced by its reflection A' in the vertical axis, see Figure 2.

Finally one can take advantage of the invariance of the integrand under rotations to replace the region of integration $A' \cup B$ by C characterised in Figure 3.

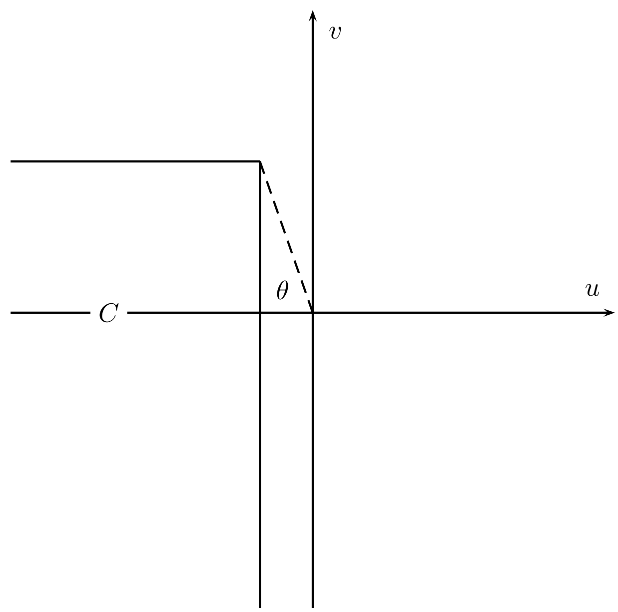

Figure 3: The region C.

The apex of the region C has coordinates

$$(-\mu/\sqrt{\alpha(\alpha+\beta)}, \mu/\sqrt{\beta(\alpha+\beta)}),$$

see (3.2); the result follows by inspection in view of the shape of the region of integration. $\square$

The above identity is remarkable in that the two parts of the integrand are not separately capable of such an explicit integration; indeed the prospects for simplification look bleak until one realizes the orthogonality between the boundaries of A' and B.

Theorem 3.1 allows us to evaluate (2.2); indeed if we denote by $I(\alpha, \beta, \mu)$ the integral in theorem 1, a moment's reflection shows that the second and fourth terms in (2.2) are nothing but

$$\sqrt{\frac{2}{\pi}}\frac{1}{t^{\frac{3}{2}}}\left\{ I\left(\sqrt{\frac{s}{t(t-s)}}, \sqrt{\frac{t-s}{st}}, \mu\right) - \frac{1}{t}\frac{\partial^2}{\partial\mu^2} I\left(\sqrt{\frac{s}{t(t-s)}}, \sqrt{\frac{t-s}{st}}, \mu\right)\right\}. \tag{3.3}$$

If suffices now to use (2.2), (2.3),(3.3) and Theorem 3.1 to obtain, after a routine regrouping of terms of a similiar nature:

Theorem 3.2: *The probability density of the time of the maximum of* $B_s + \mu s$ *over [0,t] is given by formula (1.3).*

Acknowledgements: It is a pleasure to thank Olivier Fossati for discussions on the subject matter of this article and Brien Nolan for help with the diagrams.

References

[1] Akahori, J., Some formulae for a new type of path-dependent option, *Ann. Applied Prob.* **5** (1995), 383–388.

[2] Embrechts, P., Rogers, L.C.G. and Yor, M., A proof of Dassios' representation of the α-quantile of Brownian motion with drift, *Ann. Applied Prob.* **5** (1995), 757–767.

[3] Karatzas, I. and Shreve S.E., . A decomposition of the Brownian path, *Statist. Probab. Lett.* **5** (1987), 87–94.

[4] Karatzas, I. and Shreve S.E., *Brownian Motion and Stochastic Calculus*, Springer, New York 1988.